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Mathematical Structures of Jeong Yag-yong's Gugo Wonlyu

丁若鏞의 算書 勾股源流의 數學的 構造

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Dedicated to Professor Lee Chang Ku (後山 李昌九) on his eightieth birthday with the deepest gratitude for his continuing encouragement

Since Jiuzhang Suanshu, the main tools in the theory of right triangles, known as Gougushu in East Asia were algebraic identities about three sides of a right triangle derived from the Pythagorean theorem. Using tianyuanshu up to siyuanshu, Song–Yuan mathematicians could skip over those identities in the theory. Chinese Mathematics in the 17–18th centuries were mainly concerned with the identities along with the western geometrical proofs. Jeong Yag-yong (1762–1836), a well known Joseon scholar and writer of the school of Silhak, noticed that those identities can be derived through algebra and then wrote Gugo Wonlyu (勾股源流) in the early 19th century. We show that Jeong reveals the algebraic structure of polynomials with the three indeterminates in the book along with their order structure. Although the title refers to right triangles, it is the first pure algebra book in Joseon mathematics, if not in East Asia.

Keywords: Jeong Yag-yong, Gugo Wonlyu, polynomials of three indeterminates, algebraic structure, order structure, lexicographic order, 丁若鏞(1762–1836), 勾股源流, 三元多項式, 代數的構造, 順序構造, 사전식 순서.

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1 Introduction

Since ancient times, right triangles have been used in various areas of real life, e.g., surveying and astronomical observation in East Asia besides mathematics.

We first briefly review the history of the theory of right triangles in East Asia for our motivation of this paper.

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The theory of right triangles was formalized in the last chapter Gouguzhang (勾股章) of Jiuzhang Suanshu (九章筭術). The chapter begins with the Pythagorean theorem, namely $a^2 + b^2 = c^2$, where *in this paper*, a, b, c *always* denote the shorter base, gou (勾), longer height, gu (股) and the hypotenuse, xian (弦) of a right triangle, respectively. In the East traditional mathematics, isosceles right triangles were considered as right triangles obtained by halving squares by diagonals. Thus $a < b$ is assumed. After the theorem, actual problems of the type jinyou (今有) given with specific numerical assumptions were solved essentially by the identities about $\{a, b, c\}$ i.e., those with the condition $a^2 + b^2 = c^2$, like $(c - b)(c + b) = a^2$, $c^2 - (b - a)^2 = 2ab$ and $2(c - a)(c - b) = (a + b - c)^2$. For the sake of brevity, identities about $\{a, b, c\}$ are called *simply identities* in what follows. As other mathematics books of East mathematics, Jiuzhang Suanshu does not include proofs for the identities. It contains only one general quadratic equation in Problem 20 although the second case of the above identities eventually leads to a general quadratic equation.

Wang Xiaotong (王孝通, fl. 7th C.) used right triangles in his Jigu Suanjing (緝古算經) to obtain general cubic equations without any explanations how to construct them. Zhang Dunren (張敦仁, 1754–1834) used tianyuanshu (天元術) in his Jigu Suanjing Xicao (緝古算經細艸, 1813) to Jigu Suanjing to construct the equations in the book, and Joseon mathematician Nam Byeong-gil (南秉吉) used jiegenfang (借根方) in his Jibgo Yeondan (緝古演段).

Yang Hui (楊輝) introduced ten terms in his Xiangjie Jiuzhang Suanfa (詳解九章算法, 1261) as follows:

gougujiao (勾股較, $b - a$), gouxianjiao (勾弦較, $c - a$), guxianjiao (股弦較, $c - b$),
gouguhe (勾股和, $b + a$), gouxianhe (勾弦和, $c + a$), guxianhe (股弦和, $c + b$),
xianjiaohe (弦較和, $c + (b - a)$), xianhehe (弦和和 $c + (b + a)$)
xianhejiao (弦和較, $(a + b) - c$), xianjiaojiao (弦較較, $c - (b - a)$).

The sum (difference, resp.) of five terms ending with he (和) (jiao (較), resp.) is called wuhe (五和) (wujiao (五較), resp.).

These terminologies were already used before Yang Hui's book for one can find them in Ceyuan Haijing (測圓海鏡, 1282) of Li Ye (李治, 1192–1279) which was completed in 1248 and used tianyuanshu to find the diameter of the inscribed circle in the given right triangle.

Besides Ceyuan Haijing, Zhu Shijie (朱世傑) first applied tianyuanshu to solve a right triangle in Suanxue Qimeng (算學啓蒙, 1299). Zhu put two problems with $a+c$ and $b + c$ which are obtained by solving systems of linear equations and obtained a, b, c by the identity $2(a + c)(b + c) = (a + b + c)^2$ in the first problem, where he assumed the validity of the identity as in Jiuzhang Suanshu. He then constructed the equation for the problem by tianyuanshu and the Pythagorean theorem without referring to the identity in the second problem. In Siyuan Yujian (四元玉鑑, 1303), Zhu chose problems of the right triangles to explain his theory of tianyuanshu up to siyuanshu. He set four problems in the introductory examples (四象細草假令之圖) from tianyuanshu through siyuanshu. The examples are the only part with full explanations in the whole book. These four problems deal with right triangles. Suanxue Qimeng and Siyuan Jujian are known to be the most important books about constructing equations in the history of Chinese mathematics.

Suanxue Qimeng was completely lost during the Ming dynasty (1368–1644). It was brought into Joseon in the first half of the 15th century and then into Japan in the last decade of the 16th century through Joseon. It plays the most important role for the development of mathematics in both countries.

Extending the results in Suanxue Qimeng in Joseon, Hong Jeongha (洪正夏, 1684–?) established the most advanced theory of right triangles in Gu-il jib (九一集, 1713–1724) and so did Yu Suseok (劉壽錫, fl. 18th C.) in his Gugo Sulyo (勾股術要) [2].

Since tianyuanshu was discarded and the western mathematics were introduced into China, Chinese mathematicians in the 17th century gave more attentions to the validity of identities in the theory of right triangles. The Chinese translation of the first six chapters of Euclid's Elements was published in 1607 by Ricci (利瑪竇, 1552–1610) and Xu Guangqi (徐光啓, 1562–1633). Using the Euclidean geometry and illustrative diagrams like those in gouguyuanfangtu (勾股圓方圖) of Zhoubi Suanjing (周髀算經), Xu Guangqi included the strict proofs for the identities used in his Gouguyi (勾股義). We should point out that his problems in the book are given with specific numbers, indeed always $a : b : c = 3 : 4 : 5$, but his proofs are applicable to any Pythagorean triples. The right triangles with the above ratio are called zhenggougo (正勾股). We recall that an interval in the Elements refer to an *arbitrary* number. Further, as mentioned in the above second identity of Jiuzhang Suanshu, Xu used identities to obtain the sum or difference of two numbers and their multiplications. Xu's format of presentations was retained in Gougu Chanmei (勾股闡微) [6] of Yang Zuomei (楊作枚) and Mei Wending (梅文鼎, 1633–1721), and Gougu Juyu (勾股舉隅) of Mei Wending, compiled by his grandson Mei Juecheng (梅瑴成, 1681–1763), and then eventually in Shuli Jingyun (數理精蘊, 1723).

There are two parts dealing with right triangles in Shuli Jingyun. The first part is in Chapter 12 and 13 which extends the results in Mei's books. In particular, the theory of right triangles using the ten items and identities are dealt in sections gouguxianhejiao xiangqiufa (勾股弦和較相求法) and gouguji yu gouguxianhejiao xiangqiufa (勾股積與勾股弦和較相求法) together with a few problems in Chapter 37. The second part deals with equations via jiegenfang in Chapter 35–36 which is a counter part of tianyuanshu in the theory of right triangles. As is well known, tianyuanshu represents rational polynomials, i.e., polynomials with terms with negative powers so that it is much more versatile than jiegenfang (see Problem 23 and 24 in Chapter 36 and Gu-il jib). A suitable choice of the unknown, in this case one of a, b, c reduces the processes of constructing equations but they chose mostly a as gen (根) and explained the processes in words contrary to those with the calculating rod expressions in tianyuanshu. Thus one can conclude that tianyuanshu is much better than jiegenfang in constructing equations.

In the present school mathematics, the algebraic identities can be easily proved by expansions and factorizations. For example,

$$2(c-a)(c-b) = 2(c^2 + ab - ac - bc) = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc = (a+b-c)^2.$$

Comparing the above algebraic proof with the proofs in Problem 14 in Gouguyi and Problem 10 of Chapter 12 in Shuli Jingyun, one can easily deduce that the algebraic proof with symbols or letters a, b, c is much more succinct than geometrical proof. Using tinyuanshu up to siyuanshu in Siyuan Yujian, Zhu Shijie could manipulate any multiplications of polynomials of up to four indeterminates but did not deal with their factorizations as all the mathematicians in East Asia.

Jeong Yag-yong (丁若鏞, 1762–1836) is one of the most famous scholar and writer of the school of Silhak (實學), literally practical learning in Joseon. He wrote a huge amount of books, notably Gyeongse Yupyo (經世遺表), Mokmin Simseo (牧民心書) and Heumheum Sinseo (欽欽新書) among others during the 18 years (1801–1818) of his exile related to his connection to Catholic belief in Sinyu Saok (辛酉邪獄) in 1801.

Jeong Yag-yong also wrote a mathematics book, Gugo Wonlyu (勾股源流) [3, 4]. We doubt that Jeong studied Suanxue Qimeng, more precisely tianyuanshu and recall that Siyuan Yujian was available to Joseon mathematicians after 1850 only through Siyuan Yujian Xicao (四元玉鑑細草, 1836) of Luo Shilin (羅士琳, 1784–1853). Thus his reference books should be limited, probably to Shuli Jingyun and Gougu Juyu. He noticed that the fundamentals, or wonlyu (源流) in the theory of right triangles are the identities, or relations between three sides and that they can be obtained by algebraic processes. To do so, he used gou, gu, xian themselves as a, b, c to form polynomials of indeterminates a, b, c and then obtained numerous identities by expansions and factorizations.

The purpose of this paper is to investigate mathematical structures used in the process to have those identities. Besides the algebraic structure of polynomials of three indeterminates, Jeong Yag-yong used their order structure through the inequalities and theory of combinations.

The reader may find all the Chinese sources of this paper in ZhongGuo KeXue JiShu DianJi TongHui ShuXueJuan (中國科學技術典籍通彙 數學卷) [1] and ZhongGuo LiDai SuanXue JiCheng (中國歷代算學集成) [5], and hence they will not be numbered as an individual reference.

2 Mathematical structures of Gugo Wonlyu

The book Gugo Wonlyu consists of four chapters and each chapter begins with the same heading, gugo wonlyu with the title of the book.

Jeong chose gouguxianhejiao xiangqiufa (勾股弦和較相求法) of the Shuli Jingyun as the title of the first chapter. He first quoted the ten terms of Yang's Xiangjie Jiuzhang Suanfa and then the Pythagorean theorem. For a, b, c, he first included all possible relations to obtain them with the ten terms together with wuhe and wujiao, called the twelve terms in the sequel. Jeong then included the identities for the twelve terms which result from the difference (相減) and sum with a, b, c and the twelve terms. Here the difference, sang-gam, or xiangjian (相減) *means* subtracting a smaller number from a larger number and hence sang-gam of x, y is $|x - y|$. In all, the identities except the Pythagorean theorem in the first chapter deal with sums and differences of three sides and the twelve terms, and hence involve the linear homogeneous equations of three sides. The first chapter is a complete extension of the preliminary in Gougu Juyu. Thus, Jeong's gouguxianhejiao xiangqiufa is completely different from that in Shuli Jingyun. We don't know the reason why Jeong chose the title but he probably interpreted the title literally. The important part of the chapter is that he represented linear polynomial $p(a, b, c)$ of three indeterminates using gou, gu and xian and then their algebraic operations as the following quotation indicates:

> 弦較較減勾 卽股弦較 減於股 卽二股少勾弦和 若相減恰盡 卽二股與勾弦和等

It says that $\{c-(b-a)\}-a = c-b$, $b-\{c-(b-a)\} = 2b-(c+a)$ and that if the latter is 0, then $2b = c + a$. This implication means that for two polynomials P, Q, $P = Q$ iff $P - Q = 0$. This fact is used in the last stage on constructing equations via tianyuanshu. Further, Jeong also had the condition $2c = 2a + b$ together with $2b = c + a$ in the first chapter and each of them is equivalent to the fact that the right triangle is a zhenggougu. Except these, every identity except the Pythagorean theorem in this chapter holds for *any* numbers regardless Pythagorean triples. Thus the identities are not Pythagorean identities but simply algebraic identities, and hence they reveal simply the algebraic structure of additions and subtractions of polynomials.

We quote the following:

> 勾股和與弦較和相減 卽一弦二勾之較
> 勾股和多 卽爲二勾內少一弦 弦較和多 卽爲一弦內少二勾

The difference of $a + b$ and $c + (b - a)$ is the difference of c and $2a$. Thus If

$c + (b - a) \leq a + b$, then the difference is $2a - c$ and if $a + b < c + (b - a)$, then the difference is $c - 2a$. We recognize that Jeong defined $P(a, b, c) \leq Q(a, b, c)$ for two Pythagorean polynomials $P(a, b, c)$ and $Q(a, b, c)$ if the inequality $P(x, y, z) \leq Q(x, y, z)$ holds for all Pythagorean triples (x, y, z) and we will discuss the detail of the order structure in Gugo Wonlyu in the next section. The first inequality in the above quotation holds for any zhenggougu but not for the triple $(5, 12, 13)$. We must point out that inequalities between polynomials, even more those between polynomials of three variables had never appeared in Joseon mathematics books, probably neither in traditional East Asian mathematical literatures. Furthermore, the basic field for the East Asian mathematics is the field $\mathbb{Q}$ of rational numbers and indeed the ordered field in the present context so that the trichotomy law holds and hence there have never been any difficulties on the differences, xiangjian. As the above quotation indicates, Jeong noticed that the trichotomy no longer holds for polynomials. Moreover, Jeong had two solutions for this problem. We recall that almost all traditional mathematicians in East Asia could not accept multiple solutions for a problem. We recall that the problem given with the area and sum of two sides of a rectangle were solved by two separate problems for each side.

Furthermore, Jeong arranged the sums and differences in a kind of lexicographic order with a fixed order on the set of three sides and their 12 terms of a right triangle, so that his presentation is in perfect order and hence systematic.

3 Mathematical structures of homogeneous quadratic polynomials

The title for the remaining part of the book is gouguxianmiji xiangqiufa (勾股弦冪積相求法) and divides into three chapters. This part deals with homogeneous polynomials of degree 2 about gou, gu and xian and hence squares (冪) of the ten terms together with gou, gu and xian and product (積) of two terms among them.

As we mentioned in the previous section, the linear identities were widely used in some Chinese books and the representation of polynomials about gou, gu and xian can be found in numerous problems in Chapter 12 and 13 of Shuli Jingyun as follows in problem 2 of Chapter 13

> 蓋勾股弦總和爲一勾一股一弦之共數 今加勾弦較
> 是於勾數加勾弦較 卽又得一弦矣 故爲兩弦一股也

It says that $(a + b + c) + (c - a) = \{a + (c - a)\} + b + c = c + (b + c) = 2c + b$.

Until the last decade of the 18th century when Song–Yuan mathematics, in particular Siyuan Yujian, were restored in China, Chinese mathematicians could not handle with multiplication of polynomials of more than two indeterminates in the setting of algebra but only in the geometrical setting like areas of rectangles or volumes of rectangular cuboids.

Jeong Yag-yong introduced the multiplication of Pythagorean polynomials in Gugo Wonlyu which is the most important first step to the study of the ring of polynomials with three indeterminates. Although he left off his study at quadratic polynomials, his study can be easily extended to the study of the commutative ring of polynomials, because he used all the properties of the ring like commutative law and distributive law in the three chapters. Further, Jeong also used factorizations as a main tool to obtain identities and the order structure as in the first chapter.

The second chapter begins with the area $A = \frac{ab}{2}$ of a right triangle (勾股積) and the relations

$$\begin{aligned} A &= \frac{\{c+(b-a)\}\{c-(b-a)\}}{4} = \frac{\{(a+b)+c\}\{(a+b)-c\}}{4} \\ &= \frac{(a+b)^2-c^2}{4} = \frac{c^2-(b-a)^2}{4}, \end{aligned}$$

and then adds the well known

$$a^2 = (c+b)(c-b), b^2 = (c+a)(c-a),$$
$$c^2 = (a+b)^2 - 4\Big(\frac{ab}{2}\Big) = (a-b)^2 + 4\Big(\frac{ab}{2}\Big)$$

together with $a^2b^2 = (ab)^2$, $a^2c^2 = (ac)^2$, $b^2c^2 = (bc)^2$ as a preliminary.

Jeong then included the main body of the book, namely the algebraic identities of homogeneous quadratic polynomials. For the systematic presentation, he took the set of elements 弦和和, 弦和較, 弦較和, 弦較較, 勾股和, 勾股較, 勾弦和, 勾弦較, 股弦和, 股弦較, 勾, 股, 弦 as the set of letters with the given order in it, then he arranged two letters words in the lexicographic order together with pairs of two letters words. Using this order, the multiplications of two letters and their sums and differences for pairs of two letters words, Jeong obtained identities from $(a+b+c)^2$ to $c^2 \pm (c-b)^2$, altogether $1{,}547$ items.

In the following, we quote a number of examples in Gugo Wonlyu how Jeong Yag-yong obtained the identities and then we show his mathematical perceptions in the setting of mathematical structures.

We begin with the first item.

Example A 弦和和冪 卽弦冪 勾股相乘積 勾弦相乘積 股弦相乘積 各二

It says that $(a+b+c)^2 = 2(c^2+ab+ac+bc)$.

Instead of the well used identity $(a+b+c)^2 = 2(a+c)(b+c)$ in Gougushu, Jeong included the above identity for the *factorization* by taking out a common factor as the following examples of factorizations show.

Example B 弦和和冪與 弦較和乘勾弦較積相加 卽弦乘股弦和三倍 多勾乘勾股和 相減 卽勾乘弦及股弦和二倍 多股乘弦和和,

that is,

$$(a+b+c)^2+(c+b-a)(c-a)=3c(c+b)+a(a+b),$$

and

$$(a+b+c)^2-(c+b-a)(c-a)=2a\{c+(b+c)\}+b(a+b+c).$$

For the sum, by expansion one has

$$(a^2+b^2+c^2+2ab+2ac+2bc)+(a^2+c^2-ab-2ac+bc)=(3c^2+3bc)+(a^2+ab)=3c(c+b)+a(a+b),$$

where the Pythagorean theorem is used to take out the common factor $3c$, and a is the common factor of the remaining part. For the difference, by expansion, one has

$$(2ab+2ac+2ac)+(b^2+ab+bc)=2a\{c+(b+c)\}+b(a+b+c).$$

The identity for the difference is always true, in other words, we don't use any property of right triangles. There are numerous examples like this one.

Example C 弦和較(冪)乘弦較較積與 勾股和乘股弦較積相加 卽股弦較乘三股一勾相減 卽股弦較乘勾股較,

which says that

$$(a+b-c)(c-b+a)+(a+b)(c-b)=(c-b)(3b+a),$$
$$(a+b-c)(c-b+a)-(a+b)(c-b)=(c-b)(b-a).$$

For the sum, one has

$$\begin{aligned}&(a^2-b^2-c^2+2bc)+(-b^2+bc+ac-ab)\\&=(-3b^2+3bc)+(ac-ab)=3b(c-b)+a(c-b)\\&=(c-b)(3b+a),\end{aligned}$$

where $c^2-a^2=b^2$ is used. For the difference, we have

$$\begin{aligned}&(a^2-b^2-c^2+2bc)-(-b^2+bc+ac-ab)\\&=(-b^2+bc)-(ac-ab)=b(c-b)-a(c-b)\\&=(c-b)(b-a).\end{aligned}$$

We note that in the above quotation, mi (冪) was mistakenly put by the unknown transcriber in the whole items like this example so that in the following, we will omit them in our paper.

By the above three examples, we can conclude that Jeong did have the perfect perception about the factorizations of polynomials by taking out common factors including variables because all the identities in Gugo Wonlyu can be obtained by the factorizations.

We now discuss the differences of two polynomials with their inequalities. In the above Example B, C, the differences of the two polynomials, say P and Q with $P+Q$, the differences are given by $P-Q$, i.e., $P>Q$ in the sense of pointwise order for those polynomials discussed in Section 1. Indeed, for Example B, it is clear from $c+b-a, c-a<a+b+c$ and for Example C, it is not the case but $P-Q=(c-b)(b-a)>0$ and hence one can conclude $P>Q$. We quote the following two examples:

Example D 弦和較乘勾弦較積與 弦較較乘勾股和積相加 卽弦乘弦和較二倍 多勾乘弦較較 相減 卽勾冪二倍 多勾乘弦和較

It says that $(a+b-c)(c-a)+(c-b+a)(b+a) = 2c(a+b-c)+a(c-b+a)$ and that $(c-b+a)(b+a)-(a+b-c)(c-a) = 2a^2+a(a+b-c)$. One has the identity for the sum by expansions, $a^2+b^2 = c^2$ and factorizations. The difference is not the type of $P-Q$ but the type of $Q-P$. Indeed, the final result shows that $Q-P > 0$ as in Example C.

Example E 弦較較乘勾弦和積與 勾股和乘勾弦較積相加 卽勾乘弦三倍 多勾股較冪 相減 卽弦乘三弦一勾 少股乘股弦和二倍 反減 卽多少相反

That is, $(c-b+a)(c+a)+(b+a)(c-a) = 3ac+(b-a)^2$ and $(c-b+a)(c+a)-(b+a)(c-a) = c(3c+a)-2b(b+c)$. For the reverse subtraction, the signs $+, -$ should be exchanged in the polynomial. For the sum, Jeong used $b^2+a^2-2ab = (b-a)^2$. For the difference, the right hand side is positive for zhenggougu but is negative for the Pythagorean triple $(11, 60, 61)$. Thus the two polynomials $(c-b+a)(c+a)$ and $(b+a)(c-a)$ are non comparable. Thus the differences for those non comparable P, Q are either $P-Q$ or $Q-P = -(P-Q)$ for the triples as above and hence Jeong introduced the concept, the reverse subtraction, ban-gam or Chinese fanjian (反減).

Remark

1) It is extremely difficult to solve inequalities $P(x, y, z) \geq 0$ even for the present mathematics and hence it is almost impossible task for Jeong to solve inequalities except those two cases mentioned in the above. Thus he made many wrong conclusions in xiangjian. We quote an example among others.

> 弦較和乘勾股較積與 勾股和乘勾弦和積相加 卽弦乘股弦和倍之 少股乘勾股和
> 相減 卽勾乘股弦和倍之 少股乘勾股較

For the sum, the author used $2a^2+b^2 = 2c^2-b^2$ and factorizations to obtain $(c+b-a)(b-a)+(b+a)(c+a) = 2c(b+c)-b(a+b)$. For the difference, he obtained $(b+a)(c+a)-(c+b-a)(b-a) = 2a(b+c)-b(b-a)$. Clearly, the subtraction is positive in zhenggougu but negative at the Pythagorean triple $(11, 60, 61)$. Thus the difference must be fanjian.

2) As in the first chapter, Jeong also indicated the cases of the final result $P-Q$ being 0 in the differences with the statement, 恰盡 卽多少相等, i.e., $P = Q$ in the case. But he only added this statement for conditions for the right triangle to be zhenggougu. With the two conditions in the previous section, he had also linear conditions $3a = b+c$ and $3c = a+3b$. Thus Jeong always paid his attention to

zhenggougu as a special case. Further, he might conclude the inequality just by a few substitutions of Pythagorean triples.

4 Conclusions

In Gugo Wonlyu, Jeong Yag-yong introduced a new method to represent polynomials of three sides of right triangles by taking gou, gu, xian as indeterminates and then algebraic operations. Using them and factorizations, Jeong obtained universal statements, altogether more than 3,000 statements, or algebraic identities for three sides of right triangles without referring to geometrical proofs. Among them, there are also universal statements for any triple of numbers. We must point out that Jeong did not relate the identities to the traditional gougushu. We also recall that East Asian mathematics always reveal mathematical structures through some special cases like jinyu (今有) problems. Jeong also introduced the order structure in the set of polynomials by pointwise order, although it involves really difficult problems to solve inequalities. Thus Jeong Yag-yong was able to initiate the study of algebraic and order structures of the partially ordered ring of polynomials of three indeterminates, even though he discussed only up to homogeneous quadratic polynomials in the book. Further, Jeong arranged the statements so much systematically that we can easily recover 25 items with 50 identities which were missed by the transcriber.

In all, we conclude that Gugo Wonlyu is one of the most innovative mathematics books in East Asia and is a pure mathematics book not related to Silhak (實學).

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