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OPTIMAL CONSUMPTION/INVESTMENT AND LIFE INSURANCE WITH REGIME-SWITCHING FINANCIAL MARKET PARAMETERS

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ABSTRACT. We study optimal consumption/investment and life insurance purchase rules for a wage earner with mortality risk under regime-switching financial market conditions, in a continuous time-horizon. We apply the Markov chain approximation method and suggest an efficient algorithm using parallel computing to solve the simultaneous Hamilton-Jaccobi-Bellman equations arising from the optimization problem. We provide numerical results under the utility functions of the constant relative risk aversion type, with which we illustrate the effects of regime switching on the optimal policies by comparing them with those in the absence of regime switching.

1. INTRODUCTION

Understanding an economic agent's optimal behaviour is important in the analysis of the financial market or the whole economy.

Insurance was analyzed first in a financial context by [1] by considering a consumption choice problem (without risky investment) with life insurance in a random lifetime horizon. [2] studied consumption and investment with life insurance in a discrete time horizon. Robert C. Merton first studied a continuous time version of consumption and investment problem. He provides a closed form solution in the case where the economic agent exhibits constant relative risk aversion (CRRA) or constant absolute risk aversion (CARA) [3, 4]. [5] extended [3, 4] by introducing life insurance which was considered as one of the assets of the economic agent. The agent's lifetime in Richard's framework is random and bounded by a fixed planning horizon. [6] studied optimal life insurance purchase and consumption (without risky investment) for a wage earner with random and unbounded lifetime but with a fixed retirement time. They provided an explicit solution assuming that the utilities from consumption, bequest, and terminal wealth are the same CRRA function. [7, 8] apply martingale and dynamic programming method, respectively, to solve an optimal life insurance purchase and consumption/investment problem. [9] introduces a numerical method, Markov chain approximation (MCA), to solve the same problem.

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These aforementioned studies do not consider regime switching of financial market and can neglect abrupt changes of an economic situation observed in reality over a long time horizon.

[10] considered the fair valuation of a participating life insurance policy with surrender options, adopting regime switching Esscher transform. [11] investigated the optimal portfolio under regime switching and value-at-risk constraint by involving regime switching Hamilton-Jacobi-Bellman (HJB) equation. [12] studied the cost optimization of an insurance company, where the surplus of insurance companies is modeled by a controlled regime-switching diffusion. However, life insurance purchase and consumption/investment of an individual(wage earner) demanding for the life insurance under regime switching environment has not studied, although the study can provides his or her optimal policies for them considering possible structural changes in economic condition and can help insurance companies to analyze their target market.

In this paper, we study optimal consumption/investment and life insurance purchase rules for an economic agent (a wage earner) facing mortality risk with regime-switching financial market parameters, in a continuous time-horizon where the agent's fixed retirement time is given exogenously. Thus the agent's wealth evolves according to a Markov-modulated diffusion (regime switching) process. We apply the Markov chain approximation(MCA) method and suggest an efficient algorithm using parallel computing to solve the simultaneous Hamilton-Jaccobi-Bellman(HJB) equations arising from the optimization problem. The MCA is one of the most powerful method to solve stochastic control problems [13]. Over the past decade, G. Yin and his colleague have developed the MCA approach for a Markov modulated diffusion, based on the Kushner's idea. They suggested the local consistency condition for a Markov modulated diffusion, and showed weak convergence of the MCA under the condition. Subsequently, they indicated the usefulness of the method, applying to several realistic model in [14, 15, 16, 17, 18]. We provide numerical results under the utility functions of the constant relative risk aversion (CRRA) type, with which we illustrate the effects of regime switching on the optimal policies by comparing them with those in the absence of regime switching. The optimal consumptions with the possibility of regime switching into a better (resp. worse) investment opportunity state are larger (resp. less) than those in the absence of regime switching. Such effect of regime switching on the optimal consumption is strong when the agent is young, while the effect is negligible when his age is near the retirement time since the regime switching opportunity decrease as the agent gets older. However, the effect of regime switching on the risky investment is negligible although the agent is young. The intuitive reason for this is because the risky investment depends only on the total wealth and the time to retirement does not separately affect it in the absence of regime switching as shown in [9]. Without separate time effect on the risky investment, the potential regime switching effect is negligible similarly to consumption near retirement. The optimal insurance purchases with the possibility of regime switching into a better (resp. worse) investment opportunity state are larger (resp. less) than those in the absence of regime switching. Such effect of regime switching on the optimal insurance purchases is strong when the agent is young, while the effect is negligible when his age is near the retirement time as in the optimal consumptions. There has been some inconsistency between empirical findings and theoretical results for the insurance demand. Empirical studies show that the wealth has a positive effect on purchasing insurance (See [19] which review twenty six published empirical studies.), while theories suggest that negative [6, 20] or no relationship [21, 22, 23]. Our result on the optimal insurance purchases shows the regime switching effect is consistent with the empirical findings in the sense that the insurance purchases are larger (resp. less) when the agent expect regime switching into a better (resp. worse) investment opportunity state than in the absence of such an expectation.

The rest of the paper is structured as follows. In section 2, we model the optimization problem. Section 3 derives the simultaneous HJB equations arising from the optimization. In section 4, we apply the Markov chain approximation(MCA) method and suggest the algorithm using parallel computing to solve the simultaneous Hamilton–Jaccobi–Bellman(HJB) equations. Section 5 provides numerical results under the utility functions of the constant relative risk aversion (CRRA) and illustrates the effects of regime switching on the optimal policies by comparing them with those in the absence of regime switching. Section 6 concludes.

2. Model setup

We investigate the optimal life insurance purchase, consumption and portfolio strategies for an economic agent who is a wage earner, subject to mortality risk in a continuous-time economy consisting of an insurance market and a financial market whose parameters change stochastically according to business cycle, that is, whose regime changes stochastically.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space. We denote the time-t state(regime) of the financial market by $\alpha(t)$ for $t \geq 0$, where $\big(\alpha(t)\big)_{t\geq 0}$ is a continuous-time Markov chain with the finite state space $\mathcal{M} = \{1, 2, ..., n\}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote the generator of the Markov chain by $\mathcal{Q} = (q_{ij}) \in \mathbb{R}^{n\times n}$ so that $q_{ij} > 0$ for $i \neq j$, and $\sum_{j=1}^{n} q_{ij} = 0$ for $i \in \mathcal{M}$. There are one riskless asset and one risky asset in the financial market. The price $S_0(t)$ of the riskless asset evolves according to

$$dS_0(t) = r(\alpha(t))S_0(t)dt \quad \text{for} \quad t \geq 0 \quad \text{with} \quad S_0(0) = s_0.$$

The price $S_1(t)$ of the risky asset evolves according to

$$dS_1(t) = S_1(t)[\mu(\alpha(t))dt + \sigma(\alpha(t))dW(t)] \quad \text{for} \quad t \geq 0 \quad \text{with} \quad S_1(0) = s_1,$$

where $(W(t))_{t=0}^{\infty}$ is a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We let $(\mathcal{F}_t)_{t=0}^{\infty}$ be the augmentation under P of the natural filtration generated by the standard Brownian motion $(W(t))_{t=0}^{\infty}$ and assume that $\alpha(t)$ is independent of $\mathcal{F}_t$ for $t \geq 0$.

The agent is alive at $t = 0$ and we denote his lifetime by τ which is an exogenously given nonnegative random variable independent of $\big(\alpha(t)\big)_{t\geq 0}$ and $(\mathcal{F}_t)_{t=0}^{\infty}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The hazard function $\lambda(t)$(the instantaneous death rate for the agent surviving to time t) defined by $\lambda(t) \triangleq \lim_{\delta t\to 0} \frac{\mathbb{P}(t\leq\tau<t+\delta t|\tau\geq t)}{\delta t}$ is assumed to be a given deterministic Borel measure function such that $\int_0^{\infty} \lambda(t)dt = \infty$, as in [6, 7, 8, 9].

Let $f(s,t)$ and $\overline{F}(s,t)$ denote the conditional probability density for death and the conditional probability to be alive, respectively, at time s, conditional upon being alive at time $t \leq s$.

As is shown in [6], we have

$$f(s,t) = \lambda(s)e^{-\int_t^s \lambda(u)du} \text{ and } \overline{F}(s,t) = e^{-\int_t^s \lambda(u)du}. \tag{2.1}$$

We consider a finite time horizon $[0, T]$ where $T < \infty$ is interpreted as the agent's retirement time, as in [6, 7, 8, 9]: the agent will receive income with rate $I(t)$ at each time t during the period $[0, T \wedge \tau]$, where $T \wedge \tau \triangleq \min\{T, \tau\}$ and $I(t) : [0, \mathrm{T}] \to \mathbb{R}^+$ is a given deterministic function such that $\int_0^T I(t)dt < \infty$.

A life insurance contract is offered continuously as in [1, 6, 7, 8, 9] etc., and the agent enters it by paying premium at rate $p(t)$ at each time t. In compensation, if the wage earner dies at time t, then the insurance company pays an insurance amount $\frac{p(t)}{\eta(t)}$, where $\eta : [0, T] \to \mathbb{R}^+$ is a continuous and deterministic pre-specified function called the insurance premium-payout ratio. That is, we consider term insurance where the length of the term is infinitesimally small as mentioned in [6]. Negative insurance is allowed as in the existing literature considering term insurance such as the above papers.

Let $c(t) \geq 0$ and $\theta(t)$ be the consumption rate and the amount of money invested in the risky asset at time t, respectively. Thus, the agent' wealth process $\{X(t)\}_0^{T\wedge\tau}$ until $T \wedge \tau$ described by

$$\begin{aligned} dX(t) = {} & [X(t)r(\alpha(t)) + (\mu(\alpha(t)) - r(\alpha(t)))\theta(t) - c(t) - p(t) + I(t)]dt \\ & + \theta(t)\sigma(\alpha(t))dW(t), X(0) = x. \end{aligned} \tag{2.2}$$

The agent's total legacy(bequest) $Z(\tau)$ at death time τ is the wealth plus insurance amount. That is, the function $Z(t)$ is defined by

$$Z(t) = X(t) + \frac{p(t)}{\eta(t)} \text{ for } t \geq 0. \tag{2.3}$$

Let

$$b(t) \triangleq \int_t^T I(s)e^{-\int_t^s [r(\alpha(v))+\eta(v)]dv}ds. \tag{2.4}$$

Then, $b(t)$ represents the present value of the agent's future income and $W(t) \triangleq X(t) + b(t)$ can be regarded as his total available wealth, at time t, respectively, as explained in [8].

The agent endowed with the initial wealth $X(0) = x$, given the initial financial market regime $\alpha(0) = i \in \mathcal{M}$, chooses a consumption/insurance/investment policy $(c, p, \theta) \triangleq (c_t, p_t, \theta_t)_0^{T\wedge\tau} \in \mathcal{A}$ to maximize

$$J(x, i; c, p, \theta) \triangleq \mathbb{E}\Big[\int_0^{T\wedge\tau} U_C(c(t), t)dt + U_B(Z(\tau), \tau)\mathbf{1}_{\{\tau\leq T\}} + U_L(X(T), T)\mathbf{1}_{\{\tau>T\}}\Big],$$

where U_C, U_B, and U_L are the instantaneous utility function from consumption, bequest, and terminal wealth, respectively, and $\mathcal{A} = \mathcal{A}(x, i)$, called the admissible control set, denotes the set of all policies under which $c(t) \geq 0$ and $Z(t) \geq 0$ for all $t \in [0, T]$, $X(T) \geq 0$, and the above expectation is well defined. The three conditions $c(t) \geq 0$ and $Z(t) \geq 0$ for all $t \in [0, T]$, $X(T) \geq 0$ are satisfied only if $W(t) = X(t) + b(t) \geq 0$ for all $t \in [0, T]$ as mentioned in [8]. Therefore we consider the case where $X(0) = x \geq -b(0)$ (The case where

$X(0) = x = -b(0)$ is omitted since the solution is trivial in this case.). Let (c^*, p^*, θ^*) and $V(x,i)$ be an optimal strategy and the value function, respectively, of the agent endowed with the initial wealth $X(0) = x > -b(0)$, given the initial financial market regime $\alpha(0) = i \in \mathcal{M}$. That is, let

$$V(x,i) = \max_{(c,p,\theta)\in\mathcal{A}} J(x,i;c,p,\theta) = J(x,i;c^*,p^*,\theta^*) \text{ for } x > -b(0) \text{ and } i \in \mathcal{M}.$$

3. Simultaneous Hamilton–Jaccobi–Bellman equations

For $0 \le t < T$, $x \ge -b(t)$, and $i \in \mathcal{M}$, let

$$\begin{aligned} J(x,t,i;c,p,\theta) = \mathbb{E}\Big[\int_t^{T\wedge\tau} & U_C(c(s),s)ds + U_B(Z(\tau),\tau)\mathbf{1}_{\{\tau\le T\}} \\ & + U_L(X(T),T)\mathbf{1}_{\{\tau>T\}} \big| \tau > t, X(t) = x, \alpha(t) = i\Big], \end{aligned} \tag{3.1}$$

and let the value function at time t be given by

$$V(x,t,i) = \max_{(c,p,\theta)\in\mathcal{A}} J(x,t,i;c,p,\theta). \tag{3.2}$$

Thus, we have $J(x,i;c,p,\theta) = J(x,0,i;c,p,\theta)$ and $V(x,i) = V(x,0,i)$, for $x > -b(0)$ and $i \in \mathcal{M}$.

By integrating over the random time τ which is independent of $\big(\alpha(t)\big)_{t\ge0}$ and $(\mathcal{F}_t)_{t=0}^{\infty}$, and by using (2.1), we obtain the equivalent form of $J(x,t,i;c,p,\theta)$ as follows (for example, see [8]):

$$\begin{aligned} J(x,t,i;c,p,\theta) &= \mathbb{E}\Big[\int_t^T [f(s,t)U_B(Z(s),s) + \overline{F}(s,t)U_C(c(s),s)]ds \\ &\qquad + \overline{F}(T,t)U_L(X(T),T) \big| X(t) = x, \alpha(t) = i\Big] \\ &= \mathbb{E}\Big[\int_t^T [\lambda(s)e^{-\int_t^s \lambda(u)du}U_B(Z(s),s) + e^{-\int_t^s \lambda(u)du}U_C(c(s),s)]ds \\ &\qquad + e^{-\int_t^T \lambda(u)du}U_L(X(T),T) \big| X(t) = x, \alpha(t) = i\Big], \end{aligned} \tag{3.3}$$

where the second equality comes from (2.1).

Therefore the corresponding system of Hamilton–Jaccobi–Bellman(HJB) equations for $0 \le t < T$ and $x > -b(t)$ is given by

$$\begin{aligned} \lambda(t)V(x,t,i) = \max_{c\ge0,p\ge-\eta(t)x,\theta} & \Big[\lambda(t)U_B(x+\frac{p}{\eta(t)},t) + U_C(c,t) + V_t(x,t,i) \\ & + [xr(i) + (\mu(i)-r(i))\theta - c - p + I(t)]V_x(x,t,i) \\ & + \frac{1}{2}\theta^2\sigma^2(i)V_{xx}(x,t,i) + \sum_{j\ne i, j\in\mathcal{M}} q_{ij}(V(x,t,j) - V(x,t,i))\Big],\ i \in \mathcal{M}, \end{aligned} \tag{3.4}$$

with the terminal conditions $V(x,T,i) = U_L(x,T)$.

4. Numerical Method: Markov Chain Approximation Method

Mathematical models, which describe today's very complex financial system in the real world, are in many cases difficult to obtain explicit solutions, so the use of numerical methods has become more and more attractive and important in finance. The development of algorithms as well as computational resources enable us to explore more realistic problems with various uncertain factors or economic constraints. The Markov Chain Approximation Method (MCAM) is an efficient approach to obtain the numerical solution with the terminal condition [13]. The basic idea of this technique is to approximate the original continuous Markov process by a discrete-time, finite-state, controlled Markov chain which satisfies the *local consistency* condition. Then, the approximated chain weakly converges to the original continuous Markov process as an interval between grids approaches zero. The MCAM was extended to address Markov modulated diffusion process [14], and then applied to a wide variety of fields. Thus, we use the MCAM to solve numerically the system (3.4) of HJB equations with the terminal condition which is not solved explicitly. The convergences and existence of the interpolated wealth process, value function and controls are found in [13, 14, 16].

4.1. **Approximation.** To approximate the continuous Markov process, we use a finite difference method. That is, the first and the second derivatives of $\widetilde{V}(\cdot,\cdot,i)$ are approximated in following forms:

$$\begin{aligned}
\widehat{V}(u,t,i) &\longrightarrow \widehat{V}^{h,\delta}(u,t,i), \\
\widehat{V}_t(u,t,i) &\longrightarrow \frac{\widehat{V}^{h,\delta}(u,t,i)-\widehat{V}^{h,\delta}(u,t-\delta,i)}{h}, \\
\widehat{V}_u(u,t,i) &\longrightarrow \frac{\widehat{V}^{h,\delta}(u+h,t,i)-\widehat{V}^{h,\delta}(u,t,i)}{h} \qquad \text{for } b(u,t,i)\ge 0, \\
\widehat{V}_u(u,t,i) &\longrightarrow \frac{\widehat{V}^{h,\delta}(u,t,i)-\widehat{V}^{h,\delta}(u-h,t,i)}{h} \qquad \text{for } b(u,t,i)< 0, \\
\widehat{V}_{uu}(u,t,i) &\longrightarrow \frac{\widehat{V}^{h,\delta}(u+h,t,i)-2\widehat{V}^{h,\delta}(u,t,i)+\widehat{V}^{h,\delta}(u-h,t,i)}{h^2},
\end{aligned} \tag{4.1}$$

where h and δ are a step-size for wealth u and for time t, respectively; and $b(u,t,i)=\overline{r}(t,i)+(\mu(i)-r(i))\widehat{\theta}-\widehat{c}-\eta(t)\widehat{Z}-\frac{1}{2}\sigma^2(i)\widehat{\theta}^2$. After calculations, we obtain

$$\begin{aligned}
\widehat{V}^{h,\delta}(u,n\delta,i) = \max_{\widehat{c}\ge 0,\widehat{Z}\ge 0,\theta} &\Big\{\widehat{V}^{h,\delta}(u,n\delta+\delta,i)\big[1-\sigma^2(i)\theta^2\frac{\delta}{h^2} \\
&-|b(u,n\delta,i)|\frac{\delta}{h}-\lambda(n\delta)\delta+q_{ii}\delta\big]+\widehat{V}^{\delta,h}(u+h,n\delta+\delta,i)\big[b^+(u,n\delta,i)\frac{\delta}{h}\big] \\
&+\widehat{V}^{\delta,h}(u-h,n\delta+\delta,i)\big[b^-(u,n\delta,i)\frac{\delta}{h}\big]+\lambda(n\delta)U_B(e^u\widehat{Z},n\delta)\delta+U_C(e^u\widehat{c},n\delta)\delta \\
&+\sum_{j\neq i,j\in\mathcal{M}} q_{ij}\widehat{V}^{h,\delta}(u,n\delta+\delta,j)\delta\Big\},\ i\in\mathcal{M},
\end{aligned} \tag{4.2}$$

where $b^+(u, n\delta, i)$ (resp. $b^-(u, n\delta, i)$) is the positive (resp. negative) part of $b(u, n\delta, i)$ so that $b^+(u, n\delta, i) + b^-(u, n\delta, i) = |b(u, n\delta, i)|$. Then, we obtain the transition probabilities of the wealth process satisfying the *local consistency*:

$$\begin{aligned}
p^{h,\delta}((u,i),(u,i),n\delta|\widehat{c},\widehat{Z},\widehat{\theta}) &= \frac{[1-(\sigma(i)\widehat{\theta})^2(\delta/h^2)-|b(u,n\delta,i)|(\delta/h)]}{\widetilde{G}},\\
p^{h,\delta}((u,i),(u+h,i),n\delta|\widehat{c},\widehat{Z},\widehat{\theta}) &= \frac{[((\sigma(i)\widehat{\theta})^2/2)(\delta/h^2)+b^+(u,n\delta,i)(\delta/h)]}{\widetilde{G}},\\
p^{h,\delta}((u,i),(u-h,i),n\delta|\widehat{c},\widehat{Z},\widehat{\theta}) &= \frac{[((\sigma(i)\widehat{\theta})^2/2)(\delta/h^2)+b^-(u,n\delta,i)(\delta/h)]}{\widetilde{G}},
\end{aligned} \tag{4.3}$$

where $\widetilde{G} = 1 - \lambda(n\delta)\delta + q_{ii}\delta$. In policy iteration, we can obtain optimal controls c^*, Z^* and θ^* by solving the equation (4.2) and (4.3).

4.2. **Algorithm.** Because the computational method to solve (4.2) and (4.3) is based on the dynamic programming principle (DPP), it suffers from the so-called "*curse of dimensionality*" that means the extraordinarily rapid growth in the difficulty of problems as the number of variables (or the dimension) increases. Furthermore, in multiple regimes, it would be an impractical choice due to computing time or lack of memory. Since nowadays a multi-core processor PC (or a large cluster) is well equipped and popular, parallel computation is easily available, and so can be a useful alternative to solve the HJB equation as described in Algorithm 1. For parallel computing, we used the message passing interface (MPI) in C on a Linux cluster

Algorithm 1 Algorithm for solving (4.2) and (4.3)

```
Each regime i is allocated to each processor i (or node)
Set initial starting values
while Increase in the value function less than a prescribed tolerance do
    Each processor i obtains optimal controls c*, Z* and θ* by computing the equation
    (4.2) and (4.3)
    The value function (4.2) is updated with the obtained optimal controls
    Every processors send their own value function to the other processors and receives
    those from the others.
end while
```

and implemented $MPI\_Sendrecv$ routine for communications. Because the amount of communicating information among processors is relatively less (i.e., just updated value functions), it is very efficient methodology for solving the simultaneous HJB equations.

5. Numerical results

We consider the utility functions of constant relative risk aversion(CRRA) type which is well accepted in finance. That is,

$$U_C(c,t) = \frac{e^{-\rho t}}{\gamma_C} c^{\gamma_C}, \quad U_B(Z,t) = \frac{e^{-\rho t}}{\gamma_B} Z^{\gamma_B}, \quad U_L(x,T) = \frac{e^{-\rho t}}{\gamma_L} x^{\gamma_L}, \tag{5.1}$$

where $\rho > 0$ is the subjective discount factor, and $0 \neq \gamma$ and $\gamma_k < 1$ for $k = C, B, L$. We consider systems consisting of two regimes ($\alpha(t) \in \mathcal{M} = \{1,2\}$) with the generator,

$$\mathcal{Q} = q_{ij} = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}. \tag{5.2}$$

Since the investment opportunity in the financial market becomes better as the value of $\left(\frac{\mu(\alpha(t))-r(\alpha(t))}{\sigma(\alpha(t))}\right)^2$ increases (See Remark 2.1 in [24].), we consider, without loss of generality, the cases where the drift term of the price of the risky asset depends on the market regime but the volatility and the risk free rate do not change. That is, we consider the case where $\mu(1) \neq \mu(2)$, but $r(1) = r(2) = r$ and $\sigma(1) = \sigma(2) = \sigma$. Let the parameters except $\mu(1)$ and $\mu(2)$ be given as in the following: the initial time $t_0 = 25$ (i.e., a 25 years old wage earner); the retirement time $T = 35$; $\lambda(t) = 0.001 + \frac{1}{10.54}\exp(\frac{t-87.24}{10.54})$ (Gompertz-Makeham hazard function), $\eta(t) = \lambda(t)$, and $I(t) = 10000\exp(0.03t)$ for $t \geq t_0$; $\gamma_C = \gamma_B = \gamma_L = -1$; $r = 0.04$, $\sigma = 0.18$, and $\rho = 0.03$.

We illustrate the effects of regime switching on the optimal policies by comparing them, at three different ages 25, 30, and 34.75, with those in the absence of regime switching which are calculated in the same way as in [9].

In Figures 1, 2, 3, the two regimes are given by $\mu(1) = 0.09$ (middle regime) and $\mu(2) = 0.13$ (high regime).

Figure 1 compares the optimal consumptions at middle regime in our model (at coupled middle regime) with those at middle regime in the absence of regime switching (at uncoupled middle regime) and at high regime in the absence of regime switching, as a function of the total wealth at each age. As expected, the optimal consumptions at middle regime in our model are lager (resp. less) than those at middle (resp. high) regime in the absence of regime switching. Such effect of regime switching on the optimal consumption is strong when the agent is young, while the effect is negligible when his age is near the retirement time since the regime switching opportunity decrease as he gets older.

Figure 2 compares the optimal investments in the risky asset in the three cases as in the comparison of consumption, at each age. Differently from the consumption policy, the effect of regime switching on the risky investment is negligible although the agent is young. The intuitive reason for this is because the risky investment depends only on the total wealth and the time to retirement does not separately affect it in the absence of regime switching (See [9]). Without separate time effect on the risky investment, the potential regime switching effect is negligible similarly to consumption near retirement.

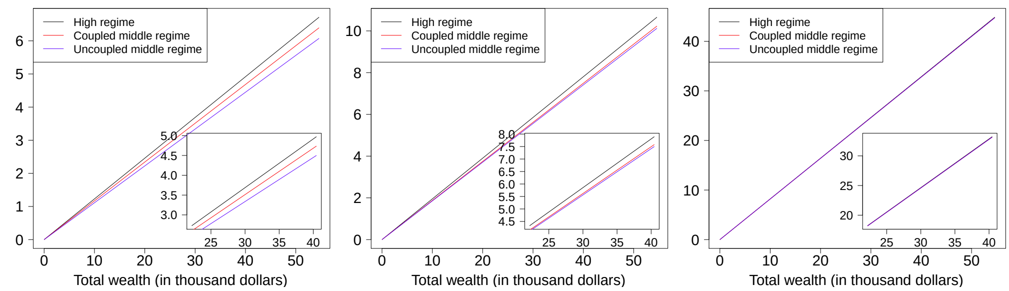

FIGURE 1. Optimal consumptions at age 25 (top-left), 30 (top-right), 34.75 (bottom). In this figure, y axis is the optimal consumption. "Coupled middle regime" shows the consumptions in the middle regime coupled to the high one.

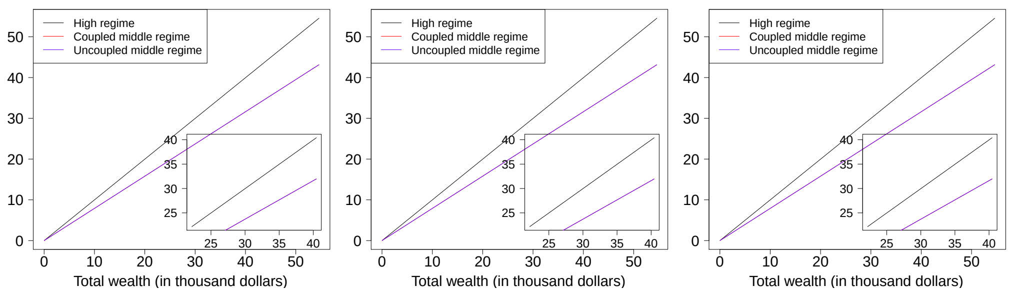

FIGURE 2. Optimal investment at age 25 (top-left), 30 (top-right), 34.75 (bottom). In this figure, y axis is the optimal investment. "Coupled middle regime" shows the investments in the middle regime coupled to the high one.

Figure 3 compares the insurance premiums in the three cases as in the comparison of consumption and risky investment, at each age. The optimal insurance premiums at middle regime in our model are lager (resp. less) than those at middle (resp. high) regime in the absence of regime switching. Such effect of regime switching on the optimal insurance premiums is strong when the agent is young, while the effect is negligible when his age is near the retirement time since the regime switching opportunity decrease as he gets older. As mentioned in Section 1, there has been some inconsistency between empirical findings and theoretical results for the insurance demand. Empirical studies show that the wealth has a positive effect on purchasing insurance (See [19] which review twenty six published empirical studies.), while theories suggest that negative [6, 20] or no relationship [21, 22, 23]. Figure 3 shows the regime switching effect is consistent with the empirical findings in the sense that the premiums are larger when the the regime is higher or can switch into higher one, although the wealth effect is negative as in the theoretical results.

In Figures 4, 5, 6, the two regimes are given by $\mu(1) = 0.09$ (middle regime) and $\mu(2) = 0.05$ (low regime).

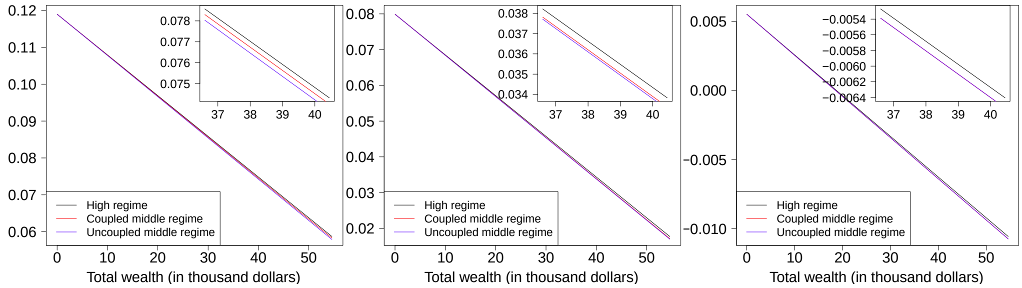

FIGURE 3. Optimal premium payment at age 25 (top-left), 30 (top-right), 34.75 (bottom). In this figure, y axis is the optimal premium payment payment. "Coupled middle regime" shows the payments in the middle regime coupled to the high one.

Figure 4 compares the optimal consumptions at middle regime in our model (at coupled middle regime) with those at middle regime in the absence of regime switching (at uncoupled middle regime) and at low regime in the absence of regime switching, as a function of the total wealth at each age. As expected, the optimal consumptions at middle regime in our model are less (resp. larger) than those at middle (resp. low) regime in the absence of regime switching. Such effect of regime switching on the optimal consumption is strong when the agent is young, while the effect is negligible when his age is near the retirement time in the same reason as in Figure 1.

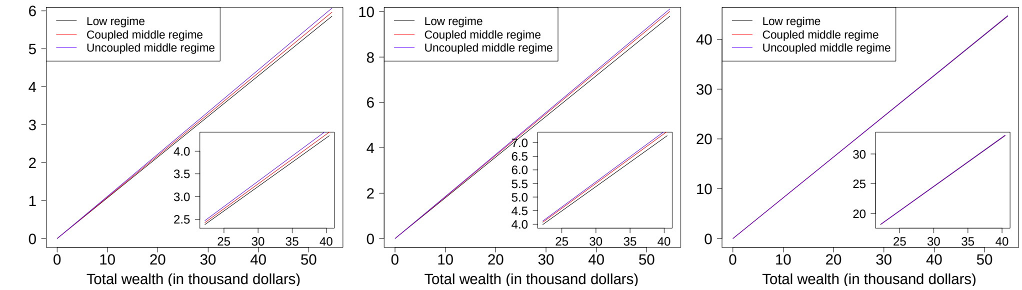

FIGURE 4. Optimal consumptions at age 25 (top-left), 30 (top-right), 34.75 (bottom). In this figure, y axis is the optimal consumption. "Coupled middle regime" shows the consumptions in the middle regime coupled to the low one.

Figure 5 compares the optimal investments in the risky asset in the three cases as in the comparison of consumption, at each age. The effect of regime switching on the risky investment is negligible although the agent is young. The reason for this is the same as in Figure 2.

Figure 6 compares the insurance premiums in the three cases as in the comparison of consumption and risky investment, at each age. The optimal insurance premiums at middle regime in our model are less (resp. larger) than those at middle (resp. low) regime in the absence of

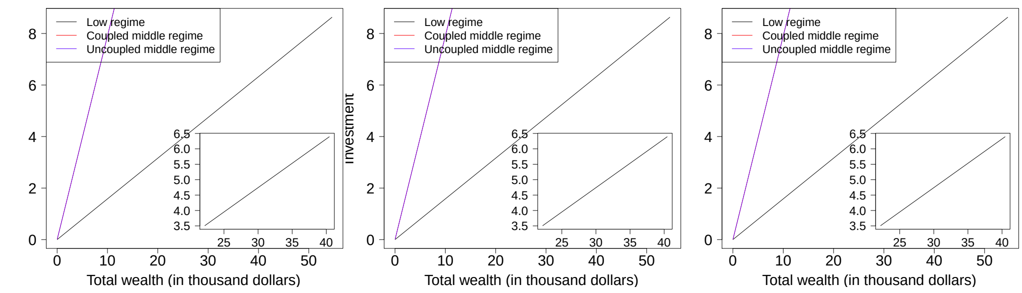

FIGURE 5. Optimal investment at age 25 (top-left), 30 (top-right), 34.75 (bottom). In this figure, y axis is the optimal investment. "Coupled middle regime" shows the investments in the middle regime coupled to the low one.

regime switching. Such effect of regime switching on the optimal insurance premiums is strong when the agent is young, while the effect is negligible when his age is near the retirement time in the same reason as in Figure 3. Figure 6 shows the regime switching effect is consistent with the empirical findings, as in Figure 3, in the sense that the premiums are less when the regime is lower or can switch into a lower one, although the wealth effect is negative as in the theoretical results.

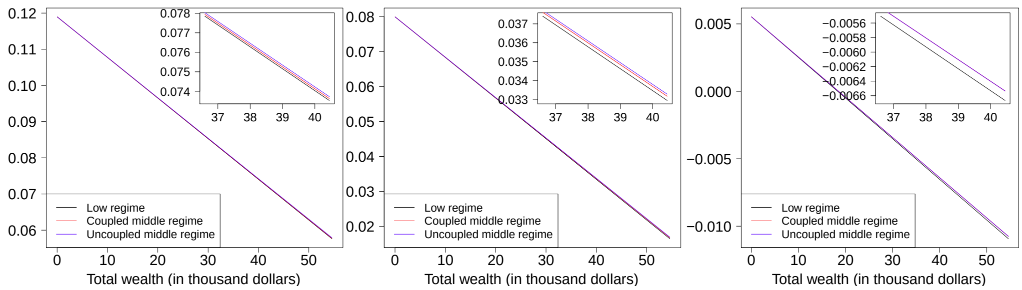

FIGURE 6. Optimal premium payment at age 25 (top-left), 30 (top-right), 34.75 (bottom). In this figure, y axis is the optimal premium payment. "Coupled middle regime" shows the payments in the middle regime coupled to the low one.

6. CONCLUSION

We have studied optimal consumption/investment and life insurance purchase rules for an economic agent facing mortality risk with regime-switching financial market parameters which cause the agent's wealth to evolve according to a Markov-modulated diffusion (regime switching) process , in a continuous time-horizon where the agent's fixed retirement time is given exogenously. We have applied the Markov chain approximation(MCA) method and suggested an efficient algorithm using parallel computing to solve the simultaneous Hamilton-Jaccobi-Bellman(HJB) equations arising from the optimization problem. We have provided numerical results under the utility functions of the constant relative risk aversion (CRRA) type, with which we illustrate the effects of regime switching on the optimal policies by comparing them with those in the absence of regime switching. The optimal consumptions with the possibility of regime switching into a better (resp. worse) investment opportunity state are larger (resp. less) than those in the absence of regime switching. Such effect of regime switching on the optimal consumption is strong when the agent is young, while the effect is negligible when his age is near the retirement time since the regime switching opportunity decrease as the agent gets older. However, the effect of regime switching on the risky investment is negligible although the agent is young, since the risky investment depends only on the total wealth without separate time effect. The optimal insurance purchases with the possibility of regime switching into a better (resp. worse) investment opportunity state are larger (resp. less) than those in the absence of regime switching. Such effect of regime switching on the optimal insurance purchases is strong when the agent is young, while the effect is negligible when his age is near the retirement time as in the optimal consumptions. There has been some inconsistency between empirical findings and theoretical results for the insurance demand. Empirical studies show that the wealth has a positive effect on purchasing insurance, while theories suggest that negative or no relationship. Our result on the optimal insurance purchases shows the regime switching effect is consistent with the empirical findings in the sense that the insurance purchases are larger (resp. less) when the agent expect regime switching into a better (resp. worse) investment opportunity state than in the absence of such an expectation.

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