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OPTIMAL PORTFOLIO SELECTION UNDER STOCHASTIC VOLATILITY AND STOCHASTIC INTEREST RATES

MI-HYUN KIM[1], JEONG-HOON KIM[2], AND JI-HUN YOON[3†]

[1]KIS PRICING, 35-4 YOIDO-DONG, YOUNGDEUNGPO-KU, SEOUL 150-885, KOREA
E-mail address: mh2501@naver.com

[2]DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, SEOUL 120-749, KOREA
E-mail address: jhkim96@yonsei.ac.kr

[3]DEPARTMENT OF MATHEMATICS, PUSAN NATIONAL UNIVERSITY, PUSAN 609-735, KOREA
E-mail address: yssci99@pusan.ac.kr

ABSTRACT. Although, in general, the random fluctuation of interest rates gives a limited impact on portfolio optimization, their stochastic nature may exert a significant influence on the process of selecting the proportions of various assets to be held in a given portfolio when the stochastic volatility of risky assets is considered. The stochastic volatility covers a variety of known models to fit in with diverse economic environments. In this paper, an optimal strategy for portfolio selection as well as the smoothness properties of the relevant value function are studied with the dynamic programming method under a market model of both stochastic volatility and stochastic interest rates.

1. INTRODUCTION

Portfolio optimization arises from the well-known fact that an investor dynamically allocates wealth between risky and riskless assets while he or she chooses a consumption rate with the goal of maximizing the total expected discounted utility of consumption. It is known as the Merton problem since the seminal works given by Merton [1, 2]. For a utility function of hyperbolic average risk aversion (HARA) type, the problem has a simple explicit solution. The optimal investment strategy is to put the fixed ratio of the total wealth onto the risky asset while the optimal consumption rate is a constant times the wealth. See, for example, [3, 4] for general reference.

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† Corresponding author.

Unlike the constant interest rate in the classical Merton model, the interest rate may fluctuate from time to time in our real world. In addition, the interest rate fluctuation can be strongly correlated with the price fluctuation of a risky asset. Therefore, there are studies on portfolio optimization considering the random character of the interest rate for a riskless asset. Some examples of the relevant study include Lioui and Poncet [5], Korn and Kraft [6], Fleming and Pang [7], Detemple and Rindisbacher [8], Liu [9], Li and Wu [10], Noh and Kim [11], Chang and Chang [12], and Shen and Siu [13].

Another drawback of the Merton model comes from the constant volatility of a risky asset. Since the concept of stochastic volatility has been used effectively to explain many well-known empirical findings such as the volatility smile, the volatility clustering and the heavy-tailed nature of return distributions, it is desirable for stochastic volatility to enter into the risky asset price dynamics representing a complex economic factor. Zariphopoulou [14], Fleming and Hernandez-Hernandez [15], Chacko and Viceira [16], Liu [9], and Noh and Kim [11] are examples of study on the optimal investment and consumption problems under stochastic volatility models.

So, this article considers both stochastic interest rates and stochastic volatility for the dynamics of a risky asset. One recent study [11] covers both stochastic interest rates and stochastic volatility for a given underlying asset. However, it is limited because it assumes a specific ergodic Markov diffusion process, i.e., fast mean reversion, for stochastic volatility. In fact, there are other stochastic volatility models that can not be covered by [11]. See [17, 18, 19] for instance. The main goal of this paper is to remove this assumption and extend the work by using the dynamic programming method. We verify that the corresponding value function has desired smooth properties and obtain the optimal investment strategies.

The rest of this paper is organized as follows. In Section 2, based on the correlation structure among the underlying risky asset and its stochastic volatility and stochastic interest rates, a stochastic portfolio optimization problem is formulated as a nonlinear partial differential equation by the dynamic programming principle. Theorems on the value function and optimal control policy are derived in Section 3 and 4, respectively. Section 5 provides concluding remarks.

2. Formulation

Considering an investor who can allocate his or her wealth to a riskless asset and a risky asset at each time t, we assume as in [15] that the unit price P_t of the risky asset fluctuates randomly according with a diffusion process given by the stochastic differential equation

$$dP_t = \mu P_t dt + \sigma(Y_t) P_t dB_t^p, \quad t > 0, \tag{2.1}$$

and the riskless asset denoted by β_t is given by $d\beta_t = r_t \beta_t dt$, where μ is a positive constant and B_t^p is a standard one-dimensional Brownian motion. Here, Y_t represents an economic factor with a strong ergodic property and its dynamics, unlike the Merton model, are given by

$$\begin{aligned} dY_t &= g(Y_t)dt + \beta d\hat{B}_t, \quad t > 0, \\ d\hat{B}_t &= \rho_{py} dB_t^p + \sqrt{1-\rho_{py}^2} dB_t^y, \end{aligned} \tag{2.2}$$

where $\rho_{py} \in [-1,1]$ is the correlation coefficient of B_t^p and $\hat{B}_t$, B_t^y and B_t^p are independent Brownian motions, β is a positive constant, g is a smooth function and the functions σ and g satisfy $\sigma, g \in \mathbf{C}^1(\mathbb{R})$, $\sigma_l \leq \sigma(\cdot) \leq \sigma_u$, and $g^{'} \leq -k$, where σ_u and σ_l are positive constants with $\sigma_u > \sigma_l$ and k is a positive constant.

A typical example of stochastic interest rate r_t is given by the Vasicek model [20] in which r_t fluctuates around a certain value at most of the time. In this paper, we take a general Vasicek model for the interest rate which is correlated with the risky asset price process. Instead of constant interest rate in [15], we consider a range of interest rate $r_l \leq r_t \leq r_u$ for some positive constants r_l and r_u, where r_t is randomly fluctuating in such a way that

$$\begin{aligned} dr_t &= f(r_t)dt + \hat{\sigma} d\tilde{B}_t, \quad t > 0, \\ d\tilde{B}_t &= \rho_{pr} dB_t^p + \sqrt{1-\rho_{pr}^2} dB_t^r \end{aligned} \tag{2.3}$$

where $\hat{\sigma}$ is a positive constant, $\rho_{pr} \in [-1,1]$ is the correlation coefficient of B_t^p and $\tilde{B}_t$, B_t^p and B_t^r are independent Brownian motions and the function f satisfies $f(r) \in \mathbf{C}^2(\mathbb{R})$, $|f^{''}(r)| \leq K(1+|r|^\nu)$, and $c_2 \leq f^{'}(r) \leq c_1$, where K and ν are positive constants and c_1 and c_2 are some constants. Also, we assume that the Brownian motions B_t^y and B_t^r are mutually independent.

Let us denote X_t as the total wealth of the agent at time t. Suppose at time t a fraction u_t of his or her money is invested on the risky asset and the consumption rate is C_t. Thus a fraction of $1-u_t$ will be invested on the riskless asset. Then, the dynamic of X_t is described by

$$dX_t = u_t X_t \frac{dP_t}{P_t} + (1-u_t) X_t \frac{d\beta_t}{\beta_t} - C_t dt.$$

To factorize the right side of this equation into X_t and the rest, we use $c_t := C_t/X_t$ instead of C_t. Given these assumptions, one can get a stochastic differential equation for X_t as follows:

$$dX_t = X_t[(r_t + (\mu - r_t)u_t - c_t)dt + u_t \sigma(Y_t) dB_t^p], \quad t > 0. \tag{2.4}$$

Here, the control variables are u_t and c_t. It is assumed that u_t takes values in a finite interval. A negative u_t stands for the disinvestment or short selling. We also assume that there is no transaction cost and the risky asset can be traded any time.

We define the concept of admissible strategy (u_t, c_t) as follows:

Definition 2.1. *A pair (u,c) is admissible strategy if u_t and c_t are $\mathcal{F}_t$-progressively measurable process such that*

$$\mathbf{P}[\forall t > 0, |u_t| \leq A_1, 0 \leq c_t \leq A_2] = 1 \tag{2.5}$$

for some constants A_1 and A_2 which may depend on the strategy, where the probability measure $\mathbf{P}$ and the family $\{\mathcal{F}_t\}$ of σ-algebras are given. We denote the set of admissible strategies as $\mathcal{A}$.

Using the well-known Ito lemma [21], one can solve (2.4) and obtain

$$X_t = x \exp\left\{\int_0^t \left[r_s + (\mu - r_s)u_s - c_s - \frac{1}{2}u_s^2\sigma^2(Y_s)\right] ds + \int_0^t u_s\sigma(Y_s)dB_s^p\right\} \tag{2.6}$$

for $(u, c) \in \mathcal{A}$, where $X_0 = x$.

In this paper, we consider a utility function of the HARA type given by $U(C) = \frac{1}{\gamma}C^\gamma$, $0 < \gamma < 1$, and intend to maximize the expected discounted utility of consumption on the infinite horizon defined by $J(x, y, r; c, u) := \mathbf{E}\int_0^\infty \frac{1}{\gamma}e^{-\alpha t}(c_t X_t)^\gamma dt$ over the set of admissible strategies $\mathcal{A}$. From (2.6) we have

$$J(x, y, r; c, u) = \frac{x^\gamma}{\gamma}\mathbf{E}\int_0^\infty c_t^\gamma e^{-\alpha t+\gamma\int_0^t [r_s+(\mu-r_s)u_s-c_s-\frac{1}{2}u_s^2\sigma^2(Y_s)]ds+\gamma\int_0^t u_s\sigma(Y_s)dB_s^p} dt. \tag{2.7}$$

To reduce the Brownian motion term, let us change the measure $\mathbf{P}$ using the Girsanov transformation (cf. [22]) with the Radon-Nikodym derivative $\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} \mid \mathcal{F}_t = e^{\gamma\int_0^t u_s\sigma(Y_s)dB_s^p - \frac{1}{2}\gamma^2\int_0^t u_s^2\sigma^2(Y_s)ds}$. The process $B_t^p - \gamma\int_0^t u_s\sigma(Y_s)ds$ is a Brownian motion adapted to $\mathcal{F}_t$ under the new probability measure $\tilde{\mathbf{P}}$. Also, under the new measure, the dynamics of Y_t is given by $dY_t = [g(Y_t) + \beta\gamma\rho_{pr}u_t\sigma(Y_t)]dt + \beta dZ_t$ for some Brownian motion Z_t. Then the objective function $J(x, y, r; c, u)$ can be written as

$$J(x, y, r; c, u) = \frac{x^\gamma}{\gamma}\tilde{\mathbf{E}}\int_0^\infty c_t^\gamma e^{-\alpha t+\gamma\int_0^t [r_s+(\mu-r_s)u_s-c_s]ds+\frac{\gamma(\gamma-1)}{2}\int_0^t u_s^2\sigma^2(Y_s)ds} dt. \tag{2.8}$$

In terms of a new process $z_t := -\alpha t + \gamma\int_0^t [r_s + (\mu - r_s)u_s - c_s]ds + \frac{\gamma(\gamma-1)}{2}\int_0^t u_s^2\sigma^2(Y_s)ds$, the objective function becomes

$$J(x, y, r; c, u) = \frac{x^\gamma}{\gamma}\tilde{\mathbf{E}}\int_0^\infty c_t^\gamma e^{z_t} dt := \frac{x^\gamma}{\gamma}\tilde{J}(y, r; c, u). \tag{2.9}$$

Now, let us define the value function $V(x, y, r)$ by $V(x, y, r) := \sup_{c,u} J(x, y, r; c, u)$. Using the dynamic programming principle (cf. [22, 23]), one can write a differential equation for $V(x, y, r)$ as follows:

$$\begin{aligned}\alpha V =& \sup_u\left\{[r + (\mu - r)u]xV_x + \frac{1}{2}u^2\sigma^2(y)x^2V_{xx} + \beta\rho_{py}u\sigma(y)xV_{xy} + \hat{\sigma}\rho_{pr}u\sigma(y)xV_{xr}\right\} \\ &+ \sup_c\left\{-cxV_x + \frac{1}{\gamma}(cx)^\gamma\right\} + g(y)V_y + \frac{\beta^2}{2}V_{yy} + f(r)V_r + \frac{\hat{\sigma}^2}{2}V_{rr}.\end{aligned} \tag{2.10}$$

Alternatively, in terms of the value function $W(y,r)$ defined by $W(y,r) := \sup_{c,u} \tilde{J}(y,r;c,u)$, a differential equation for $W(y,r)$ is given by

$$\alpha W = \sup_u \gamma \left\{ [r + (\mu - r)u]W + \frac{1}{2}u^2\sigma^2(y)(\gamma - 1)W + \beta\rho_{py}u\sigma(y)W_y + \hat{\sigma}\rho_{pr}u\sigma(y)W_r \right\} + \sup_c \{-c\gamma W + c^\gamma\} + g(y)W_y + \frac{\beta^2}{2}W_{yy} + f(r)W_r + \frac{\hat{\sigma}^2}{2}W_{rr}. \tag{2.11}$$

If $W(y,r) > 0$ and it is smooth enough, then the potential optimal control policy is given by

$$u^*(y,r) := \frac{(\mu - r)W + \beta\rho_{py}\sigma(y)W_y + \hat{\sigma}\rho_{pr}\sigma(y)W_r}{(1-\gamma)\sigma^2(y)W}, \qquad c^*(y,r) := W^{\frac{1}{\gamma-1}}. \tag{2.12}$$

In Section 4, this will be verified to be the actual optimal control policy.

From now on, we find a suitable solution of (2.11), say $\tilde{W}(y,r)$, and then verify that $\tilde{W}(y,r)$ is equal to the value function $W(y,r)$ defined by (2.10) when the optimal control (u^*, c^*) is applied.

3. Boundedness and continuity of the value function

We consider a finite time horizon problem with $T > 0$ first. For each $(u,c) \in \mathcal{A}$, we define a finite horizon functional $J(x,y,r;c,u,T)$ by

$$\begin{aligned} J(x,y,r;c,u,T) &:= \mathbf{E}\int_0^T \frac{1}{\gamma}e^{-\alpha t}(c_t X_t)^\gamma dt \\ &= \frac{x^\gamma}{\gamma}\mathbf{E}\int_0^T c_t^\gamma e^{-\alpha t + \gamma\int_0^t [r_s + (\mu - r_s)u_s - c_s - \frac{1}{2}u_s^2\sigma^2(Y_s)]ds + \gamma\int_0^t u_s\sigma(Y_s)dB_s^p} dt. \end{aligned} \tag{3.1}$$

Here, the second equality follows from (2.6).

To reduce the Brownian motion term in (3.1), let us change the measure $\mathbf{P}$ using the Girsanov transformation with the above Radon-Nikodym derivative $\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}$. Then under the measure $\tilde{\mathbf{P}}$ the process $B_t^P - \gamma\int_0^t u_s\sigma(Y_s)ds$ is a Brownian motion adapted to $\mathcal{F}_t$ and the functional $J(x,y,r;c,u,T)$ becomes

$$J(x,y,r;c,u,T) = \frac{x^\gamma}{\gamma}\tilde{\mathbf{E}}\int_0^T c_t^\gamma e^{-\alpha t + \gamma\int_0^t [r_s + (\mu - r_s)u_s - c_s]ds + \frac{\gamma(\gamma-1)}{2}\int_0^t u_s^2\sigma^2(Y_s)ds} dt. \tag{3.2}$$

In terms of z_t, the functional $J(x,y,r;c,u,T)$ becomes

$$J(x,y,r;c,u,T) = \frac{x^\gamma}{\gamma}\tilde{\mathbf{E}}\int_0^T c_t^\gamma e^{z_t} dt := \frac{x^\gamma}{\gamma}\tilde{J}(y,r;c,u,T).$$

If we define the value function $W(y,r,T)$ by $W(y,r,T) := \sup_{c,u}\tilde{J}(y,r;c,u,T)$ and the function $\bar{W}(y,r)$ by $\bar{W}(y,r) := \lim_{T\uparrow\infty} W(y,r,T)$, then we expect that the function $\bar{W}(y,r)$ coincides with the original value function $W(y,r)$ given by (2.11).

Now, we shall investigate main properties of the value functions $W(y,r,T)$ and $W(y,r)$. From (2.11) the corresponding dynamic programming equation associated with $W(y,r,T)$ is given by

$$
\begin{aligned}
-W_T + \alpha W &= g(y)W_y + \frac{\beta^2}{2}W_{yy} + f(r)W_r + \frac{\hat{\sigma}^2}{2}W_{rr} \\
&\quad + \sup_u \gamma\{[r+(\mu-r)u]W + \frac{1}{2}u^2\sigma^2(y)(\gamma-1)W + \beta\rho_{py}u\sigma(y)W_y + \hat{\sigma}\rho_{pr}u\sigma(y)W_r\} \\
&\quad + \sup_c\{-c\gamma W + c^\gamma\}. \qquad (3.3)
\end{aligned}
$$

Then an equation for $W(y,r)$ is given by (3.3) without the W_T term.

To prove the existence of the solution $W(y,r)$ in the next section, it is required that the value function $W(y,r)$ is bounded and Lipschitz continuous. These properties are necessary conditions for the Arzela-Ascoli theorem (cf. [24]) to be used in Section 4 as a main tool. So, this section is devoted to prove the boundedness and Lipschitz continuity of $W(y,r)$.

Lemma 3.1. *Let us define* $\lambda_l := \frac{(\mu-r_a)^2}{2(1-\gamma)\sigma_l^2} + r_u$ *and* $\bar{\alpha} := \alpha - \gamma\lambda_l$*, where* r_a *is* r_l *or* r_u *maximizing* $|\mu - r_a|$*. Suppose that* $\bar{\alpha} > 0$ *holds. Then we have* $K_1 \le W(y,r,T), \bar{W}(y,r) \le K_2$*, where*

$$
K_1 = \left(\frac{\alpha - \gamma r_l}{1-\gamma}\right)^{\gamma-1}, \qquad K_2 = \left(\frac{\bar{\alpha}}{1-\gamma}\right)^{\gamma-1}. \qquad (3.4)
$$

Proof Let us first obtain the upper bound K_2. Since

$$
r_t + (\mu - r_t)u_t + \frac{\gamma-1}{2}\sigma^2(y)u_t^2 \le \frac{(\mu-r_a)^2}{2(1-\gamma)\sigma_l^2} + r_u = \lambda_l
$$

holds for each admissible consumption process $c_t \ge 0$, we have $z_t \le -\bar{\alpha}t - \gamma\int_0^t c_s ds$ and so

$$
\tilde{J}(y,r;c,u,T) \le \tilde{\mathbf{E}}\int_0^T c_t^\gamma e^{-\bar{\alpha}t - \gamma\int_0^t c_s ds}dt = \tilde{\mathbf{E}}\int_0^T e^{-\bar{\alpha}t}c_t^\gamma \zeta_t^\gamma dt, \qquad (3.5)
$$

where ζ_t satisfies $d\zeta_t = -c_t\zeta_t dt$ with the initial condition $\zeta_0 = 1$.

Now, we define an auxiliary pure consumption maximizing problem in terms of $v(\zeta,T) := \max_c \int_0^T e^{-\bar{\alpha}t}c_t^\gamma\zeta_t^\gamma dt$. Then the corresponding dynamic programming equation becomes $v_T + \bar{\alpha}v = \max_c[-c\zeta v_\zeta + (c\zeta)^\gamma]$. By adapting the same idea as in solving the classical Merton problem to this problem, it can be seen that $v(\zeta,T) = \zeta^\gamma w(T)$, where w solves the ordinary differential equation $w_T + \bar{\alpha}w = (1-\gamma)w^{\frac{\gamma}{\gamma-1}}$. Moreover, we note that $w(T) \to \left(\frac{\bar{\alpha}}{1-\gamma}\right)^{\gamma-1} = K_2$ as $T \to \infty$. Since $W(y,r,T) \le v(1,T) = w(T)$ holds from (3.5), the above argument implies that there exists $T_1 > 0$ such that $W(y,r,T) \le K_2$ for $T \ge T_1$.

Next, by taking $u_t = 0$, applying the inequality $r_l \le r_u$ and maximizing $\tilde{J}$ with respect to admissible $c_t \ge 0$, we obtain a positive lower bound K_1 satisfying $K_1 \le W(y,r,T), \bar{W}(y,r)$.
□

From now on, we shall assume the existence of c_1 and c_u such that $c_1 < c_u$, $K_2^{\frac{1}{\gamma-1}} \le c_u$, $c_l \le K_1^{\frac{1}{\gamma-1}}$ and the supremum with respect to c in (3.3) is achieved at $c^* = W^{\frac{1}{\gamma-1}} \in [c_l, c_u]$. Now, we verify that the value function $W(y,r,T)$ and $\bar{W}(y,r)$ are Lipschitz continuous.

Lemma 3.2. *Suppose that there exists a constant M independent of T such that $M \ge A_1$ and $k - M\beta\gamma|\rho_{py}|\|\sigma^{'}\| > 0$ and $c_l - M\hat{\sigma}\gamma|\rho_{pr}|\|\sigma^{'}\| > 0$. Then $W(y,r,T)$ and $\bar{W}(y,r)$ are Lipschitz continuous for each fixed T. In addition, $W(y,r,T) \to \bar{W}(y,r)$ uniformly in compact sets as $T \to \infty$.*

Proof Given the initial condition (y,r), let Y_t and r_t be the corresponding solutions of the stochastic differential equations (2.2) and (2.3), respectively. Also, given the initial condition $(\hat{y},\hat{r})$, let $\hat{Y}_t$ and $\hat{r}_t$ be the corresponding solutions of (2.2) and (2.3), respectively, with $\hat{z}_t$ corresponding to z_t. Then

$$\tilde{J}(\hat{y},\hat{r};c,u,T) - \tilde{J}(y,r;c,u,T) = \tilde{\mathbf{E}}\int_0^T c_t^{\gamma}(e^{\hat{z}_t} - e^{z_t})dt \le \tilde{\mathbf{E}}\int_0^T c_t^{\gamma}e^{\hat{z}_t}(\hat{z}_t - z_t)dt. \quad (3.6)$$

On the other hand, since $|u_t| \le A_1$ from the admissibility condition (2.5),

$$\begin{aligned}\hat{z}_t - z_t &= \gamma\int_0^t ((\hat{r}_s - r_s) + (r_s - \hat{r}_s)u_s)\,ds + \frac{\gamma(\gamma-1)}{2}\int_0^t u_s^2(\sigma^2(\hat{Y}_t) - \sigma^2(Y_s))ds \\ &\le (A_1+1)\gamma\int_0^t |\hat{r}_s - r_s|ds + {A_1}^2\sigma_u\gamma(1-\gamma)\|\sigma^{'}\|\int_0^t |\hat{Y}_s - Y_s|ds,\end{aligned} \quad (3.7)$$

where $\|\sigma^{'}\|$ denotes the supremum norm of the derivative $\sigma^{'}$. Since the sample paths of $\hat{Y}_t - Y_t$ are continuously differentiable, $d|\hat{Y}_t - Y_t|^2 = 2(\hat{Y}_t - Y_t)[g(\hat{Y}_t) - g(Y_t) + \beta\gamma\rho_{py}u_t(\sigma(\hat{Y}_t) - \sigma(Y_t))]dt$ holds and so we have

$$\begin{aligned}|\hat{Y}_t - Y_t|^2 &= |\hat{y} - y|^2 + \int_0^t 2(\hat{Y}_s - Y_s)(g(\hat{Y}_s) - g(Y_s))ds \\ &\quad + \int_0^t 2\beta\gamma\rho_{py}u_s(\hat{Y}_s - Y_s)(\sigma(\hat{Y}_s) - \sigma(Y_s))ds \\ &\le |\hat{y} - y|^2 - 2(k - A_1\beta\gamma|\rho_{py}|\|\sigma^{'}\|)\int_0^t |\hat{Y}_s - Y_s|^2 ds.\end{aligned}$$

Then, from the Gronwall inequality [25], we obtain

$$|\hat{Y}_t - Y_t| \le |\hat{y} - y|e^{-(k - A_1\beta\gamma|\rho_{py}|\|\sigma^{'}\|)t}. \quad (3.8)$$

Similarly, since the sample paths of $\hat{r}_t - r_t$ are continuously differentiable, one can obtain

$$|\hat{r}_t - r_t| \le |\hat{r} - r|e^{-(c_l - A_1\hat{\sigma}\gamma|\rho_{pr}|\|\sigma^{'}\|)t}. \quad (3.9)$$

The above results (3.7), (3.8) and (3.9) are put together and provide

$$\begin{aligned}|\hat{z}_t - z_t| &\leq \gamma(A_1+1)\int_0^t |\hat{r}-r|e^{-(c_l - A_1\hat{\sigma}\gamma|\rho_{pr}|\|\sigma'\|)s}ds \\ &+ \gamma(1-\gamma){A_1}^2\sigma_u\|\sigma'\|\int_0^t |\hat{y}-y|e^{-(k-A_1\beta\gamma|\rho_{py}|\|\sigma'\|)s}ds\end{aligned}$$

which leads to $|\hat{z}_t - z_t| \leq L_1(t)|\hat{r}-r| + L_2(t)|\hat{y}-y|$, where

$$L_1(t) := \frac{\gamma(A_1+1)}{2(c_l - A_1\hat{\sigma}\gamma|\rho_{pr}|\|\sigma'\|)}(1-e^{-(c_l-A_1\hat{\sigma}\gamma|\rho_{pr}|\|\sigma'\|)t}),$$

$$L_2(t) := \frac{\gamma(1-\gamma){A_1}^2\sigma_u\|\sigma'\|}{2(k - A_1\beta\gamma|\rho_{py}|\|\sigma'\|)}(1-e^{-(k-A_1\beta\gamma|\rho_{py}|\|\sigma'\|)t}).$$

Thus, from (3.6), we have $\tilde{J}(\hat{y},\hat{r};c,u,T) - \tilde{J}(y,r;c,u,T) \leq L(|\hat{r}-r|+|\hat{y}-y|)\tilde{\mathbf{E}}\int_0^T c_t^\gamma e^{\hat{z}_t}dt$, where $L := \max\{L_1, L_2\}$. Here, as pointed out by [5], L does not depend on T from the hypothesis. Then it follows that

$$|W(\hat{y},\hat{r},T) - W(y,r,T)| \leq K_2 L(|\hat{r}-r| + |\hat{y}-y|). \tag{3.10}$$

Subsequently, $W(y,r,T)$ is Lipschitz continuous for each fixed T. Moreover, since $W(y,r,T)$ is bounded by Lemma 3.1, it converges uniformly to $\bar{W}(y,r)$ in compact sets as $T\to\infty$. □

4. Optimal portfolio selection

Once Lemma 3.1 and Lemma 3.2 are established, it is ready to verify that the value function $W(y,r,T)$ is the unique bounded solution of (3.3) and $\bar{W}(y,r)$ is equal to the value function $W(y,r)$.

Theorem 4.1. *$W(y,r,T)$ is the unique bounded classical solution of* (3.3) *with the initial condition $W(y,r,0)=0$.*

Proof The proof of this theorem is a straightforward extension of the argument in the book by Fleming and Rishel [26] and so we do not repeat the proof here. □

Next, we prove that $\bar{W}(y,r)$ is the solution of (2.11).

Theorem 4.2. *Suppose that $\bar{\alpha} > 0$ and the assumption of Lemma 3.2 hold. Then $\bar{W}(y,r) \in \mathbf{C}^{2,2}(\mathbb{R}^2)$ and it is a classical solution of* (2.11) *with $\bar{W}(y,r)$ and $\bar{W}^{-1}(y,r)$ bounded.*

Proof First, we shall estimate the partial derivative $W_T(y,r,T)$. Since the inequality $z_t \leq -\bar{\alpha}t$ holds, for each $(c,u)\in\mathcal{A}$ and $T' > T$ we have

$$\tilde{J}(y,r;c,u,T') - \tilde{J}(y,r;c,u,T) = \tilde{\mathbf{E}}\int_T^{T'} c_t^\gamma e^{z_t}dt \leq c_u^\gamma e^{-\bar{\alpha}T}(T'-T).$$

This implies that $W(y,r,T') - W(y,r,T) \le c_u^\gamma e^{-\bar{\alpha}T}(T'-T)$ or equivalently $W(y,r,T+h) - W(y,r,T) \le c_u^\gamma e^{-\bar{\alpha}T}h$, where $T' = T+h$, $h>0$. Then dividing this inequality by h and letting $h \to 0$ yield

$$0 \le W_T(y,r,T) \le c_u^\gamma e^{-\bar{\alpha}T}. \tag{4.1}$$

Next, we shall estimate the first and second partial derivatives W_y, W_{yy}, W_r, W_{rr} and W_{yr}. Taking $\hat{y} = y+h$, $h>0$, from the inequality (3.10), we have $|W(\hat{y},r,T) - W(y,r,T)| \le K_2Lh$. Then dividing this inequality by h and letting $h \to 0$, we obtain

$$|W_y(y,r,T)| \le K_2L. \tag{4.2}$$

Similarly, we have

$$|W_r(y,r,T)| \le K_2L. \tag{4.3}$$

In addition, Lemma 3.1 and (4.1)-(4.3) yield that $|W_{yy}(y,r,T)|$, $|W_{rr}(y,r,T)|$ and $|W_{yr}(y,r,T)|$ are also uniformly bounded by a constant independent of T.

Moreover, since $W(y,r,T)$ is a classical solution of (3.3) and the part

$$\sup_u \gamma \left\{ [r + (\mu - r)u + \frac{1}{2}u^2\sigma^2(y)(\gamma-1)]W + \beta\rho_{py}u\sigma(y)W_y + \hat{\sigma}\rho_{pr}u\sigma(y)W_r \right\}$$

of (3.3) is locally Lipschitz, the well-known Arzela-Ascoli theorem implies that there is a subsequence T_n which goes to infinite such that $W(y,r,T_n)$, as well as its first and the second derivatives, converges uniformly to $\bar{W}(y,r)$ on compact sets. Hence, $\bar{W}(y,r) \in \mathbf{C}^{2,2}(\mathbb{R}^2)$ and it is a solution of (2.11). □

Now, we obtain optimal consumption and investment controls as follows.

Theorem 4.3. *(Verification Theorem) Suppose that $\bar{\alpha} > 0$ and the assumption of Lemma 3.2 holds and $W(y,r)$ is the unique solution of the Hamilton-Jacobi-Bellman equation (2.11). Then $\bar{W}(y,r) = W(y,r)$ and the strategy (u^*, c^*) given by*

$$u^*(y,r) := \frac{(\mu - r)W + \beta\rho_{py}\sigma(y)W_y + \hat{\sigma}\rho_{pr}\sigma(y)W_r}{(1-\gamma)\sigma^2(y)W}, \tag{4.4}$$

$$c^*(y,r) := W^{\frac{1}{\gamma-1}} \tag{4.5}$$

is the optimal policy in $\mathcal{A}$.

Proof Given (y,r), the inequality $\bar{W}(y,r) \le W(y,r)$ holds clearly by definition. Given an admissible strategy $(u,c) \in \mathcal{A}$, equation (3.3) applied by the Feynman-Kac formula (cf. [22]) yields

$$\bar{W}(y,r) \ge \tilde{\mathbf{E}} \int_0^T c_t^\gamma e^{z_t} dt + \tilde{\mathbf{E}}[\bar{W}(Y_T, r_T)e^{z_T}] \tag{4.6}$$

Letting $T \to \infty$ leads to $\bar{W}(y,r) \ge \tilde{\mathbf{E}} \int_0^\infty c_t^\gamma e^{z_t} dt$ since the second term on the right side of (4.6) goes to zero as $T \to \infty$. So, we obtain the inequality $\bar{W}(y,r) \ge W(y,r)$.

Furthermore, the feed back controls given by

$$u^*(y,r) = \text{argmax}_u \gamma \left\{ [r + (\mu - r)u + \frac{1}{2}u^2\sigma^2(y)(\gamma - 1)]\bar{W} + \beta\rho_{py}u\sigma(y)\bar{W}_y + \hat{\sigma}\rho_{pr}u\sigma(y)\bar{W}_r \right\},$$

$$c^*(y,r) := \bar{W}^{\frac{1}{\gamma-1}},$$

make inequality (4.6) become equality. Note that u^* and c^* are bounded and locally Lipschitz. Therefore, we obtain the desired optimality of the policies u^* and c^*. □

Figure 1 shows the behavior of the optimal strategy (u^*, c^*) with respect to the stochastic volatility level y and the interest rate r for some fixed parameters and functions. It shows that the optimal strategy should be more sensitive to the volatility level as the interest rate goes up.

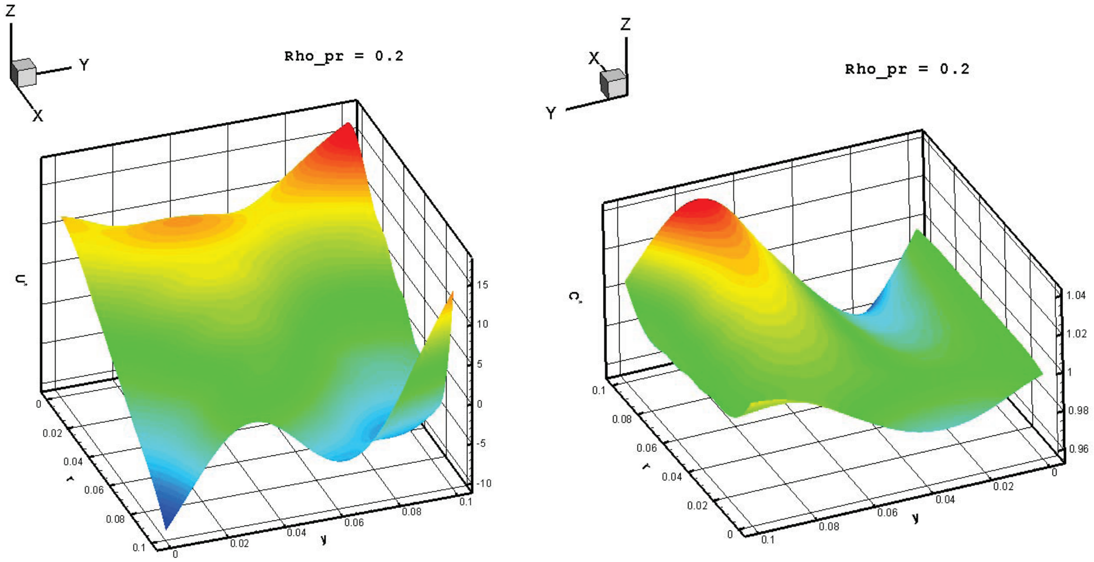

FIGURE 1. Typical surfaces for the optimal investment policy (u^*, c^*) are drawn, where the parameters and functions used are given by $\gamma = 0.4$, $\sigma(y) = \frac{1}{10}e^{-y}$, $\rho_{pr} = 0.2$, $\mu = 0.08$, $\alpha = 1$, $\rho_{py} = -0.05$, $g(y) = \alpha'(m - y)$, $f(r) = \kappa(m' - r)$, $\beta = 5$, $\hat{\sigma} = 0.5$, $\alpha' = 100$, $m = 0$, $m' = 0$ and $\kappa = 5$.

Figure 2 represents cross sections of the optimal policy given by fixing the volatility variable for some choices of the correlation coefficient ρ_{pr} between the price of a risky asset and the interest rate. It indicates that the optimal policy should be more irregular (careful) with respect to the interest rate as the correlation becomes far away from zero.

5. CONCLUSION

By studying the consumption and investment controls maximizing the total expected discounted HARA utility of consumption under the mixture of stochastic volatility and stochastic

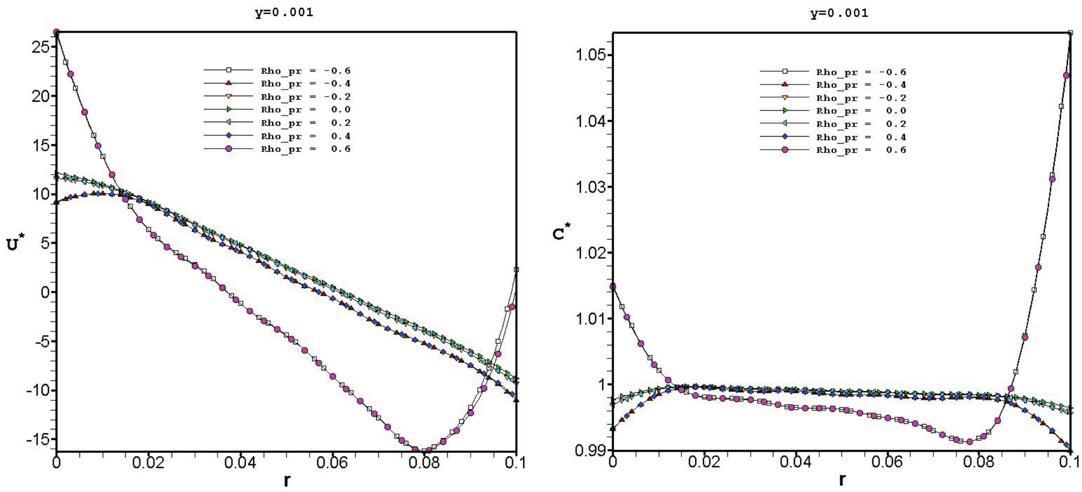

FIGURE 2. For a fixed level of stochastic volatility, the graphs of the optimal policy (u^*, c^*) are drawn with respect to the interest rate for several choices of the correlation between the price of a risky asset and the interest rate, i.e., $\rho_{pr} = 0, \pm 0.2, \pm 0.4, \pm 0.6$. The parameters and functions used in this figure are given by $\gamma = 0.4$, $\sigma(y) = \frac{1}{10}e^{-y}$, $\mu = 0.08$, $\alpha = 1$, $\rho_{py} = -0.05$, $g(y) = \alpha'(m - y)$, $f(r) = \kappa(m' - r)$, $\beta = 5$, $\hat{\sigma} = 0.5$, $\alpha' = 100$, $m = 0$, $m' = 0$ and $\kappa = 5$.

interest rates, we verify that the associated Hamilton-Jacobi-Bellman equation has a unique solution and the solution becomes a value function with the desired smooth properties. Moreover, we obtain explicitly the dependence structure of the optimal policy on the correlation between the price of an underlying risky asset and stochastic volatility or stochastic interest rates. As a result, it requires more careful selection of a portfolio as the correlation between the underlying asset and the interest rate becomes far away from zero. A possible next step of research work on this type of mission is an extension to underlying models where interest rate and volatility are operated not only by a continuous Ito diffusion but also by a discontinuous process.

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