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FIXED POINT THEOREMS FOR DIGITAL IMAGES

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Abstract. In this paper, as a survey paper, we review many works related to fixed point theory for digital spaces using Lefschetz fixed point theorem, Banach fixed point theorem, Nielsen fixed point theorem and so forth. Besides, we refer some properties of the fixed point property of a digital k-retract.

1. Introduction

In mathematics, there are many theorems for studying fixed point theory such as Brouwer fixed point theorem, Lefschetz fixed point theorem, Banach fixed point theorem, Schauder's fixed point theorem, Nielsen fixed point theorem and so forth [2, 25, 26]. Indeed, using these theorems, we can recognize the existence of a fixed point of a compact mapping in terms of traces of the induced mappings on the algebraic topological tools such as homology groups of X. Owing to the usage of homology groups, it is well known that the *fixed point property* (*FPP* for short) is both a topological and a homotopy invariant [26].

Digital topology has a focus on studying digital topological properties of nD digital images [27, 11, 23, 24], which has contributed to some areas of computer sciences such as computer graphics, image processing, mathematical morphology and so forth. Thus an establishment of a digital version of the ordinary Lefschetz number can be so meaningful. Thus the recent works [19, 20, 21] studied Brouwer, Nielsen and Banach fixed point theorems from the viewpoint of digital topology which corrects the papers [7, 8, 9] written by Ege *et al.*. Furthermore, the recent works [18, 19, 20, 21] precede many fixed point properties related to

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fixed point theory for digital images. Besides, Ege *et al.* [7, 8, 9] tried to formulate digital versions of the ordinary Lefschetz fixed theorem, Banach fixed point theorem and Schauder's fixed point theorem. However, these approaches are due to invoke wrong results (see [5, 18, 19, 20, 21]). Before referring the work, first of all, we say that a digital image (X, k) has the *FPP* [27] if every k-continuous map $f : (X, k) \to (X, k)$ has a fixed point $x \in X$, *i.e.* $f(x) = x$.

Motivated by the fixed point theories of [6, 25], Ege *et al.* [8] tried to study the Lefschetz number from the viewpoint of digital homotopy theory. To work this out in detail, Ege *et al.* [8] used a digital k-homotopy [3], relative digital homotopy [11, 13], digital surface theory [13, 14], the digital homology group proposed in [1, 8] and so forth. Unlike the homotopy invariant property of the ordinary Lefschetz number, using A counterexample to the digital homotopy invariant property of the digital Lefschetz number (see Example 4.5 of the current paper), we prove that a digital version of the Lefschetz number in [8] is not a digital homotopy invariant (see Proposition 4.4 in the present paper).

Even though the works [5, 18, 19, 20, 21] correct many errors in [7, 8, 9], the current paper explains them in details and corrects some further errors. Besides, the present paper is based on the presentation of 11th ICFPTA (2015) [19] which firstly pointed out many errors in [7, 9], Banach fixed point theorem for digital images [20], fixed point theorems for digital images [21].

Indeed, the posting of an abstract to the international conference held at the 11th ICFPTA (July 20-25, 2015, Galatasaray University, Istanbul, Turkey) [19] started on the date(June 30, 2015). Hence the author of the present work came to conclusion that the works of [7, 8, 9] have lots of errors and some usages of several mathematical models already developed in [13, 14] were proceeded without any citation.

Recently, the works [5, 18, 19, 20, 21] fixed many errors in [7, 8, 9]. Even though the recent paper [9] studied Banach fixed point theorem for digital images, the recent paper [20] also corrected some errors in [9] and improved it. Despite of this works, we need to further comment on the topic. In addition, we need to refer some property of the fixed point property of a digital k-retract in [7].

The rest of the paper is organized as follows: Section 2 provides basic notions from digital topology. Section 3 investigates some properties of digital k-contractibility inherited from a digital k-homotopy and studies digital homotopy axiom associated with the digital homology in [1, 4, 8]. Namely, we study that the digital homology does not have the digital homotopy property. Section 4 proves that a digital version of the Lefschetz number in [8] is not a digital homotopy invariant. Owing to this finding, we correct some errors in [7, 8, 9] and improves the paper (O. Ege, I. Karaca in The Bulletin of the Belgian Mathematical Society-Simon Stevin, Applications of the Lefschetz Number to Digital Images, Vol.21(5) (2014), 823-839). Besides, we need to point out that Ege *et al.* [8] used some of figures such as MSC_4 [11], MSS_6 [13] and MSS'_{18} [13, 14] and a digital wedge [11] without citation. Section 5 concludes the paper with some remarks.

In this paper all digital images (X, k) are assumed to be k-connected and $|X| \geq 2$.

2. Preliminaries

To study the *FPP* of digital images from the viewpoint of digital topology, we need to recall some basic notions from digital topology such as digital k-connectivity, a digital k-neighborhood, digital continuity and so forth [11, 24, 27]. Let $\mathbf{N}$ and $\mathbf{R}$ represent the sets of natural numbers and real numbers, respectively. Let $\mathbf{Z}^n$ be the set of points in the Euclidean nD space with integer coordinates, $n \in \mathbf{N}$.

To study nD digital images, we will say that two distinct points $p, q \in \mathbf{Z}^n$ are k-(or $k(m,n)$-)adjacent if they satisfy the following [11] (see also [15, 16]):
For a natural number m, $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n,$$

are $k(m,n)$-(k-, for brevity)adjacent if

$$\text{at most } m \text{ of their coordinates differs by } \pm 1, \text{ and all others coincide.} \tag{2.1}$$

Concretely, these $k(m,n)$-adjacency relations of $\mathbf{Z}^n$ are determined according to the numbers $m, n \in \mathbf{N}$ [11] (see also [13]).

In terms of the operator (2.1), the k-adjacency relations of $\mathbf{Z}^n$ are obtained [11] (see also [15, 16]) as follows:

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \tag{2.2}$$

where $C_i^n = \frac{n!}{(n-i)!\ i!}$.
For instance, [27, 11, 15]

$$(n,m,k) \in \left\{ \begin{array}{l} (2,2,8),(2,1,4); \\ (3,3,26),(3,2,18),(3,1,6); \\ (4,4,80),(4,3,64),(4,2,32),(4,1,8). \end{array} \right\}$$

A. Rosenfeld [27] called a set $X \subset \mathbf{Z}^n$ with a k-adjacency a digital image, denoted by (X,k). Indeed, to follow a graph theoretical approach of studying nD digital images [11, 22, 27], both the k-adjacency relations of $\mathbf{Z}^n$ of (2.2) and a digital k-neighborhood are used to study digital images [11, 12]. More precisely, using the k-adjacency relations of $\mathbf{Z}^n$ of (2.2), we say that a digital k-neighborhood of p in $\mathbf{Z}^n$ is the set [27] $N_k(p) := \{q \mid p \text{ is } k\text{-adjacent to } q\}$. Furthermore, we often use the notation [24]

$$N_k^*(p) := N_k(p) \cup \{p\}.$$

For $a, b \in \mathbf{Z}$ with $a \lneq b$, the set $[a,b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \mid a \le n \le b\}$ with 2-adjacency is called a digital interval [24]. Besides, for a k-adjacency relation of $\mathbf{Z}^n$, a simple k-path with $l+1$ elements in $\mathbf{Z}^n$ is assumed to be an injective sequence $(x_i)_{i \in [0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\, i - j \,| = 1$ [24]. If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x,y)$, is the number l. A simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$ [24, 11] (see Figure 1(a)), is the simple k-path $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if either $j = i+1 (mod\, l)$ or $i = j+1 (mod\, l), l \ge 4$ [24]. For a digital image (X,k), as a generalization of $N_k^*(p)$ [24] the digital k-neighborhood of $x_0 \in X$ with radius 1 is defined in X to be the following subset of X [11, 14]

$$N_k(x,1) = N_k^*(x) \cap X. \tag{2.3}$$

In Section 4, in relation to the study of Lefschetz fixed point theorem, we use the notions of both a digital simplicial complex derived from a digital image (X,k) [12, 14, 1, 8] and a digital homology group $H_q^k(X)$ introduced in [1] (for more details, see [1, 7, 8]). In terms of this approach, the following q-th digital simplicial group is established in [1, 8].

$$H_q^k(X) := Z_q^k(X)/B_q^k(X) \tag{2.4}$$

Lemma 2.1. [1, 4] *For the digital images* $([0,l]_{\mathbf{Z}}, 2)$, $SC_k^{n,l}$ *and a singleton, we obtain the following:*

(1)

$$H_q^2([0,l]_{\mathbf{Z}}) = \begin{cases} \mathbf{Z}, & q = 0; \\ 0, & q \neq 0. \end{cases}$$

(2)

$$H_q^k(SC_k^{n,l}) = \begin{cases} \mathbf{Z}, & q \in \{0,1\}; \\ 0, & q \notin \{0,1\}. \end{cases}$$

(3) $H_q^k(\{x_0\})$ *is isomorphic to* $\mathbf{Z}$ *if* $q = 0$*, and it is trivial if* $q \neq 0$.

3. Some properties of digital k-contractibility and digital homology

To study fixed point theory for digital images, the present paper follows the Rosenfeld model [27]. Since digital continuity is an essential notion in digital topology, the digital continuity of a map $f : (X, k_0) \to (Y, k_1)$ was established in [27] by saying that f maps every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1). Motivated by this approach, since the digital k-neighborhood of (2.3) is very useful in digital topology, the digital continuity of maps between digital images was represented with the following version, which can be substantially used to study digital images (X, k) in $\mathbf{Z}^n$.

Proposition 3.1. [11, 14] *Let* (X, k_0) *and* (Y, k_1) *be digital images in* $\mathbf{Z}^{n_0}$ *and* $\mathbf{Z}^{n_1}$*, respectively. A function* $f : (X, k_0) \to (Y, k_1)$ *is digitally* (k_0, k_1)*-continuous if and only if for every* $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.

Hereafter, we will use the term "(k_0, k_1)-continuous" for short instead of "digitally (k_0, k_1)-continuous". In Proposition 3.1 in case $n_0 = n_1$ and $k_0 = k_1 := k$, the map f is called a "k-continuous" map instead of a "(k, k)-continuous" map.

According to Proposition 3.1, we see that the point $y \in N_{k_0}(x, 1)$ is mapped into the point $f(y) \in N_{k_1}(f(x), 1)$, which implies that for the points x, y which are k_0-adjacent a (k_0, k_1)-continuous map f has the property

$$f(x) = f(y) \text{ or } f(y) \in N_{k_1}(f(x)) \cap Y.$$

Since an nD digital image (X, k) is viewed as a set $X \subset \mathbf{Z}^n$ with one of the k-adjacency relations of (2.2) (or a digital k-graph in [11, 14]), in relation to the classification of nD digital images, we use the term a (k_0, k_1)-isomorphism as in [12] (see also [22]) rather than a (k_0, k_1)-homeomorphism as in [6].

Definition 1. [3, 22] *(see also* [12, 14]*) Consider two digital images* (X, k_0) *and* (Y, k_1) *in* $\mathbf{Z}^{n_0}$ *and* $\mathbf{Z}^{n_1}$*, respectively. Then a map* $h : X \to Y$ *is called a* (k_0, k_1)*-isomorphism if* h *is a* (k_0, k_1)*-continuous bijection and further,* $h^{-1} : Y \to X$ *is* (k_1, k_0)*-continuous.*

In Definition 2, in case $n_0 = n_1$ and $k_0 = k_1 := k$, we call it a *k-isomorphism* [11]. Furthermore, we denote by $X \approx_k Y$ a k-isomorphism from X to Y [22] (see also [12]).

To study a digital topological invariant of the digital homology referred in Section 2, the following was established.

Proposition 3.2. [1, 8] *If* $h : (X, k_0) \to (Y, k_1)$ *is a* (k_0, k_1)*-isomorphism, then the induced homomorphism* $h_* : H_q^{k_0}(X) \to H_q^{k_1}(Y), q \geq 0$ *is an isomorphism.*

To study the notion of k-contractibility of a digital image (X, k), the following digital homotopy was used in [8].

Definition 2. [23, 3] *Let* (X, k_0) *and* (Y, k_1) *be digital images. Let* $f, g : X \to Y$ *be* (k_0, k_1)*-continuous functions. Suppose there exist* $m \in \mathbf{N}$ *and a function* $H : X \times [0, m]_{\mathbf{Z}} \to Y$ *such that*
• *for all* $x \in X, H(x, 0) = f(x)$ *and* $H(x, m) = g(x)$*;*
• *for all* $x \in X$*, the induced function* $H_x : [0, m]_{\mathbf{Z}} \to Y$ *given by* $H_x(t) := H(x, t)$ *for all* $t \in [0, m]_{\mathbf{Z}}$ *is* $(2, k_1)$*-continuous; and*
• *for all* $t \in [0, m]_{\mathbf{Z}}$*, the induced function* $H_t : X \to Y$ *given by* $H_t(x) := H(x, t)$ *for all* $x \in X$ *is* (k_0, k_1)*-continuous.*
Then we say that H *is a* (k_0, k_1)*-homotopy between* f *and* g*.*

When f and g are (k_0, k_1)-homotopic in Y, we denote by $f \simeq_{(k_0,k_1)} g$ the homotopic relation [3]. In addition, if $n_0 = n_1$ and $k_0 = k_1$, then we say that f and g are k_0-homotopic in Y and use the notation $f \simeq_{k_0} g$.

To study the *FPP* of digital images, the following k-contractibility was used in [8].

Definition 3. [3] *Let* (X, k_0) *and* (Y, k_1) *be digital images. A* (k_0, k_1)*-continuous map is digitally nullhomotopic if* f *is* (k_0, k_1)*-homotopic in* (Y, k_1) *to a constant map. A digital image* (X, k) *is a* k*-contractible if its identity map is digitally nullhomotopic.*

To study a digital version of the Lefschetz number for digital images (X, k) such as digital k-curves and digital images in $\mathbf{Z}^n, n \in \{2, 3\}$, we need the following:

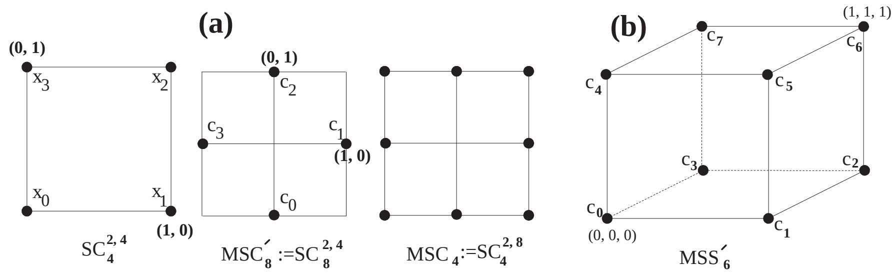

FIGURE 1. (a) Simple closed 4-curves $SC_4^{2,4}$ [8] and $SC_4^{2,8}$ [11], and a simple closed 8-curve $MSC_8 := SC_8^{2,4}$ [3]; (b) The digital image $(MSS_6', 6)$ [8].

Theorem 3.3. *(1)* $MSS_6' := \{c_i \mid i \in [0,7]_{\mathbf{Z}}\}$ *is* 6*-contractible.*
(2) $MSC_4 := SC_4^{2,8}$ *is not* 4*-contractible* [11].
(3) Both $SC_4^{2,8}$ *and* $SC_8^{2,4} := MSC_8'$ *are* 8*-contractible* [3, 11].

Proof: Let us prove 6-contractibility of MSS_6' as follows: consider the map $H : MSS_6' \times [0,3]_{\mathbf{Z}} \to MSS_6'$ (see Figure 2) given by

$$\begin{cases} H(c_i, 0) = c_i, \ i \in [0,7]_{\mathbf{Z}}; \\ H(c_1, 1) = c_0, \ H(c_2, 1) = c_3, \\ H(c_5, 1) = c_4, \ H(c_6, 1) = c_7, \\ H(c_i, 1) = c_i, \ i \in \{0, 3, 4, 7\}; \\ H(c_i, 2) = c_0, \ i \in \{0, 1, 2, 3\}, \\ H(c_i, 2) = c_4, \ i \in \{4, 5, 6, 7\}; \\ H(c_i, 3) = c_0, \ i \in [0,7]_{\mathbf{Z}}. \end{cases}$$

Then it is clear that the map H is a 6-homotopy between $1_{MSS_6'}$ and the constant map $C_{\{c_0\}}$, which implies that MSS_6' is 6-contractible and finally completes the proof.
In addition, using the method similar to the proof of the 6-contractibility of MSS_6', we see that $SC_8^{2,4}$ is also 8-contractible [6, 11]. The other cases (2) and (3) were already proved in [3, 11, 13, 14]. □

To study the non-homotopy property of digital homology, we need to recall homology groups of a digital image $MSS_6' \subset \mathbf{Z}^3$, as follows:

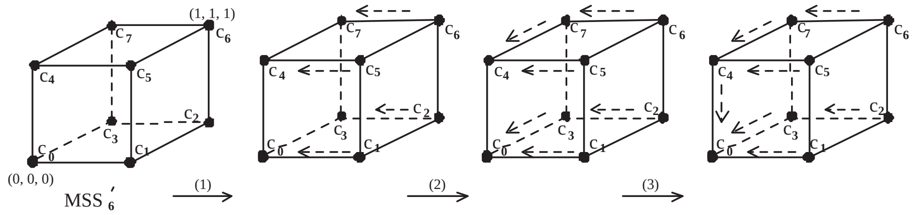

FIGURE 2. Explanation of a process of 6-contractibility of MSS'_6 [13].

Lemma 3.4. [1, 8] *For the digital image $MSS'_6 \subset \mathbf{Z}^3$, we obtain*

$$H_q^6(MSS'_6) = \begin{cases} \mathbf{Z}, & q = 0; \\ \mathbf{Z}^5, & q = 1; \\ 0, & q \notin \{0, 1\}. \end{cases}$$

Theorem 3.5. [21] *The digital homology does not have the digital homotopy property.*

Before proving this theorem, we say "the digital homotopy property related to the digital homology" as follows: for two (k_0, k_1)-continuous map $f, g : (X, k_0) \to (Y, k_1)$ if f is (k_0, k_1)-homotopic to g, then the induced homomorphisms $f_*, g_* : H_*^{k_0}(X) \to H_*^{k_1}(Y)$ coincide.

Proof: It suffices to suggest examples explaining that the digital homology does not have the digital homotopy property in a way different from that of [21].
Owing to the 6-contractibility of $MSS'_6 := (c_i)_{i \in [0,7]_{\mathbf{Z}}}$ (see Figure 2) (see Theorem 3.3(1)), we see that $1_{MSS'_6}$ is 6-homotopic to the constant map $C_{\{c_0\}}$. However, by Lemmas 2.1 and 3.4, we obtain

$$H_q^6(MSS'_6) \neq H_q^6(\{c_0\}).$$

As a result, while $1_{MSS'_6}$ is 6-homotopic to the constant map $C_{\{c_0\}}$, their digital homology groups of MSS'_6 and the singleton $\{c_0\}$ are not equal. □

4. Digital non-homotopy invariant property of the Lefschetz number for digital images and Banach fixed point theorem for digital spaces

To study the non-homotopy invariant of Lefschetz number, we need to recall the notion of a digital wedge firstly introduced in [11] with a compatible k-adjacency of a digital wedge sum, as follows.

Definition 4. [11] *(see also* [17]*) For pointed digital images* $((X, x_0), k_0)$ *in* $\mathbf{Z}^{n_0}$ *and* $((Y, y_0), k_1)$ *in* $\mathbf{Z}^{n_1}$, *the wedge sum of* (X, k_0) *and* (Y, k_1), *written* $(X \vee Y, (x_0, y_0))$, *is the digital image in* $\mathbf{Z}^n$, $n = max\{n_0, n_1\}$,

$$\{(x, y) \in X \times Y| \quad x = x_0 \quad or \quad y = y_0\} \tag{4.1}$$

with the following compatible $k(m, n)$*(or* k*)-adjacency relative to both* (X, k_0) *and* (Y, k_1), *and the only one point* (x_0, y_0) *in common such that*

(W1) the $k(m, n)$ *(or* k*)-adjacency is determined by the numbers* m *and* n *with* $m = max\{m_0, m_1\}$ *satisfying* $(W1 - 1)$ *below, where the numbers* m_i *are taken from the* k_i*(or* $k(m_i, n_i)$*)-adjacency relations of the given digital images* $((X, x_0), k_0)$ *and* $((Y, y_0), k_1)$, $i \in \{0, 1\}$.

(W 1-1) In view of (4.1), induced from the projection maps, we can consider the natural projection maps

$$W_X : (X \vee Y, (x_0, y_0)) \to (X, x_0) \quad and \quad W_Y : (X \vee Y, (x_0, y_0)) \to (Y, y_0).$$

In relation to the establishment of a compatible k*-adjacency of the digital wedge sum* $(X \vee Y, (x_0, y_0))$, *the following restriction maps of* W_X *and* W_Y *on* $(X \times \{y_0\}, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ *and* $(\{x_0\} \times Y, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ *satisfy the following properties, respectively:*

$$\begin{cases} (1)\, W_X|_{X \times \{y_0\}} : (X \times \{y_0\}, k) \to (X, k_0) \text{ is a } (k, k_0)\text{-isomorphism; and} \\ (2)\, W_Y|_{\{x_0\} \times Y} : (\{x_0\} \times Y, k) \to (Y, k_1) \text{ is a } (k, k_1)\text{-isomorphism.} \end{cases}$$

(W2) Any two distinct elements $x(\neq x_0) \in X \subset X \vee Y$ *and* $y(\neq y_0) \in Y \subset X \vee Y$ *are not* $k(m, n)$ *(or* k*)-adjacent to each other.*

It is obvious that $(X \vee Y, k)$ is k-isomorphic to $(Y \vee X, k)$.

Although Ege *et al.* [8] studied a digital version of the Lefschetz number for digital images, this section makes some errors in the paper fixed and improves the paper. Indeed, Lefschetz [25] introduced the Lefschetz number of a map and proved that if the number is nonzero, then the map has a fixed point. For a formal statement of the theorem, let $f : X \to X$ be a continuous map from a compact triangulable space X to itself. Define the Lefschetz number $\lambda(f)$ of f by

$$\lambda(f) := \Sigma_{k\geq 0}(-1)^k Tr(f_* \,|\, H_k(X, \mathbf{Q})) \tag{4.2}$$

the alternating (finite) sum of the matrix traces of the linear maps induced by f on the $H_k(X, \mathbf{Q})$, the singular homology of X with rational coefficients. A simple version of the Lefschetz fixed-point theorem states: if $\lambda(f) \neq 0$, then f has at least one fixed point, *i.e.* there exists at least one point $x \in X$ such that $f(x) = x$.
Motivated by the property of (4.2), Ege *et al.* [7] proposed a formula (see also Definition 3.3 of [8]) using the digital homology proposed in [1, 8], as follows:

Definition 5. [7] *For a k-continuous map $f : (X, k) \to (X, k)$, where (X, k) is a digital image whose digital homology groups are finitely generated and vanish above some dimension, the Lefschetz number $\lambda(f)$ is defined as follows:*

$$\lambda(f) = \Sigma_{i=0}^{\infty}(-1)^i tr(f_*), \tag{4.3}$$

where $f_* : H_i^k(X) \to H_i^k(X)$ is the induced homomorphism by the given map f, where X is the digital simplex inherited from the digital image (X, k).

Using the digital simplex (see Definition 1) from a digital image (X, k), Ege *et al.* [7] established the following which are essential parts of [7, 8] (see Theorems 4.1 and 4.2 below).

Theorem 4.1. *(1) (Theorem 3.3 of the paper* [7] *and Theorem 3.4 of* [8]*) If (X, k) is a finite digital simplicial complex, or the retract of some finite digital simplicial complex, and $f : (X, k) \to (X, k)$ is a k-continuous map with $\lambda(f) \neq 0$, then f has a fixed point.*

(2) (Theorem 3.5 of [8]*) Let (X, k) be a digital image. If a k-continuous map $f : (X, k) \to (X, k)$ has $\lambda(f) \neq 0$, then any k-continuous map k-homotopic to f has a fixed point.*

However, this theorem is invalid (see [5, 19, 21]. Besides, Ege *et al.* [7, 8] studied the *FPP* of a digital k-retract (see Theorem 3.11 of [7]), as follows:

Theorem 4.2. [7] *Let (A, k) be a k-retract of (X, k). If (X, k) has the FPP, then (A, k) has also the FPP.*

Basically this assertion is very trivial. However, this case can be true only the trivial case, as follows:

Remark 4.3. *Due to the study of* [27], *it is clear that only the digital image (X, k) with $|X| = 1$ has the FPP. Hence, to support Theorem 4.2, we see that both (A, k) and (X, k) are all singletons.*
However, we see that in another digital topological category instead of the Rosenfeld model this property may have nontrivial cases.

Unlike the homotopy invariant property of the ordinary fixed point theorem, we obtain the following:

Proposition 4.4. *The non-homotopy property of the digital homology (see Theorem 3.5) implies invalidity of Theorem 4.1.*

Let us now confirm the invalidity of Theorem 4.1 by using a counterexample to Theorem 4.1 (see Example 4.5), which guarantees Proposition 4.4.

Example 4.5. *Let us consider* $(X, 8)$ *which is a wedge sum with an compatible* 8*-adjacency (see Figure 3), where* $X := \{x_i \,|\, i \in [0, 7]_{\mathbf{Z}}\}$. *Then, by Theorem 3.3(3), it is obvious that* $(X, 8)$ *is* 8*-contractible. This implies that the identity map* 1_X *is* 8*-homotopic to the constant map* $C_{\{x_0\}}$, *where* $x_0; = (0, 0)$.
Let us now investigate the Lefschetz number of 1_X. *Indeed, we see that*

$$\lambda(1_X) = \Sigma_{i=0}^{\infty}(-1)^i tr((1_X)_*) = 1 - 0 + 2 + \cdots = 3. \qquad (4.4),$$

because $H_q^8(X)$ *is isomorphic to* $\mathbf{Z} \times \mathbf{Z}$ [10]. *Furthermore, consider the* 8*-continuous self-map* g *of* $(X, 8)$ *in such a way:*

$$g(x_i) = x_0, i \in [1, 7]_{\mathbf{Z}} \text{ and for } g(x_0) = x_1 := (1, 1).$$

Thus and we obtain

$$1_X \simeq_8 g, \qquad (4.5)$$

which satisfies the hypothesis of Theorem 4.1.
However, it is clear that

$$\lambda(g) = \Sigma_{i=0}^{\infty}(-1)^i tr(g_*) = 1 - 0 + \cdots = 1. \qquad (4.6).$$

In addition, the map g *has no fixed point, by (4.2)-(4.6) contrary to Theorem 4.1.*

As a result we obtain the following:

Proposition 4.6 (Invalidity of a digital homotopy invariant of the digital Lefschetz number proposed in [8])**.** *In view of Theorem 3.5, Proposition 4.4 and Example 4.5, it turns out that the digital Lefschetz number is not a digital homotopy invariant, correcting an error in* [7, 8].

Remark 4.7. *In view of Proposition 4.4, although the recent papers* [7, 8] *studied Lefschetz fixed point theorem and Nielsen fixed point theorem for digital images, this kind of approach is invalid. Besides, although the recent paper* [9] *also studied Banach fixed point theorem for digital images, the recent work* [18] *corrected some errors in* [7, 8, 9] *and improved it.*

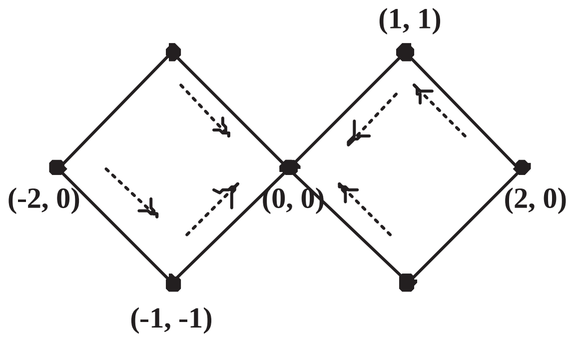

FIGURE 3. Explanation of the 8-contractibility of the wedge sum $(X, 8)$.

5. Further remarks and works

We have proved the non-homotopy invariant property of digital homology. Owing to the result, it turns out that a digital version of the ordinary Lefschetz number is insufficient for studying the *FPP* of digital images and further, the digital homology introduced in [1, 8] is not sufficient for studying the *FPP* of digital images.
As a further work, using various properties of digital surfaces, we need to further study the *FPP* of digital k-surfaces. In addition, after developing a new digital topological structure, we need to study its *FPP*. In spite of the study of fixed point theory for contractible space in digital topological sense, we need to further study non-trivial cases.

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