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REMARKS ON HOMOTOPIES ASSOCIATED WITH KHALIMSKY TOPOLOGY

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Abstract. Several kinds of homotopies have been substantially used to study topological properties of digital spaces. The present paper, as a survey article, studies some recent results in the field of homotopy theory associated with Khalimsky topology. In particular, Khalimsky topological properties of digital products related to the establishment of the homotopies are mainly treated.

1. Introduction

In digital homotopy theory, we have followed several approaches to find some digital topological properties of digital spaces. First of all, we need to recall the notion of a digital space [22]: we say that a *digital space* is a pair (X, R), where X is a nonempty set and R is a binary symmetric relation on X such that X is R-connected. The term "R-connected" means that for any two elements x and y of X there is a finite sequence $(x_i)_{i \in [0,l]_{\mathbf{Z}}}$ of elements in X such that $x = x_0$, $y = x_l$ and $(x_j, x_{j+1}) \in R$ for $j \in [0, l-1]_{\mathbf{Z}}$.

To study digital spaces, we have often used homotopic methods to do homotopic thinning, calculations of digital fundamental groups, and so forth [2, 3, 4, 5, 21, 23, 25, 27]. One of the typical tools is a digital homotopy and its applications. Thus the papers [2, 3, 21, 25] proposed digital homotopies to study digital fundamental groups of digital images. For some digital images, their digital fundamental groups need not be equal to each other [12] and further, they have their own features [2, 3, 21, 25]. Furthermore, we have used various tools such as a digital isomorphism [3, 6, 19], a digital homotopy equivalence [4, 11, 13, 20], a digital covering space [5, 7, 9, 10, 11, 12, 13], an elementary k-deformation [25, 27]

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and so forth. More precisely, the notion of a digital fundamental group was originated by Khalimsky [23]. Motivated by this notion, several kinds of digital homotopies [2, 3, 11, 13, 21, 25, 27] as well as a relative k-homotopy [5, 7] were established. Besides, recently Han [11, 13] introduced homotopies in the categories $KDTC$ [11] and CTC [13] which are different from both the k-homotopy in DTC [3, 5, 7] and a relative k-homotopy [5]. These notions have contributed to the study of digital homotopic properties of digital images (X, k). By using both the k-homotopic thinning [9, 13] and Han's digital covering theory [5, 7, 10, 12], we have often calculated digital fundamental groups of digital images.

The paper focuses on comparing two homotopies in the categories $KDTC$ [11] and CTC [13] (see Sections 4 and 5) and it is organized as follows:
Section 2 provides basic notions. Section 3 recalls some properties of homotopies associated with Khalimsky topology. Section 4 compares the roles of digital products in establishing homotopies in the categories $KDTC$ and CTC. Section 5 refers some difficulty in developing a homotopy in the category of Marcus Wyse (M- for brevity) topological spaces. Finally, Section 6 concludes the paper with a summary and further works.

2. Preliminaries

In the present paper we will often use the following notation: for $a, b \in \mathbf{Z}$ with $a \lneq b$, the set $[a, b]_{\mathbf{Z}} := \{n \in \mathbf{Z} \,|\, a \leq n \leq b\}$ is called a digital interval [3, 26]. In order to study a digital space in $\mathbf{Z}^n, n \in \mathbf{N}$, we have often used the following adjacency relations of $\mathbf{Z}^n$. Consequently, as a generalization of k-adjacency relations of a low dimensional digital space in $\mathbf{Z}^2$ and $\mathbf{Z}^3$ [26, 28], the adjacency relations of $\mathbf{Z}^n$ were established [5] (for more details, see [9, 11]), as follows:
for a natural number m, $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n,$$

are $k(m, n)$-(k-, for brevity)adjacent if

$$\text{at most } m \text{ of their coordinates differs by } \pm 1, \text{ and all others coincide.} \tag{2.1}$$

Hereafter, we may use the notation $k := k_m$ or $k(m, n)$, where $k := k_m := k(m, n)$ is the number of points q which are k-adjacent to a given point p.

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \tag{2.2}$$

where $C_i^n = \frac{n!}{(n-i)!\ i!}$.

As usual, a (binary) digital space (or a digital image) (X, k) is considered in a digital picture $(\mathbf{Z}^n, k, \bar{k}, X)$ with $k \neq \bar{k}$ in [26, 28]. But in the paper we will concern with only the k-adjacency of (X, k). For an adjacency relation k of $\mathbf{Z}^n$, a simple k-path with l elements on $\mathbf{Z}^n$ is assumed to be a sequence $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\,i - j| = 1$ [26]. Then the number l is called the length of a simple k-path. Furthermore, a simple closed k-curve with l elements in $\mathbf{Z}^n$ is a simple k-path $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}}$ with $x_0 = x_l$, where x_i and x_j are k-adjacent if and only if $j = i + 1 (mod\, l)$ or $i = j + 1 (mod\, l)$ [26]. By $SC_k^{n,l}$ we denote a simple closed k-curve with l elements on $\mathbf{Z}^n$ [7]. In addition, for $x \in \mathbf{Z}^n$ we follow the notations $N_k(x) := \{y \in \mathbf{Z}^n \,|\, x \text{ is } k \text{ -adjacent to } y\}$ and $N_k^*(x) := N_k(x) \cup \{x\}$ [26].

The following notion has been often used to study digital k-curve and digital k-surface theory.

Definition 1. [5] *(see also* [7]*) For a digital space (X, k) in $\mathbf{Z}^n$, the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset of X $N_k(x_0, \varepsilon) = \{x \in X \mid l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}$, where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$.*

Let us move onto the study of Khalimsky (K-, for short, if there is no ambiguity) topology. Motivated by the Alexandroff space [1], the *Khalimsky line topology* on $\mathbf{Z}$ is induced by the set $\{[2n-1, 2n+1]_{\mathbf{Z}} : n \in \mathbf{Z}\}$ as a subbase [1]. Furthermore, the product topology on $\mathbf{Z}^n$ induced by $(\mathbf{Z}, \kappa)$ is called the *Khalimsky product topology* on $\mathbf{Z}^n$ (or *Khalimsky nD space*) which is denoted by $(\mathbf{Z}^n, \kappa^n)$. A point $x = (x_1, x_2, \cdots, x_n) \in \mathbf{Z}^n$ is *pure open* if all coordinates are odd; and it is *pure closed* if each of the coordinates is even [24]. The other points in $\mathbf{Z}^n$ are called *mixed* [24].

For a point $p := (p_1, p_2)$ in $(\mathbf{Z}^2, \kappa^2)$, its smallest open neighborhood $SN_K(p)$ is obtained [8, 24].

$$SN_K(p) := \left\{ \begin{array}{l} \{p\} \text{ if } p \text{ is pure open,} \\ \{(p_1-1,p_2),p,(p_1+1,p_2)\} \text{ if } p \text{ is closed-open,} \\ \{(p_1,p_2-1),p,(p_1,p_2+1)\} \text{ if } p \text{ is open-closed,} \\ N_8^*(p) \text{ if } p \text{ is pure closed,} \end{array} \right\} \quad (2.3)$$

where the point $p := (p_1, p_2)$ is called *closed-open* (resp. *open-closed*) if p_1 is even (resp. odd) and p_2 is odd (resp. even).

In this paper each space $X \subset \mathbf{Z}^n$ related to K-topology is considered to be a subspace (X, κ_X^n) induced by $(\mathbf{Z}^n, \kappa^n)$ [8, 24].
Let us now recall the structure of $(\mathbf{Z}^n, \kappa^n)$. In each of the spaces of Figure 1, a black jumbo dot means a pure open point and further, the symbols ■ and • mean a pure closed point and a mixed point, respectively. In relation to the further statement of a pure point and a mixed point, we can say that a point x is open if $SN_K(x) = \{x\}$, where $SN_K(x)$ means the smallest neighborhood of $x \in \mathbf{Z}^n$. Many studies have examined various properties of a K-continuous map, connectedness, K-adjacency, a K-homeomorphism [8, 10, 23, 24], KD-k-continuity associated with K-topology [11], k-continuity associated with K-topology [13] and so forth.

Let us recall the following terms to study K-topological spaces.

Definition 2. [8] *Let $(X, \kappa_X^n) := X$ be a K-topological space. We say that two distinct points $x, y \in X$ are K-adjacent if $x \in SN_K(y)$ or $y \in SN_K(x)$. Then we define the following:*
A simple K-path in X is the injective sequence $(x_i)_{i\in[0,l]_{\mathbf{Z}}}$ such that x_i and x_j are K-adjacent if and only if $|\, i-j \,| = 1$.
Furthermore, we say that a simple closed K-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_K^{n,l}, l \geq 4$, is a simple K-path $(x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are K-adjacent if and only if $j = i+1 (mod\, l)$ or $i = j+1 (mod\, l)$.

Third, let us now recall basic concepts from M-topology. The M-topology on $\mathbf{Z}^2$, denoted by $(\mathbf{Z}^2, \gamma)$, is induced by the set $\{U\}$ in (2.4) as a base [29], where for each point $p = (x, y) \in \mathbf{Z}^2$

$$U := SN_M(p) = \left\{ \begin{array}{l} U(p) := N_4(p) \cup \{p\} \text{ if } x+y \text{ is even, and} \\ \{p\} : \text{ else.} \end{array} \right\} \quad (2.4)$$

In relation to the further statement of a point in $\mathbf{Z}^2$, in the paper we call a point $p = (x_1, x_2)$ *double even* if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; *even* if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and *odd* if $x_1 + x_2$ is an odd number.

After considering an M-topological adjacency, the recent paper [21] developed the notion of an MA-homotopy associated with M-topology to classify M-topological spaces with M-adjacency.

3. Some properties of homotopies associated with Khalimsky topology

This section investigates various properties of homotopies associated with K-topology which will be used in Section 4. Let us now recall some properties of digital spaces in a graph theoretical approach. To map every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1), the paper [28] established the notion of digital continuity of maps between digital images.

As special kind of digital neighborhood of x in (X, k) in Definition 1, for a digital space (X, k) on $\mathbf{Z}^n$ we can consider

$$N_k(x, 1) = N_k^*(x) \cap X. \tag{3.1}$$

Since every point $x \in X$ has $N_k(x, 1) \subset (X, k)$, motivated by both the digital continuity of [28], we can establish the following version.

Proposition 3.1. [7] *(see also* [13]*) Let (X, k_0) and (Y, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

Since a digital image can be recognized to be a digital k-graph, as discussed above, we can represent a (k_0, k_1)-homeomorphism in [3], as follows: for two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism [19] (see also [8]) if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then, we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-isomorphism and use the notation $X \approx_{k_0} Y$.

In Proposition 3.1, in case $k_1 = k_2$, the map f is called a k_1-continuous map.

By using this concept, we establish a digital topological category, denoted by *DTC*, consisting of two sets [5] (see also [7]):
• For any set $X \subset \mathbf{Z}^n$, the set of objects (X, k) in $\mathbf{Z}^n$ as objects of *DTC*;
• For every ordered pair of objects $(X_i, k_i), i \in \{1, 2\}$, the set of all (k_0, k_1)-continuous maps between these objects as morphisms of *DTC*.
In *DTC*, in case $k_0 = k_1 := k$, we will particularly use the notation *DTC(k)* [21].

Based on the pointed digital homotopy in [23, 3], the notion of a k-homotopy relative to a subset $A \subset X$ [5, 7] is often used to study a k-homotopic thinning and to classify digital images (X, k) in $\mathbf{Z}^n$ [9]. Using this relative k-homotopy in [5, 7], we can establish a strong deformation retract of a digital space pair [7, 9].

Let us now recall three kinds of continuities related to the study of map between Khalimsky nD spaces such as K-continuity, KD-(k_0, k_1)-continuity, (k_0, k_1)-continuity associated with K-topology [8, 11, 13]. For the Khalimsky nD space $(\mathbf{Z}^n, \kappa^n)$ and a subset $X \subset \mathbf{Z}^n$, we obtain the subspace (X, κ_X^n) induced from $(\mathbf{Z}^n, \kappa^n)$, where $\kappa_X^n = \{O \cap X \mid O \in \kappa^n\}$. In K-topology, the digital interval $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b\}$ is often assumed to be a subspace of $(\mathbf{Z}, T)$ induced by the K-topology $(\mathbf{Z}, T)$. Based on the above terminology, the typical K-continuity of maps between Khalimsky nD spaces is considered in such a way. For two spaces $(X, \kappa_X^{n_0}) := X$ and $(Y, \kappa_Y^{n_1}) := Y$, a function $f : X \to Y$ is said to be K-continuous at a point $x \in X$ if for any open set $O_{f(x)} \in \kappa_Y^{n_1}$ there is $O_x \in \kappa_X^{n_0}$ such that $f(O_x) \subset O_{f(x)}$. Further, we say that a map $f : X \to Y$ is K-continuous if it is K-continuous at every point $x \in X$.

In terms of the K-continuity, we obtain the *K-topological category*, briefly *KTC*, consisting of the following two classes [8]:
(1) the set of objects (X, κ_X^n);
(2) For every ordered pair of objects $(X, \kappa_X^{n_0})$ and $(Y, \kappa_Y^{n_1})$ the set of all K-continuous maps $f : (X, \kappa_X^{n_0}) \to (Y, \kappa_Y^{n_1})$ as morphisms.

Let us move onto the study of a KD-(k_0, k_1)-continuous map.

Definition 3. [8] *We say that a space (X, κ_X^n) with k-adjacency is a (computer topological) space and use the notation $(X, k, \kappa_X^n) := X_{n,k}$.*

Definition 4. [11] *For two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, we say that a function $f : X \to Y$ is KD-(k_0, k_1)-continuous at a point $x_0 \in X$ if*

(1) f is continuous at the point x_0 in KTC; and

(2) for any $N_{k_1}(f(x_0), \varepsilon) \subset Y$, there is $N_{k_0}(x_0, \delta) \subset X$ such that $f(N_{k_0}(x_0, \delta)) \subset N_{k_1}(f(x_0), \varepsilon)$, where $\varepsilon, \delta \in \mathbf{N}$.

Furthermore, we say that a map $f : X \to Y$ is KD-(k_0, k_1)-continuous if the map f is KD-(k_0, k_1)-continuous at any point $x \in X$.

For $X_{n,k}$ we observe that if $k = 3^n - 1$, then $N_k^*(x, 1) = N_k(x, 1)$; and if $k \neq 3^n - 1$, then $N_k^*(x, 1)$ need not be equal to $N_k(x, 1)$ [8].
Let us consider the *KD-topological category*, denoted by *KDTC*, consisting of two sets:

- the set of objects $X_{n,k}$;

• For every ordered pair of objects X_{n_0,k_0} and Y_{n_1,k_1} the set of KD-(k_0,k_1)-continuous maps as morphisms.

Let us now consider a computer topological k-neighborhood which is different from $N_k(x,\varepsilon)$ of Definition 1.

Definition 5. [8] *Consider $X_{n,k}$, $x,y \in X$, and $\varepsilon \in \mathbf{N}$.*

(1) A subset V of X is called a neighborhood of x if there exists $O_x \in \kappa_X^n$ such that $x \in O_x \subseteq V$.

(2) If a digital k-neighborhood $N_k(x,\varepsilon)$ is a K-topological neighborhood of x in (X,κ_X^n), then this set is called a computer topological k-neighborhood of x with radius ε and we use the notation $N_k^*(x,\varepsilon)$.

Let us move onto the study of (k_0,k_1)-continuity between X_{n_0,k_0} and Y_{n_1,k_1} which is different from the KD-(k_0,k_1)-continuity.

In order to study spaces $X_{n,k}$, we need to use another continuity instead of K-continuity and KD-k-continuity, as follows.

Definition 6. [13] *For two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, a function $f : X \to Y$ is said to be (k_0,k_1)-continuous at a point $x \in X$ if for any $N_{k_1}^*(f(x),\varepsilon) \subset Y$, there is $N_{k_0}^*(x,\delta) \subset X$ such that*

$$f(N_{k_0}^*(x,\delta)) \subset N_{k_1}^*(f(x),\varepsilon),$$

where for some $\varepsilon \in \mathbf{N}$, $N_{k_1}^*(f(x),\varepsilon)$ is assumed to be existed. Furthermore, we say that a map $f : X \to Y$ is (k_0,k_1)-continuous if the map f is (k_0,k_1)-continuous at every point $x \in X$.

In Definition 6 if $k_0 = k_1$ and $n_0 = n_1$, then we use the terminology "k_0-continuous" instead of "(k_0,k_1)-continuous". As a representation of (k_0,k_1)-continuity of Definition 6, we obtain the following:

Remark 3.2. [13] *The (k_0,k_1)-continuity of Definition 6 is equivalent to the following:*

$$f(N_{k_0}^*(x,r)) \subset N_{k_1}^*(f(x),s),$$

where the number r is the least element of $\mathbf{N}$ such that $N_{k_0}^*(x,r)$ contains an open set including the point x (so $N_{k_0}^*(x,r) = N_{k_0}(x,r)$) and s is the least element of $\mathbf{N}$ such that $N_{k_1}^*(f(x),s)$ contains an open set including the point $f(x)$ (so $N_{k_1}^*(f(x),s) = N_{k_1}(f(x),s)$).

In Definition 6 if we replace $N_{k_1}^*(f(x),\varepsilon) \subset Y$ (resp. $N_{k_0}^*(x,\delta) \subset X$) by $N_{k_1}(f(x),\varepsilon) \subset Y$ (resp. $N_{k_0}(x,\delta) \subset X$), then the (k_0,k_1)-continuity of Definition 6 is equivalent to the digital (k_0,k_1)-continuity in DTC (see Proposition 3.1) [13].

Thus we establish a category [13], denoted by CTC, consisting of the

two sets:
• the set of objects $X_{n,k}$ and
• For every ordered pair of objects X_{n_0,k_0} and Y_{n_1,k_1} the set of (k_0, k_1)-continuous maps between these objects as morphisms.

Comparing the KD-(k_0, k_1)-continuity and (k_0, k_1)-continuity, they have their own features and merits [8].

In order to study some KD-k-homotopic properties of spaces $X_{n,k}$ in *KDTC*, we introduce the notion of a KD-k-homotopy as follows:

Definition 7. [11] *For three spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, and a subspace $A_{n_0,k_0} := A \subset X_{n_0,k_0}$, let $f, g : X \to Y$ be KD-(k_0, k_1)-continuous functions. Let $[0, m]_{\mathbf{Z}}$ be considered as a subspace of the Khalimsky line with 2-adjacency. Suppose there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that*

(•1) *for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;*

(•2) *for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is KD-$(2, k_1)$-continuous;*

(•3) *for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ for all $x \in X$ is KD-(k_0, k_1)-continuous.*

Then we say that F is a KD-(k_0, k_1)-homotopy between f and g, and f and g are KD-(k_0, k_1)-homotopic in Y. And we use the notation $f \simeq_{KD\cdot(k_0,k_1)} g$.

(•4) *If, further, for all $t \in [0, m]_{\mathbf{Z}}$, then the induced map F_t on A is a constant which is the prescribed function from A to Y. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbf{Z}}$. Then, we say that the homotopy is a KD-(k_0, k_1)-homotopy rel.A and denote it $f \simeq_{KD\cdot(k_0,k_1)rel.A} g$. In particular, if $A = \{x_0\} \subset X$, then we say that F is a pointed KD-(k_0, k_1)-homotopy.*
If $X = [0, m_X]_{\mathbf{Z}}$, for all $t \in [0, m]_{\mathbf{Z}}$, we have $F(0, t) = F(0, 0)$ and $F(m_X, t) = F(m_X, 0)$, then we say that F holds the endpoints fixed.

As the computer topological analogs of the notions of *pointed k-contractibility* and *k-nullhomotopic* in [3], we say that the space $X_{n,k} := X$ is *pointed KD-k-contractible* if $1_X \simeq_{KD\cdot k} c_{\{x_0\}}$, where $c_{\{x_0\}}$ is a constant map for some point $x_0 \in X$. We say that a KD-(k_0, k_1)-continuous function $f : X_{n_0,k_0} \to Y_{n_1,k_1} := Y$ is *KD-k_1-nullhomotopic* in Y if f is KD-k_1-homotopic in Y to a constant function $c_{\{y_0\}}$ for some $y_0 \in Y$ [11].

Unlike the digital k-homotopy equivalence in *DTC* [4, 20], in *CTC* we have used a (k_0, k_1)-homotopy to study spaces $X_{n,k}$ (see Definition 8). For a space X_{n_0,k_0} and its subspace A_{n_0,k_0}, consider a space pair $(X_{n_0,k_0}, A_{n_0,k_0}) := (X, A)_{n_0,k_0}$. For two space pairs $(X, A)_{n_0,k_0}$

and $(Y, B)_{n_1,k_1}$, we say that $f : (X, A)_{n_0,k_0} \to (Y, B)_{n_1,k_1}$ is (k_0, k_1)-continuous if $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is (k_0, k_1)-continuous and $f(A_{n_0,k_0}) \subset B_{n_1,k_1}$. In *CTC*, as a K-topological analog of the (k_0, k_1)-*homotopy rel.A* in *DTC* [5], we can establish a new (k_0, k_1)-*homotopy rel.A* in terms of the (k_0, k_1)-continuity of Definition 6, as follows.

Definition 8. [13] *In CTC, for four spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, a subspace $A_{n_0,k_0} := A \subset X_{n_0,k_0}$, and a Khalimsky interval $[0, m]_{\mathbf{Z}}$, let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that*

($\star$1) *for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;*

($\star$2) *for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is $(2, k_1)$-continuous;*

($\star$3) *for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ for all $x \in X$ is (k_0, k_1)-continuous.*

Then we say that F is a (k_0, k_1)-homotopy between f and g, and f and g are (k_0, k_1)-homotopic in Y. And we use the notation $f \simeq_{(k_0,k_1)} g$.

($\star$4) *If, further, for all $t \in [0, m]_{\mathbf{Z}}$, then the induced map F_t on A is a constant which is the prescribed function from A to Y. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbf{Z}}$. Then, we say that the homotopy is a (k_0, k_1)-homotopy rel.A and denote it $f \simeq_{(k_0,k_1)rel.A} g$. In particular, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0, k_1)-homotopy.*
If $X = [0, m_X]_{\mathbf{Z}}$, for all $t \in [0, m]_{\mathbf{Z}}$, we have $F(0, t) = F(0, 0)$ and $F(m_X, t) = F(m_X, 0)$, then we say that F holds the endpoints fixed.

In *CTC*, as an analogous version of the notion of k-*contractibility* of [3] in *DTC*, we say that a space $X_{n,k}$ is *pointed k-contractible* if the identity map 1_X is pointed k-homotopic *rel.*$\{x_0\}$ in X to a constant map with the space consisting of some $x_0 \in X$. In Definition 8 if $k_0 = k_1$ and $n_0 = n_1$, then we use the terminology k_0-*homotopy* instead of (k_0, k_1)-*homotopy.* In order to study this problem, we now establish the notion of homotopy equivalence suitable for studying $X_{n,k}$. Thus in *CTC*, for two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, if there is a (k_0, k_1)-continuous map $h : X \to Y$ and a (k_1, k_0)-continuous map $l : Y \to X$ such that $l \circ h$ is k_0-homotopic to 1_X and $h \circ l$ is k_1-homotopic to 1_Y, then the map $h : X \to Y$ is called a (k_0, k_1)-homotopy equivalence [13]. Then we use the notation, $X \simeq_{(k_0,k_1)\cdot h\cdot e} Y$. Furthermore, if $k_0 = k_1$ and $n_0 = n_1$, we call h a k_0-homotopy equivalence and we use the notation $X \simeq_{k_0\cdot h\cdot e} Y$.

4. Remarks on homotopies in *KDTC* and *CTC*

In section 3 we have studied two notions of homotopies such as a *KD*-k-homotopy in *KDTC* (see Definition 7) and a k-homotopy in *CTC* (see Definition 8). Since there are some differences between KD-(k_0, k_1)-continuity in *KDTC* and (k_0, k_1)-continuity in *CTC* [8], the above two homotopies have their own features. In both Definitions 7 and 8, the homotopies require the existence of

$$\text{both "}m \in \mathbf{N} \text{ and a function } F : X \times [0, m]_{\mathbf{Z}} \to Y\text{"}, \tag{4.1}$$

in *KDTC* and *CTC*. At this moment, since many researchers recently have the following queries:
(Q1) Is there a difference between the product $X \times [0, m]_{\mathbf{Z}}$ in *DTC* and that in *KDTC* or that in *CTC*?
(Q2) In *KDTC* and *CTC*, what is the topological structure of the Cartesian product $X \times [0, m]_{\mathbf{Z}}$?
(Q3) What is a difference between the KD-(k_0, k_1)-homotopy in *KDTC* and the (k_0, k_1)-homotopy in *CTC*?

This section mainly address these issues.

Remark 4.1. *First of all, let us consider the Cartesian product* $X \times [0, m]_{\mathbf{Z}}$ *in DTC* [3, 5]. *In relation to the digital adjacency of the product, several kinds of tools have been studied such as a normal adjacency* [5], *the properties* L_C [10] *and* L_S [12] *and their various properties* [14, 15, 18]. *But in relation to the study of the Cartesian product* $X \times [0, m]_{\mathbf{Z}}$ *in DTC, we need not consider any digital* k*-connectivity of this product. This means that the set* $X \times [0, m]_{\mathbf{Z}}$ *is assumed to be a disjoint union* $\cup_{i \in [0,m]_{\mathbf{Z}}} X \times \{i\}$ *without any topological structure. Therefore, since the Cartesian product* $X \times [0, m]_{\mathbf{Z}}$ *in DTC does not have any topological structure, we see that this is quite different from those in both KDTC and CTC, which answers the query of the question (1) above.*

Let us move onto the second issue above.

Proposition 4.2. *The topological structure of the product* $X \times [0, m]_{\mathbf{Z}}$ *in Definition 7 is same as that in Definition 8.*

Proof: To address the following query "In *KDTC* and *CTC*, what is the topological structure in the Cartesian product $X \times [0, m]_{\mathbf{Z}}$?", first of all, we need to study the K-topological structure. It is well known that the Khalimsky nD topological space is a box product of

the Khalimsky line space $(\mathbf{Z}, \kappa)$. Put $\mathbf{Z}^n \times \{i\} := \mathbf{Z}_i^n, i \in \mathbf{Z}$. Then we may consider $\mathbf{Z}_i^n$ as the subspace of $(\mathbf{Z}_i^n, \kappa_{\mathbf{Z}_i^n}^{n+1})$ induced by $(\mathbf{Z}^{n+1}, \kappa^{n+1})$. Then for any $i, j \in 2\mathbf{Z}$ or $i, j \in \{2n+1 | n \in \mathbf{Z}$ we see that $(\mathbf{Z}_i^n, \kappa_{\mathbf{Z}_i^n}^{n+1})$ is K-homeomorphic to $(\mathbf{Z}_j^n, \kappa_{\mathbf{Z}_j^n}^{n+1})$. But, if $n \in \{1, 2\}$, then we have a very nice situation. For instance, consider the spaces in Figure 1. To be specific, in Figure 1(a-1) consider the K-interval $([0,3]_{\mathbf{Z}}, \kappa_{[0,3]_{\mathbf{Z}}})$ as the space $(X, \kappa_X^{n_0})$ in Definition 7. Then consider the Cartesian product $(X, \kappa_X^{n_0}) \times [0,m]_{\mathbf{Z}} := [0,3]_{\mathbf{Z}} \times [0,m]_{\mathbf{Z}}$ with $m = 2$. Then we have $[0,3]_{\mathbf{Z}} \times \{i\} := X_i, i \in [0,2]_{\mathbf{Z}}$ such that $(X_i, \kappa_{X_i}^2)$ is K-homeomorphic to $(X_j, \kappa_{X_j}^2), i, j \in [0,2]_{\mathbf{Z}}$.
Similarly, in Figure 1(a-2) consider the K-interval $([1,4]_{\mathbf{Z}}, \kappa_{[1,4]_{\mathbf{Z}}})$ as the space $(X, \kappa_X^{n_0})$ in Definition 7. Then consider the Cartesian product $(X, \kappa_X^{n_0}) \times [0,m]_{\mathbf{Z}} := [1,4]_{\mathbf{Z}} \times [0,m]_{\mathbf{Z}}$ with $m = 2$. Then we have $[1,4]_{\mathbf{Z}} \times \{i\} := X_i, i \in [0,2]_{\mathbf{Z}}$ such that $(X_i, \kappa_{X_i}^2)$ is K-homeomorphic to $(X_j, \kappa_{X_j}^2)$.
Besides, in Figure 1(b) consider the K-topological space (X, κ_X^2) as the space $(X, \kappa_X^{n_0})$ in Definition 7. Then consider the Cartesian product $(X, \kappa_X^{n_0}) \times [0,m]_{\mathbf{Z}} := X \times [0,m]_{\mathbf{Z}}$ with $m = 2$. Then we have $X \times \{i\} := X_i, i \in [0,2]_{\mathbf{Z}}$ such that $(X_i, \kappa_{X_i}^2)$ is K-homeomorphic to $(X_j, \kappa_{X_j}^2), i, j \in [0,2]_{\mathbf{Z}}$.
Furthermore, we see that $(\mathbf{Z}^n, \kappa^n)$ is a proper subspace of $(\mathbf{Z}^{n+1}, \kappa^{n+1})$ with the relative topology on $\mathbf{Z}^n$ induced by $(\mathbf{Z}^{n+1}, \kappa^{n+1})$, $n \in \mathbf{N}$ [17]. Based on this review, the Cartesian product $X \times [0,m]_{\mathbf{Z}}$ in Definitions 7 and 8 should be basically considered as an element in $Ob(KTC)$. $\square$

Let us move onto to the third issue above.

Proposition 4.3. *The KD-(k_0, k_1)-homotopy in $KDTC$ and the (k_0, k_1)-homotopy in CTC have their own features.*

Proof: In view of the categories $KDTC$ and CTC, the role of the product $X \times [0,m]_{\mathbf{Z}}$ in Definitions 7 and 8 are same. However, let us compare the properties (•2) and (⋆2). Indeed, it is obvious that $[0,m]_{\mathbf{Z}}$ is a subspace of $(\mathbf{Z}, \kappa)$, by the property (2.3), each point $p \in \mathbf{Z}$ has $SN_k(p) = \{p\}$ or $\{p-1, p, p+1\}$. Hence for any point $p \in ([0,m]_{\mathbf{Z}}, _{[0,m]_{\mathbf{Z}}})$ we have $N_2^*(p,1) = N_2(p,1)$. Therefore, while the role of $([0,m]_{\mathbf{Z}}, _{[0,m]_{\mathbf{Z}}})$ in Definitions 7 and 8 are same, there can be some different situations in $Y_{n_1,k_1} := Y$ depending on the viewpoints of $KDTC$ and CTC. THus we conclude that the roles of (•2) and (⋆2) have their own features. Namely, owing to the difference between the KD-$(2, k_1)$-continuity and the $(2, k_1)$-continuity in CTC [8], these two properties (•2) and (⋆2) are

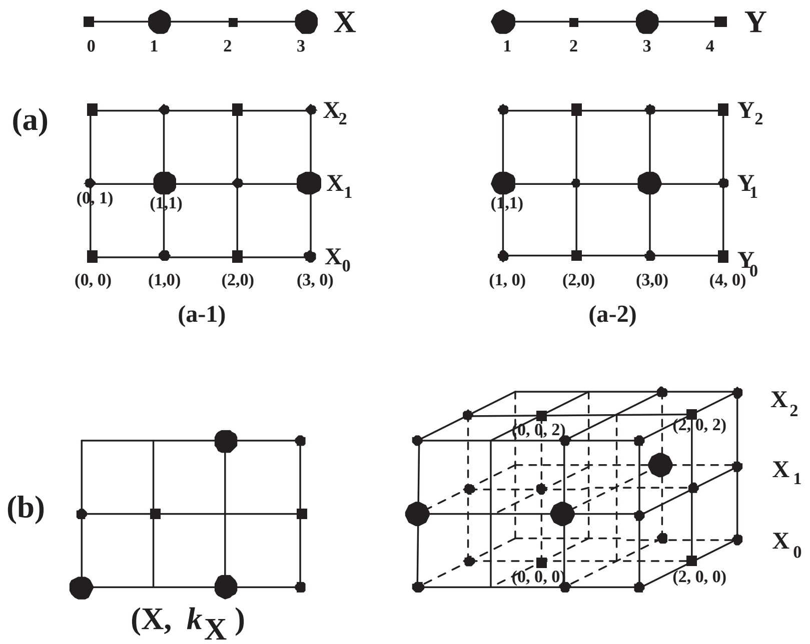

FIGURE 1. Explanation of some features of the product $X \times [0, m]_{\mathbf{Z}}$ in Definitions 7 and 8.

different from each other. Besides, by using the approach similar to the comparison between ($\bullet$2) and ($\star$2), we can see some difference between the properties ($\bullet$3) in Definition 7 and ($\star$3) in Definition 8. $\square$

5. Difficulty in developing a homotopy in M-topological spaces

Let us recall the notions of an M-adjacent relation for any two points in $\mathbf{Z}^2$ and an MA-neighborhood of a point $x \in \mathbf{Z}^2$ and further, an equivalence between M-adjacency and digital 4-connectivity.

Definition 9. [16] *In* $(\mathbf{Z}^2, \gamma)$, *we say that two distinct points* x,y *in* $\mathbf{Z}^2$ *are* M*-adjacent if* $y \in O(x)$ *or* $x \in O(y)$, $O(q)$ *means the smallest open set containing the point* $q \in \mathbf{Z}^2, q \in \{x, y\}$.

In other words, for a point $p \in \mathbf{Z}^2$ the set of M-adjacent points of p, denoted by $MA(p)$, can be represented as follows [21]:

$$MA(p) = N_4(p). \tag{5.1}$$

Remark 5.1. [16] *(1) Under* $(\mathbf{Z}^2, \gamma)$ *the notions of* M*-adjacency and* M*-connectedness are equivalent.*

(2) Under $(\mathbf{Z}^2, \gamma)$ *take a point* $p \in \mathbf{Z}^2$. *For any point* $q \in N_4(p)$ *the subspace* $(\{p, q\} := X_1, \gamma_{X_1})$ *is both* M*-connected and* MA*-connected.*

In view of Remark 5.1, we obtain the following:

Lemma 5.2. [16] *Under* $(\mathbf{Z}^2, \gamma)$ *for any two points* $p, q \in \mathbf{Z}^2$ *the subspace* $(\{p, q\} := X_1, \gamma_{X_1})$ *is connected if and only if the two points* p, q *are* 4*-adjacent, i.e.* $p \in N_4(q)$ *or* $q \in N_4(p)$.

For a space $(X, \gamma_X) := X$ we now recall an MA*-relation* of a point $p \in X$ as follows.

Definition 10. [16] *For* $(X, \gamma_X) := X$ *put* $MA_X(p) := MA(p) \cap X$. *We say that for two distinct points* $p, q \in X$ *they are* M*-adjacent to each other if* $q \in MA_X(p)$ *or* $p \in MA_X(q)$.

Definition 11. [16] *For a space* $(X, \gamma_X) := X$ *and a point* $p \in X$ *we define an* MA*-neighborhood of* p *in* X *to be the set* $MA_X(p) \cup \{p\} := MN_X(p)$.

Hereafter, in (X, γ_X) we use the notation $MN(p)$ instead of $MN_X(p)$ if there is no danger of ambiguity. In view of Definition 10 and (5.1), we conclude that in (X, γ_X) [21]

$$MN(p) = N_4(p, 1). \tag{5.2}$$

As referred in Remark 5.1, given an M-topological space (X, γ_X), since we have an M-adjacency on the space, we may use the notation (X, γ_X) again for a *space* (X, γ_X) *with an* M*-adjacency* if there is no ambiguity. For a space $(X, \gamma_X) := X$ and each point $x \in X$, owing to the Alexandroff topological structure of (X, γ_X), it is clear that [21] each point $x \in X$ always has $MN(x) \subset X$ so that we now establish a map sending $MN(x)$ into $MN(f(x))$ as follows:

Definition 12. [16] *For two spaces* $(X, \gamma_X) := X$ *and* $(Y, \gamma_Y) := Y$, *we say that a function* $f : X \to Y$ *is an* MA*-map at a point* $x \in X$ *if*

$$f(MN(x)) \subset MN(f(x)).$$

Furthermore, we say that a map $f : X \to Y$ *is an* MA*-map if the map* f *is an* MA*-map at every point* $x \in X$.

In view of Definition 12, we observe the following [16]:

Remark 5.3. *(1) An M-continuous map is an MA-map. But the converse does not hold* [11].
(2) An MA-map is an M-connectedness preserving map [11].
(3) For a given bijective MA-map, its inverse map need not be an MA-map [21].

Using MA-maps, we establish an *MA-category* [11], denoted by *MAC*, consisting of two sets.

(1) For any set $X \subset \mathbf{Z}^2$, the set of (X, γ_X) (or (X, γ_X) with M-adjacency) as objects of *MAC*,

(2) For every ordered pair of objects (X, γ_X) and (Y, γ_Y), the set of all MA-maps $f : (X, \gamma_X) \to (Y, \gamma_Y)$ as morphisms of *MAC*.

The recent paper [21] a new homotopy to study digital space from the viewpoint of *MAC*. However, to develop a homotopy in *MTC*, let us follow the methods presented in Definitions 7 and 8, then we have some difficult in proceeding the work.

Let us now recall the M-topological category and an M-homeomorphism as follows:

Definition 13. [29] *For two M-topological spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, a function $f : X \to Y$ is said to be M-continuous at a point $x \in X$ if f is continuous at the point x from the viewpoint of M-topology. Furthermore, we say that a map $f : X \to Y$ is M-continuous if it is M-continuous at every point $x \in X$.*

Using M-continuous maps, we establish an *M-topological category*, denoted by MTC [21], consisting of two sets [16, 21].

(1) For any set $X \subset \mathbf{Z}^2$ the set of objects (X, γ_X) denoted by $Obj(MTC)$,

(2) For every ordered pair of objects (X, γ_X) and (Y, γ_Y), the set of all M-continuous maps $f : (X, \gamma_X) \to (Y, \gamma_Y)$ as morphisms of *MTC*.

Remark 5.4. *To establish the notion of an M-homotopy, if we follow the method similar to those of Definitions 7 and 8, then we have a serious difficulties. Let us follow the method in Definitions 7 and 8 to develop the notion of an M-homotopy. To be specific, in MTC, for spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$ and an M-interval $([0, m]_{\mathbf{Z}}, \gamma_{[0,m]_{\mathbf{Z}}})$ as a subspace induced by the M-topology $(\mathbf{Z}^2, \gamma)$. Suppose $f, g : X \to Y$ be M-continuous functions. Suppose there exist $m \in \mathbf{N}$ and a function $G : X \times [0, m]_{\mathbf{Z}} \to Y$ such that satisfying the properties similar to those from ($\bullet$1) to ($\star$1) in Definitions 7 and 8.*

At this moment, we have some difficulty in having the product $X \times [0,m]_{\mathbf{Z}}$ *in MTC because MTC consists of all subspaces of 2D digital spaces. For instance, consider the space* $X := [0,3]_{\mathbf{Z}}$ *in Figure 2. If we have the product* $X \times [0,2]_{\mathbf{Z}}$ *(see Figure 2(a-1)), then the subspaces* $X \times \{i\} := X_i, i \in [0,2]_{\mathbf{Z}}$ *have the following property:* (X_0, γ_{X_0}) *is not M-homeomorphic to* (X_1, γ_{X_1})*, which makes difficulties in establishing the product* $X \times [0,m]_{\mathbf{Z}}$ *for an M-homotopy.*
As another example consider the space $X := [1,4]_{\mathbf{Z}}$ *in Figure 2. If we have the product* $X \times [0,2]_{\mathbf{Z}}$ *(see Figure 2(a-2)), then the subspaces* $Y \times \{i\} := Y_i, i \in [0,2]_{\mathbf{Z}}$ *have the following property:* (Y_0, γ_{Y_0}) *is not M-homeomorphic to* (Y_1, γ_{Y_1})*, which makes difficulties in establishing the product* $Y \times [0,m]_{\mathbf{Z}}$ *for an M-homotopy.*

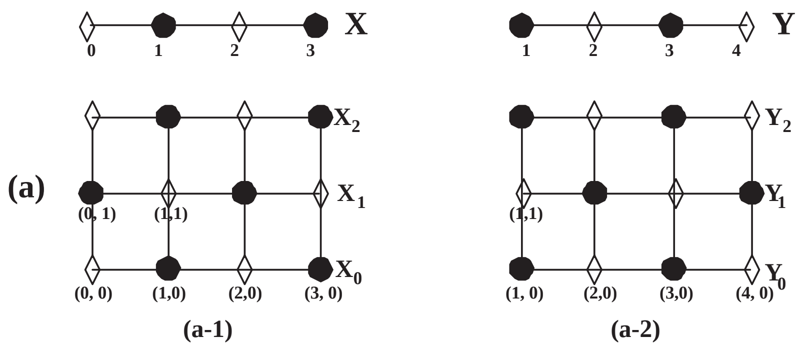

FIGURE 2. Explanation of some difficulties in establishing an M-homotopy.

6. Summary and further works

We have reviewed the motivations of the KD-(k_0, k_1)-homotopy in *KDTC* and the (k_0, k_1)-homotopy in *CTC*. Comparing the situations among *KDTC*, *CTC* and *MTC*, we need to develop a homotopy in *MTC* by using some different approach from those in Definitions 7 and 8.

References

[1] P. Alexandroff, Diskrete Räume, *Mat. Sb.* **2** (1937) 501-518.

[2] R. Ayala, E. Domínguez, A.R. Francés, and A. Quintero, Homotopy in digital spaces, *Discrete Applied Math*, **125(1)** (2003) 3-24.

[3] L. Boxer, A classical construction for the digital fundamental group, *Jour. of Mathematical Imaging and Vision* **10** (1999) 51-62.

[4] S.E. Han, On the classification of the digital images up to digital homotopy equivalence, *Jour. Comput. Commun. Res.* **10**(2000) 207-216.

[5] S.E. Han, Non-product property of the digital fundamental group, *Information Sciences* **171** (1-3) (2005) 73-91.

[6] S.E. Han, On the simplicial complex stemmed from a digital graph, *Honam Mathematical Journal* **27** (1)(2005) 115-129.

[7] S.E. Han, S.E. Han, Equivalent (k_0, k_1)-covering and generalized digital lifting, *Information Sciences* **178**(2) (2008) 550-561.

[8] S.E. Han, Continuities and homeomorphisms in computer topology and their applications, J. Korean Math. Soc. 45 (2008) 923–952.

[9] S.E. Han, The k-homotopic thinning and a torus-like digital image in $\mathbf{Z}^n$, *Journal of Mathematical Imaging and Vision* **31** (1) (2008) 1-16.

[10] S.E. Han, Cartesian product of the universal covering property *Acta Applicandae Mathematicae* **108** (2009) 363-383.

[11] S.E. Han, KD-(k_0, k_1)-homotopy equivalence and its applications *Journal of Korean Mathematical Society* **47**(5) (2010) 1031-1054.

[12] S.E. Han, Multiplicative property of the digital fundamental group, *Acta Applicandae Mathematicae* **110**(2) (2010) 921-944.

[13] S.E. Han, Homotopy equivalence which is suitable for studying Khalimsky nD spaces, *Topology Appl.* **159** (2012) 1705-1714.

[14] S.E. Han, Study on topological spaces with the semi-$T_{1/2}$ separation axiom *Honam Mathematical Journal*, (2013) **35**(4) 707-716.

[15] S.E. Han, An equivalent property of a normal adjacency of a digita product *Honam Mathematical Journal*, (2014) **36**(3) 199-215.

[16] S.E. Han, Generalizations of continuity of maps and homeomorphisms for studying 2D digital topological spaces and their applications, *Topology Appl.*, online first press (2015).

[17] S.E. Han, Contractibility and fixed point property: The case of Khalimsky topological spaces, submitted (2015).

[18] S.E. Han and Sik Lee, Remarks on digital products with normal adjacency relations *Honam Mathematical Journal*, (2013) **35**(3) 515-424.

[19] S.E. Han and B.G. Park, Digital graph (k_0, k_1)-isomorphism and its applications, *http://atlas-conferences.com/c/a/k/b/36.htm*(2003).

[20] S.E. Han and B.G. Park, Digital graph (k_0, k_1)-homotopy equivalence and its applications, *http://atlas-conferences.com/c/a/k/b/35.htm*(2003).

[21] S.E. Han, Wei Yao, Homotopies based on Marcus Wyse topology and their applications, *Topology and its applications*, in press.

[22] G. T. Herman, Oriented surfaces in digital spaces, *CVGIP: Graphical Models and Image Processing* **55** (1993) 381-396.

[23] E. Khalimsky, Motion, deformation, and homotopy in finite spaces, *Proceedings IEEE International Conferences on Systems, Man, and Cybernetics* (1987) 227-234.

[24] E. Khalimksy, R. Kopperman, P. R. Meyer, Computer graphics and connected topologies on finite ordered sets, *Topology Appl.*, **36(1)** (1990) 1-17.

[25] T.Y. Kong, A digital fundamental group *Computers and Graphics* **13** (1989) 159-166.

[26] T.Y. Kong, A. Rosenfeld, Topological Algorithms for the Digital Image Processing, Elsevier Science, Amsterdam, 1996.

[27] R. Malgouyres, Homotopy in 2-dimensional digital images, *Theoretical Computer Science* **230** (2000) 221-233.

[28] A. Rosenfeld, Continuous functions on digital pictures, *Pattern Recognition Letters*, **4** (1986) 177-184.

[29] F. Wyse and D. Marcus et al., Solution to problem 5712, *Amer. Math. Monthly* **77**(1970) 1119.

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