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SEVERAL PROPERTIES OF QUATERNIONIC REGULAR FUNCTIONS IN CLIFFORD ANALYSIS

Han Ul Kang, Min Ji Kim and Kwang Ho Shon*

Abstract. In this paper, we research some properties of quaternionic regular functions in Clifford analysis. We investigate the corresponding Cauchy-Riemann system and find regularities of some hypercomplex valued functions.

1. Introduction

The quaternion field $\mathcal{T}$ over the field $\mathbb{R}$ is identified with $\mathbb{R}^4$ and $\mathbb{C}^2$, where $\mathbb{C}$ denotes the field of complex numbers. The field $\mathcal{T}$ is a non-commutative real four-dimensional field. In 1971, Naser [5] has shown several properties of hyperholomorphic (regular) functions over the field $\mathcal{T}$ and some theorems by using quaternionic differential operator. In 1995, Nôno [6] has given a regularity of functions with hypercomplex values. Also Nôno [6] has shown some definitions and properties of regular functions in the using complex partial differential equation. In 2011, Koriyama *et al.* [3] have constructed regular function theories on the quaternion field and corresponding Cauchy-Riemann equations for each quaternionic differential operator. In 2014, Kim *et al.* [2] have defined a modified basis associated with two basis in the ternary number system and given the properties of regular functions on the ternary number field and the reduced quaternion field in Clifford analysis. In 2015, Lim and Shon [4] have given the properties of hyperholomorphic (regular) functions with split quaternions and investigated split quaternionic mappings on $\Omega \subset \mathbb{C}^2$. Kang and Shon [1] have provided several corresponding Cauchy-Riemann systems equivalent to $D_j^* f = 0$ for each differential operator on $\mathcal{T}$. And Kang and Shon [1] have shown some

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*Corresponding author

properties of left regular functions on the generalized quaternion field. In this paper, we investigate the corresponding Cauchy-Riemann systems and properties of regular functions in Clifford analysis. Also we provide theorems for the regular functions by using Jacobian matrix on the quaternion field.

2. Preliminaries

We consider the following Pauli matrices:

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$. Then we know the matrices satisfy

$$e_1e_2 = -e_2e_1 = e_3,\ e_2e_3 = -e_3e_2 = e_1,\ e_3e_1 = -e_1e_3 = e_2$$

and

$$e_1^2 = e_2^2 = e_3^2 = -1.$$

Let $\mathcal{T}$ be a quaternion field with basis $\{e_0,\ e_1,\ e_2,\ e_3\}$ for $e_0 = id$,

$$\mathcal{T} = \{z \mid z = \sum_{j=0}^{3} e_j x_j,\ x_j \in \mathbb{R}\ (j = 0,\ 1,\ 2,\ 3)\}.$$

The quaternion z is an element of four dimensional skew field (non-commutative division ring) of real numbers and the quaternionic conjugate z^* and the absolute value $|z|$ of z are defined by

$$z^* = x_0 - \sum_{j=0}^{3} e_j x_j,\ |z|^2 = zz^* = \sum_{j=0}^{3} x_j^2.$$

And every non-zero quaternion z has a unique inverse $z^{-1} = \dfrac{z^*}{|z|^2}$. Let Ω be a bounded open set in $\mathcal{T}$ and a function $f : \Omega \to \mathcal{T}$ expressed by

$$f(z) = \sum_{j=1}^{3} e_j u_j(x_0, x_1, x_2, x_3) = e_0u_0 + e_1u_1 + e_2u_2 + e_3u_3,$$

where u_j $(j = 0,\ 1,\ 2,\ 3)$ are real valued functions.

We consider the following quaternionic differential operators:

$$\begin{aligned} D &:= \frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3}, \\ D^* &= \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}. \end{aligned}$$

Let Ω be a bounded open set in $\mathbb{C}^2$. A function $f = e_0u_0 + e_1u_1 + e_2u_2 + e_3u_3$ is called a *regular function* in Ω, if (i) $f \in C^1(\Omega)$ and (ii) $D^*f = 0$ in Ω.

Remark 2.1. *The above condition* (ii) is equivalent to the following systems:

$$\text{(i)} \qquad \frac{\partial u_0}{\partial x_0} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i},$$

$$\text{(ii)} \qquad \frac{\partial u_i}{\partial x_0} = -\frac{\partial u_0}{\partial x_i} \qquad (i = 1,\ 2,\ 3),$$

$$\text{(iii)} \qquad \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \qquad (i, j = 1,\ 2,\ 3,\ i \neq j).$$

This system is called a *corresponding Cauchy-Riemann system* in $\mathcal{T}$.

3. Properties of regualr functions

Here we show some properties of regular functions.

Lemma 3.1. *Let Ω be a bounded open set in $\mathcal{T}$. If $f = \sum_{i=0}^{3} e_iu_i$ is regular with the following conditions on Ω:*

$$\text{(1)} \qquad \frac{\partial u_i}{\partial x_i}\frac{\partial u_j}{\partial x_0} = \frac{\partial u_i}{\partial x_0}\frac{\partial u_j}{\partial x_i} \qquad (i \neq j,\ i, j = 1,\ 2,\ 3),$$

then

$$\text{(2)} \qquad \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_0}\frac{\partial u_i}{\partial x_j} = 0 \qquad (j = 1,\ 2,\ 3)$$

on Ω.

Proof. Since the function f is regular on Ω, f satisfies the system in Remark 2.1. Then we have

$$\begin{aligned}
\sum_{i=0}^{3} \frac{\partial u_i}{\partial x_0}\frac{\partial u_i}{\partial x_j} &= \frac{\partial u_0}{\partial x_0}\frac{\partial u_0}{\partial x_j} + \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_0}\frac{\partial u_i}{\partial x_j} \\
&= -\frac{\partial u_0}{\partial x_0}\frac{\partial u_j}{\partial x_0} - \sum_{i=1}^{3} \frac{\partial u_0}{\partial x_i}\frac{\partial u_j}{\partial x_i} \\
&= -\sum_{i=0}^{3} \frac{\partial u_0}{\partial x_i}\frac{\partial u_j}{\partial x_i},
\end{aligned}$$

where $j = 1,\ 2,\ 3$ on Ω. Now, it is sufficient to show that

$$\sum_{i=0}^{3} \frac{\partial u_0}{\partial x_i} \frac{\partial u_j}{\partial x_i} = 0.$$

By the equations (1), we have

$$\begin{aligned}
\sum_{i=0}^{3} \frac{\partial u_0}{\partial x_i} \frac{\partial u_j}{\partial x_i} &= \frac{\partial u_0}{\partial x_0} \frac{\partial u_j}{\partial x_0} + \sum_{i=1}^{3} \frac{\partial u_0}{\partial x_i} \frac{\partial u_j}{\partial x_i} \\
&= \frac{\partial u_0}{\partial x_0} \frac{\partial u_j}{\partial x_0} + \frac{\partial u_0}{\partial x_j} \frac{\partial u_j}{\partial x_j} + \sum_{i=1,\ i\neq j}^{3} \frac{\partial u_0}{\partial x_i} \frac{\partial u_j}{\partial x_i} \\
&= \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_0} + \frac{\partial u_0}{\partial x_j} \frac{\partial u_j}{\partial x_j} + \sum_{i=1,\ i\neq j}^{3} \frac{\partial u_0}{\partial x_i} \frac{\partial u_j}{\partial x_i} \\
&= \left(\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} - \frac{\partial u_j}{\partial x_j} \right) \frac{\partial u_j}{\partial x_0} + \sum_{i=1,\ i\neq j}^{3} \frac{\partial u_0}{\partial x_i} \frac{\partial u_j}{\partial x_i} \\
&= \sum_{i=1,\ i\neq j}^{3} \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_0} + \sum_{i=1,\ i\neq j}^{3} \frac{\partial u_0}{\partial x_i} \frac{\partial u_j}{\partial x_i} \\
&= \sum_{i=1,\ i\neq j}^{3} \left(\frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_0} - \frac{\partial u_i}{\partial x_0} \frac{\partial u_j}{\partial x_i} \right) = 0,
\end{aligned}$$

where $j = 1,\ 2,\ 3$ on Ω. □

The Jacobian matrix on the quaternion field J can be denoted by

$$J = \begin{pmatrix}
\frac{\partial u_0}{\partial x_0} & \frac{\partial u_0}{\partial x_1} & \frac{\partial u_0}{\partial x_2} & \frac{\partial u_0}{\partial x_3} \\
\frac{\partial u_1}{\partial x_0} & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\
\frac{\partial u_2}{\partial x_0} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\
\frac{\partial u_3}{\partial x_0} & \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}
\end{pmatrix}, \tag{3}$$

and $|J|$ is the Jacobian determinant such that $|J| = \dfrac{\partial(u_0, u_1, u_2, u_3)}{\partial(x_0, x_1, x_2, x_3)}$. From Lemma 3.1, we have the following theorem.

Theorem 3.2. *Let Ω be a bounded open subset in $\mathcal{T}$. Under the condition* (1) *of* Lemma 3.1, *if f is regular, then there exists a hypercomplex valued function $G(z)$ on Ω satisfying*

$$\left|J^T J\right| = G(z) \sum_{i=0}^{3} \left(\frac{\partial u_0}{\partial x_i}\right)^2 .$$

Proof. Let the function f be regular on Ω and J be the Jacobian matrix on the quaternion field. Then, J^T is a transpose Jacobian matrix expressed by

$$J^T = \begin{pmatrix} \frac{\partial u_0}{\partial x_0} & \frac{\partial u_1}{\partial x_0} & \frac{\partial u_2}{\partial x_0} & \frac{\partial u_3}{\partial x_0} \\ \frac{\partial u_0}{\partial x_1} & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_0}{\partial x_2} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_0}{\partial x_3} & \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}.$$

Since f is a regular function, f satisfies the corresponding Cauchy-Riemann system in Remark 2.1. By using (3) and Remark 2.1, we have

$$\left|J^T J\right| = \begin{vmatrix} \sum_{i=0}^{3} \left(\frac{\partial u_i}{\partial x_0}\right)^2 & \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_0}\frac{\partial u_i}{\partial x_1} & \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_0}\frac{\partial u_i}{\partial x_2} & \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_0}\frac{\partial u_i}{\partial x_3} \\ \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_0}\frac{\partial u_i}{\partial x_1} & \sum_{i=0}^{3} \left(\frac{\partial u_i}{\partial x_1}\right)^2 & \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_2} & \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_3} \\ \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_0}\frac{\partial u_i}{\partial x_2} & \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_2} & \sum_{i=0}^{3} \left(\frac{\partial u_i}{\partial x_2}\right)^2 & \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_2}\frac{\partial u_i}{\partial x_3} \\ \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_0}\frac{\partial u_i}{\partial x_3} & \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_3} & \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_2}\frac{\partial u_i}{\partial x_3} & \sum_{i=0}^{3} \left(\frac{\partial u_i}{\partial x_3}\right)^2 \end{vmatrix}.$$

Since f satisfies $\frac{\partial u_i}{\partial x_i}\frac{\partial u_j}{\partial x_0} = \frac{\partial u_i}{\partial x_0}\frac{\partial u_j}{\partial x_i}$ for $i \neq j$ and $i, j = 1,\ 2,\ 3$, by Lemma 3.1, the equations (2) hold. Thus, we have

$$\left|J^T J\right| = \begin{vmatrix} \sum_{i=0}^{3}\left(\frac{\partial u_i}{\partial x_0}\right)^2 & 0 & 0 & 0 \\ 0 & \sum_{i=0}^{3}\left(\frac{\partial u_i}{\partial x_1}\right)^2 & \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_2} & \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_3} \\ 0 & \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_2} & \sum_{i=0}^{3}\left(\frac{\partial u_i}{\partial x_2}\right)^2 & \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_2}\frac{\partial u_i}{\partial x_3} \\ 0 & \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_3} & \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_2}\frac{\partial u_i}{\partial x_3} & \sum_{i=0}^{3}\left(\frac{\partial u_i}{\partial x_3}\right)^2 \end{vmatrix}$$

$$= \sum_{i=0}^{3}\left(\frac{\partial u_i}{\partial x_0}\right)^2 \begin{vmatrix} \sum_{i=0}^{3}\left(\frac{\partial u_i}{\partial x_1}\right)^2 & \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_2} & \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_3} \\ \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_2} & \sum_{i=0}^{3}\left(\frac{\partial u_i}{\partial x_2}\right)^2 & \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_2}\frac{\partial u_i}{\partial x_3} \\ \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_3} & \sum_{i=0}^{3}\frac{\partial u_i}{\partial x_2}\frac{\partial u_i}{\partial x_3} & \sum_{i=0}^{3}\left(\frac{\partial u_i}{\partial x_3}\right)^2 \end{vmatrix}.$$

Consequently, we know there exists a hypercomplex valued function $G(z)$ such that

$$\left|J^T J\right| = G(z)\sum_{i=0}^{3}\left(\frac{\partial u_i}{\partial x_0}\right)^2 = G(z)\sum_{i=0}^{3}\left(\frac{\partial u_0}{\partial x_i}\right)^2.$$

Thus, the theorem is proved. □

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Han Ul Kang
Department of Mathematics, Pusan National University,
Busan 46241, Korea.
E-mail: hukang@pusan.ac.kr

Min Ji Kim
Department of Mathematics, Pusan National University,
Busan 46241, Korea.
E-mail: romangs034@naver.com

Kwang Ho Shon
Department of Mathematics, Pusan National University,
Busan 46241, Korea.
E-mail: khshon@pusan.ac.kr