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THE DERIVATIVE OF A DUAL QUATERNIONIC FUNCTION WITH VALUES IN DUAL QUATERNIONS

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Abstract. This paper gives the expression of dual quaternions and provides differential operators in dual quaternions. The paper also represents the derivative of dual quaternion-valued functions by using a corresponding Cauchy-Riemann system in dual quaternions.

1. Introduction

Clifford [1] introduced dual numbers that similar to the structure of complex numbers. Dual numbers are consisted of two components and they are defined as follows:

$$z = x + \varepsilon y,$$

where ε is the dual operator with $\varepsilon^2 = 0$ $(\varepsilon \neq 0)$, x and y are real numbers. The dual operator ε is used in the same way which is similar to the complex operator i in complex analysis. Dual numbers can be extended to vectors and real numbers, such as their applicability with quaternions to provide rotations and transforms.

Hamilton [2] introduced quaternions and extended complex number theory to formulas in a four dimensional space. A quaternion is defined as follows:

$$p = x_0 + x_1 i + x_2 j + x_3 k,$$

where x_r $(r = 0, 1, 2, 3)$ are real numbers, while i, j and k are the imaginary components such that

$$i^2 = j^2 = k^2 = -1$$

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and

$$ij = k,\ ji = -k,\ jk = i,\ kj = -i,\ ki = j,\ ik = -j.$$

Clifford [1] combined quaternions and obtained the number system, called the dual-quaternion which is based on the dual number theory. While an unit quaternion can represent a rotation, the unit dual-quaternion has the representation of translations and rotations. Each dual-quaternion consists of eight elements and moreover, it has two quaternions such that

$$P = p + \varepsilon q,$$

where p and q are quaternions called the real part and the dual part, respectively.

Many papers studied the properties of dual quaternions and their advantages in applications to various fields. Messelmi [16, 17] generalized the notions and properties of dual functions and developed general theories of multidual numbers and multidual functions. Kajiwara et al. [3, 4] applied the theory on a closed densely defined operator and a priori estimate for the adjoint operator in a Hilbert space and br-convex domains, by using an inhomogeneous Cauchy-Riemann system in quaternion analysis. Kim et al. [8, 9] obtained some results for the regularity of functions in Clifford analysis and Kim et al. [10, 11] researched corresponding Cauchy-Riemann systems and properties of functions with values in special quaternions such as reduced quaternion and split quaternions. Kim et al. [12, 13] investigated regular functions defined by the differential operators of special quaternion number systems. McDonald [15] gave the simple approaches of the notions of quaternions and representations of rotation matrices. Kenwright [6, 7] gave a guide to the practicality of using dual-quaternions to represent the rotations and translations in the complex 3D character space. Pham et al. [18] provided a new concept of unified controls of robot manipulators involving both translation and rotation, by using Jacobian matrix in the dual-quaternion space. Kavan [5] improved that skinning of models is used for the real-time animation of characters and similar objects. Yang-Hsing [14] studied the traditional way of coplanarity conditions and least square solutions using dual quaternions to solve a relative orientation.

This paper gives expressions of dual quaternions and differential operators in dual quaternions. The paper also represents the derivative of dual quaternion-valued functions by using a corresponding Cauchy-Riemann system in dual quaternions.

2. Preliminaries

We consider the following form:

$$\mathbb{D}_q = \{Z = p_1 + \varepsilon p_2 \mid p_r \in \mathbb{H},\ \varepsilon^2 = 0,\ r = 1, 2\},$$

which is isomorphic with $\mathbb{H}^2$ and $\mathbb{R}^8$, where ε is the dual unit that commutes with i, j and k and

$$\mathbb{H} = \{p = z_1 + z_2 j \mid z_1 = x_0 + x_1 i,\ z_2 = x_2 + x_3 i,\ x_r \in \mathbb{R}\ (r = 0, 1, 2, 3)\}$$

is the set of quaternions. Here imaginary basis elements i, j and k satisfy the following conditions:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

For two quaternions $p = z_1 + z_2 j$ and $q = w_1 + w_2 j$, the rule of addition is:

$$p + q = (z_1 + w_1) + (z_2 + w_2)j$$

and multiplication is:

$$pq = (z_1 w_1 - z_2 \overline{w_2}) + (z_1 w_2 + z_2 \overline{w_1})j.$$

From the above rules, we give the norm for a quaternion as follows:

$$|p|^2 := pp^* = z_1 \overline{z_1} + z_2 \overline{z_2}$$

and the inverse of p as follows:

$$p^{-1} = \frac{p^*}{|p|^2} \quad (p \neq 0).$$

For $Z = p_1 + \varepsilon p_2$ and $W = q_1 + \varepsilon q_2$, we have the following rules of addition on $\mathbb{D}_q$:

$$Z + W = (p_1 + q_1) + \varepsilon(p_2 + q_2)$$

and multiplication on $\mathbb{D}_q$:

$$ZW = p_1 q_1 + \varepsilon(p_1 q_2 + p_2 q_1).$$

We give a complex conjugate element of $\mathbb{D}_q$ as follows:

$$Z^* = p_1^* + \varepsilon p_2^*$$

and then, the norm of Z, denoted by $|Z|$, is described by

$$|Z|^2 = ZZ^* = Z^* Z = p_1 p_1^* + 2\varepsilon\lambda,$$

where λ is a real part of $z_1\overline{w_1} + z_2\overline{w_2}$. Since the elements of the set $\{\varepsilon p \mid p \in \mathbb{H}\}$ do not have inverse, the inverse of a dual quaternion is given by

$$Z^{-1} = \frac{Z^{\dagger}}{|p_1|^2} \quad (p_1 \neq 0),$$

where

$$Z^{\dagger} = p_1^* - \varepsilon p_1^{-1} p_2 p_1^*,$$

called the left dual conjugate of Z with $ZZ^{\dagger} = Z^{\dagger}Z = p_1 p_1^*$.

3. Hyperholomorphic function in dual quaternions

Let Ω be an open set in $\mathbb{H}^2$. A function is given by

$$F : \Omega \to \mathbb{D}_q; \ F(Z) = f_1(p_1, p_2) + \varepsilon f_2(p_1, p_2),$$

where

$$f_1 = g_1(z_1, z_2, w_1, w_2) + g_2(z_1, z_2, w_1, w_2)j$$

and

$$f_2 = h_1(z_1, z_2, w_1, w_2) + h_2(z_1, z_2, w_1, w_2)j$$

are quaternion-valued functions, g_r and h_r $(r = 1, 2)$ are complex-valued functions.

Definition 3.1. *A function F is said to be left-hyperholomorphic in $\mathbb{D}_q$ if the limit*

$$\frac{dF(Z)}{dZ} := \lim_{\zeta \to 0} \zeta^{-1}(F(Z+\zeta) - F(Z)) \tag{3.1}$$

exists, where $\zeta = \eta_1 + \varepsilon \eta_2 \to 0$ means $\eta_1 \to 0$ and $\eta_2 \to 0$, that is, each component approaches to zero.

Clearly, the properties and progresses of left-hyperholomorphic functions are equivalent to those of right-hyperholomorphic functions.

For the convenience of representations of this paper, we consider left-hyperholomorphic functions, which are called simply hyperholomorphic functions. So now then, we write the inverse form as follows:

$$\frac{dF(Z)}{dZ} := \lim_{\zeta \to 0} \frac{F(Z+\zeta) - F(Z)}{\zeta}.$$

Theorem 3.1. *A function F is hyperholomorphic in $\mathbb{D}_q$ if and only if the following conditions are held:*

$$
\text{(3.2)} \qquad \begin{cases} \lim\limits_{\substack{\eta_1 \to 0, \\ \eta_2 \to 0}} \dfrac{(F(Z+\zeta) - F(Z))}{\eta_1} & \textit{exists and} \\ \lim\limits_{\substack{\eta_1 \to 0, \\ \eta_2 \to 0}} \dfrac{f_1(p_1+\eta_1, p_2+\eta_2) - f_1(p_1, p_2)}{\eta_2} & \textit{is zero.} \end{cases}
$$

Proof. From the definition of hyperholomorphic function in $\mathbb{D}_q$, the function F has to satisfy that the following limit exists.

$$
\text{(3.3)} \qquad \lim_{\zeta \to 0} \frac{F(Z+\zeta) - F(Z)}{\zeta} = \lim_{\substack{\eta_1 \to 0, \\ \eta_2 \to 0}} \frac{(F(Z+\zeta) - F(Z))(\eta_1^* - \varepsilon \eta_2^\dagger)}{\eta_1 \eta_1^*}
$$

$$
= \lim_{\substack{\eta_1 \to 0, \\ \eta_2 \to 0}} \frac{F(Z+\zeta) - F(Z)}{\eta_1}
$$

$$
- \lim_{\substack{\eta_1 \to 0, \\ \eta_2 \to 0}} \varepsilon \frac{f_1(p_1+\eta_1, p_2+\eta_2) - f_1(p_1, p_2)}{\eta_2} \Big(\frac{\eta_2}{\eta_1}\Big)^2.
$$

Since the existence of the limit has to be independent of $\left(\frac{\eta_2}{\eta_1}\right)^2$, the following limit has to be zero, that is,

$$
\lim_{\substack{\eta_1 \to 0, \\ \eta_2 \to 0}} \frac{(f_1(p_1+\eta_1, p_2+\eta_2) - f_1(p_1, p_2))}{\eta_2} = 0.
$$

Hence, the hyperholomorphic function F in $\mathbb{D}_q$ satisfies the conditions (3.2).

Conversely, if the conditions (3.2) are held, then by the process of the equation (3.3), we obtain that the limit

$$
\lim_{\zeta \to 0} \frac{F(Z+\zeta) - F(Z)}{\zeta}
$$

exists. □

As an example, for a function $F(Z) = f_1(p_1, 0) + \varepsilon f_2(p_1, p_2)$, if the limit

$$
\lim_{\substack{\eta_1 \to 0, \\ \eta_2 \to 0}} \frac{(F(Z+\zeta) - F(Z))}{\eta_1}
$$

exists, then F is hyperholomorphic in $\mathbb{D}_q$.

Theorem 3.2. *If a function F is hyperholomorphic in $\mathbb{D}_q$, then the following equations are held:*

$$\frac{\partial F}{\partial p_1^*} = 0 \quad \textit{and} \quad \frac{\partial f_1}{\partial y_r} = 0 \ (r = 0, 1, 2, 3). \tag{3.4}$$

Proof. Since F is hyperholomorphic in $\mathbb{D}_q$, the limit

$$\lim_{\substack{\eta_1 \to 0,\\ \eta_2 \to 0}} \frac{(F(Z+\zeta) - F(Z))}{\eta_1}$$

exists and

$$\lim_{\substack{\eta_1 \to 0,\\ \eta_2 \to 0}} \frac{(f_1(p_1+\eta_1, p_2+\eta_2) - f_1(p_1, p_2))}{\eta_2} = 0.$$

By rearranging terms of the above equations, we have the following equations:

$$\frac{\partial F}{\partial x_0} = -i\frac{\partial F}{\partial x_1} = -j\frac{\partial F}{\partial x_2} = -k\frac{\partial F}{\partial x_3},$$
$$\frac{\partial f_1}{\partial y_0} = -i\frac{\partial f_1}{\partial y_1} = -j\frac{\partial f_1}{\partial y_2} = -k\frac{\partial f_1}{\partial y_3} = 0.$$

Therefore, we obtain the equations (3.4). □

In detail, the Equation (3.4) is equivalent to the following system

$$\begin{cases} \dfrac{\partial g_1}{\partial \overline{z_1}} = \dfrac{\partial g_2}{\partial \overline{z_2}}, \ \dfrac{\partial g_2}{\partial \overline{z_1}} = -\overline{\dfrac{\partial g_1}{\partial z_2}}, \\ \dfrac{\partial h_1}{\partial \overline{z_1}} = \dfrac{\partial h_2}{\partial \overline{z_2}}, \ \dfrac{\partial h_2}{\partial \overline{z_1}} = -\overline{\dfrac{\partial h_1}{\partial z_2}}, \\ \dfrac{\partial f_1}{\partial y_r} = 0 \quad (r = 0, 1, 2, 3) \end{cases} \tag{3.5}$$

called the corresponding Cauchy-Riemann system to $\mathbb{D}_q$.

We give the differential operators in $\mathbb{D}_q$.

$$D := \frac{\partial}{\partial p_1} - \varepsilon\frac{\partial}{\partial p_2}, \ D^* = \frac{\partial}{\partial p_1^*} + \varepsilon\frac{\partial}{\partial p_2},$$
$$\frac{\partial}{\partial p_1} := \frac{\partial}{\partial z_1} - j\frac{\partial}{\partial \overline{z_2}}, \ \frac{\partial}{\partial p_2} := \frac{\partial}{\partial w_1} - j\frac{\partial}{\partial \overline{w_2}},$$

$$\frac{\partial}{\partial p_1^*} = \frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}, \ \frac{\partial}{\partial p_2^*} = \frac{\partial}{\partial \overline{w_1}} + j\frac{\partial}{\partial \overline{w_2}},$$

where $\frac{\partial}{\partial \overline{z_r}}$ and $\frac{\partial}{\partial \overline{w_r}}$ $(r = 1, 2)$ are usual complex differential operators.

Definition 3.2. *Let Ω be a bounded open set of $\mathbb{D}_q$ and for $Z \in \mathbb{H}^2$, a function F is said to be hyperholomorphic in $\mathbb{D}_q$ if the following conditions are satisfied:*
(i) each component f_1 and f_2 of $F(Z)$ is continuously differentiable and
(ii) $D^*F = 0$ on $\mathbb{D}_q$.

Specially, the second condition is equivalent to the system (3.5).

Example 3.1. *For $Z \in \mathbb{D}_q$, since a function*

$$F(Z) = Z = (z_1 + z_2 j) + \varepsilon(w_1 + w_2 j)$$

satisfies the system (3.5), that is, the function F has the form

$$F(Z) = f_1(p_1, 0) + \varepsilon f_2(p_1, p_2)$$

and

$$\begin{aligned}\frac{\partial F}{\partial p_1^*} &= \frac{\partial g_1}{\partial \overline{z_1}} - \frac{\partial g_2}{\partial \overline{z_2}} + \Big(\frac{\partial g_2}{\partial \overline{z_1}} + \overline{\frac{\partial g_1}{\partial z_2}}\Big)j \\ &\quad + \varepsilon\Big\{\frac{\partial h_1}{\partial \overline{z_1}} - \frac{\partial h_2}{\partial \overline{z_2}} + \Big(\frac{\partial h_2}{\partial \overline{z_1}} + \overline{\frac{\partial h_1}{\partial z_2}}\Big)j\Big\} = 0,\end{aligned}$$

the function F is hyperholomorphic function in $\mathbb{D}_q$. By using the above calculations, $F(Z) = Z^n$ is also a hyperholomorphic function in $\mathbb{D}_q$.

Example 3.2. *Since a function*

$$F(Z) = Z^* = (\overline{z_1} - z_2 j) + \varepsilon(\overline{w_1} - w_2 j)$$

does not satisfies the system (3.5), the function $F(Z)$ is not hyperholomorphic in $\mathbb{D}_q$. Also, since the system (3.5) is not satisfied for the functions

$$F(Z) = Z^\dagger = (\overline{z_1} - z_2 j) + \varepsilon\frac{(w_1 + w_2 j)(\overline{z_1} - z_2 j)^2}{z_1\overline{z_1} + z_2\overline{z_2}}$$

and

$$F(Z) = Z^{-1} = 1 - \varepsilon\frac{(w_1 + w_2 j)(\overline{z_1} - z_2 j)}{z_1\overline{z_1} + z_2\overline{z_2}},$$

the functions $Z^\dagger$ and Z^{-1} are not hyperholomorphic in $\mathbb{D}_q$.

Theorem 3.3. *Let Ω be a bounded open set of $\mathbb{H}^2$ and for $Z \in \mathbb{D}_q$, a function F be hyperholomorphic in $\mathbb{D}_q$. Then the function F satisfies the following equation:*

$$DF(Z) = \frac{\partial F}{\partial z_1} + \frac{\partial F^*}{\partial \overline{z_1}}.$$

Proof. Since the function $F(Z)$ is hyperholomorphic in $\mathbb{D}_q$, the function F satisfies the conditions (3.2). Hence, we have $\frac{\partial f_1}{\partial y_r} = 0$ $(r = 0, 1, 2, 3)$ and

$$DF(Z) = \frac{\partial F}{\partial p_1} = \frac{\partial f_1}{\partial z_1} + \varepsilon\frac{\partial f_2}{\partial z_1} - j\frac{\partial f_1}{\partial \overline{z_2}} - j\varepsilon\frac{\partial f_2}{\partial \overline{z_2}}.$$

By replacing the terms of the system (3.5) to the above equation, we have

$$\begin{aligned} DF(Z) &= \frac{\partial F}{\partial z_1} - \frac{\partial g_2}{\partial \overline{z_1}}j + \frac{\partial \overline{g_1}}{\partial \overline{z_1}} - \varepsilon\frac{\partial h_2}{\partial \overline{z_1}}j + \varepsilon\frac{\partial \overline{h_1}}{\partial \overline{z_1}} \\ &= \frac{\partial F}{\partial z_1} + \frac{\partial F^*}{\partial \overline{z_1}}. \end{aligned}$$

Therefore, we obtain the result. □

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