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STRONG VERSIONS OF κ-FRÉCHET AND κ-NET SPACES

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Abstract. We introduce strongly κ-Fréchet and strongly κ-sequential spaces which are stronger than κ-Fréchet and κ-net spaces respectively. For convenience, we use the terminology "κ-sequential" instead of "κ-net space", introduced by R.E. Hodel in [5]. And we study some properties and topological operations on such spaces. We also define strictly κ-Fréchet and strictly κ-sequential spaces which are more stronger than strongly κ-Fréchet and strongly κ-sequential spaces respectively.

1. Introduction

The notion of κ-nets (κ an infinite cardinal) was introduced by R.E. Hodel in [5] and is based on the notion of nets. A *κ-net* in a space X is a function $\xi : \kappa^{<\omega} \to X$, where $\kappa^{<\omega} = \{F : F \text{ is a finite subset of } \kappa\}$ and is directed by the set inclusion $\subseteq$. The κ-net ξ is usually denoted by $\langle x_F : F \in \kappa^{<\omega}\rangle$, or just $\langle x_F\rangle$, where $x_F = \xi(F)$ for all $F \in \kappa^{<\omega}$. However, the first idea of κ-net appeared in [6] under the name a phalanx as defined by Tukey. By definition, a *phalanx* is a function $f : A^{<\omega} \to X$, thus, a κ-net is a phalanx with $A = \kappa$. Hodel in [5] used κ-nets in the study of convergence and cluster points and then used κ-nets to extend Fréchet and sequential spaces to higher cardinality to obtain the κ-Fréchet and κ-net spaces. This has already done by Meyer in [7] with nets whose directed set have cardinality at most κ. However Hodel in [5] showed that the two approaches give the same classes of spaces.

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A κ-net on a space certainly has the intuitive appeal of nets to approach to a general theory of convergence in topology and modern analysis. The main purpose of this paper is to introduce some concepts with respect to κ-net convergence, and discuss some properties related to strong versions of κ-Fréchet and κ-net spaces. And we also study some properties and topological operations on such spaces. More precisely, we will define strongly κ-Fréchet (resp., strictly κ-Fréchet) and strongly κ-sequential (resp., strictly κ-sequential) spaces. We consider the relationship between them.

All spaces in this paper are assumed to be Hausdorff. Our topological terminologies and notions are as in [4].

2. Definitions and basic properties

In what follows, $Card$ denotes a set of cardinals and let $\kappa, \lambda, \tau, \omega \in Card$, and $|X|$ be the cardinality of any set X. Let $\langle X, \mathcal{T} \rangle$ be a space and $x \in X$. The *character* of x in X, denoted by $\chi(x, \langle X, \mathcal{T} \rangle)$ or $\chi(x, X)$, is defined as $\chi(x, X) = \min\{|B(x)| : B(x)$ is a local base at $x\}$. The *character* of X, denoted by $\chi(X)$, is defined as $\chi(X) = \sup\{\chi(x, X) : x \in X\}$. In particular, a space X is first countable if and only if $\chi(X) \leq \omega$.

We write $t(x,\ X) = \min\{\tau \in Card :$ if $A \subseteq X$ and $x \in \overline{A}$, then there exists $B \subseteq A$ such that $|B| \leq \tau$ and $x \in \overline{B}\}$. This is called the *tightness of* X *at* x. The *tightness* of X is, denoted by $t(X)$, is defined as $t(X) = \sup\{t(x, X) : x \in X\}$. In particular, a space X has a countable tightness if and only if $t(X) \leq \omega$.

A space X is *Fréchet* if for every subset A of X and for every point $p \in \overline{A}$ there is a sequence in A which converges to p. A space X is *sequential* if for every non-closed subset A of X there is a sequence in A which converges to some $p \in \overline{A} \setminus A$.

A space X is said to be *strongly Fréchet* ([8]) if for every decreasing family $\{A_n : n \in \omega\}$ of subsets of X and for every $p \in \cap\{\overline{A_n} : n \in \omega\}$, there exists a point $x_n \in A_n$ for each $n \in \omega$ such that the sequence $\{x_n : n \in \omega\}$ converges to p.

Let X be a space and let p be a point of X. Then we say that a κ-net $\langle x_F \rangle$ *converges to* p, written by $x_F \to p$, if for any open neighborhood V of p in X, there exists $F \in \kappa^{<\omega}$ such that

$$x_G \in V \text{ for all } G \in \kappa^{<\omega} \text{ with } F \subseteq G.$$

A space X is *κ-Fréchet* ([5]) if for every $p \in \overline{A}$, there exists a κ-net $\langle x_F \rangle$ in A which converges to p.

A space X is a *κ-net space* ([5]) if every κ-net-closed subset of X is closed in X (A subset A of X is said to be *κ-net-closed* provided that if $\langle x_F \rangle$ is a κ-net in A and $x_F \to p$, then $p \in A$). For convenience, we use the terminology "κ-sequential" instead of "κ-net space".

Proposition 2.1 ([5])**.**

(1) If X is λ-Fréchet (resp., a λ-net space) and $\lambda \leq \kappa$, then X is κ-Fréchet (resp., a κ-net space);
(2) For all κ, $\chi(X) \leq \kappa \Rightarrow X$ is κ-Fréchet $\Rightarrow X$ is a κ-net space $\Rightarrow$ $t(X) \leq \kappa$, and the each converses fail;
(3) X is Fréchet $\Leftrightarrow X$ is ω-Fréchet;
(4) X is a sequential space $\Leftrightarrow X$ is an ω-net space.

A space X is said to be *radial* (resp., *pseudoradial*) if for every non-closed subset A of X and every (resp., some) point $p \in \overline{A} \setminus A$ there is a transfinite sequence $S \subseteq A$ which converges to p. Some generalized properties of strongly Fréchet spaces were studied in [2].

A space X is called *κ-radial* ([3]) if for every non-closed subset A of X and for every point $p \in \overline{A} \setminus A$, there exists a transfinite sequence $\{x_\alpha \in A : \alpha < \kappa\}$ which converges to p.

3. Main results

A family $\{A_F : F \in \kappa^{<\omega}\}$ of subsets of a set X is called a *directed decreasing family* if $A_G \subseteq A_F$ whenever $F \subseteq G$ for any $F, G \in \kappa^{<\omega}$.

Definition 3.1. A space X is called *strongly κ-Fréchet* if for every directed decreasing family $\{A_F : F \in \kappa^{<\omega}\}$ of non-closed subsets of X and for every $p \in \cap\{\overline{A_F} : F \in \kappa^{<\omega}\} \setminus \cup\{A_F : F \in \kappa^{<\omega}\}$, there exists a κ-net $\langle x_F : F \in \kappa^{<\omega}\rangle$ which converges to p.

Definition 3.2. A space X is called *strongly κ-sequential* if for every directed decreasing family $\{A_F : F \in \kappa^{<\omega}\}$ of non-closed subsets of X, there exist a point $p \in \cap\{\overline{A_F} : F \in \kappa^{<\omega}\} \setminus \cup\{A_F : F \in \kappa^{<\omega}\}$ and a point $x_F \in A_F$ for each $F \in \kappa^{<\omega}$ such that the κ-net $\langle x_F : F \in \kappa^{<\omega}\rangle$ converges to p.

The following are true by definitions:

Theorem 3.3.

(1) Every strongly κ-Fréchet space is κ-Fréchet;
(2) Every strongly κ-sequential space is κ-sequential;
(3) Every strongly κ-Fréchet space is strongly κ-sequential.

Under what conditions are the converses true in the above theorem? We will consider some conditions at the end of this section.

We now consider operations on strongly κ-Fréchet and strongly κ-sequential spaces, i.e., subspaces and mapping theorems of strongly κ-Fréchet and strongly κ-sequential spaces.

Theorem 3.4.

(1) Every subspace of a strongly κ-Fréchet space is strongly κ-Fréchet;
(2) Every closed subspace of a strongly κ-sequential space is strongly κ-sequential;
(3) Every open subspace of a strongly κ-sequential space is strongly κ-sequential.

Proof. (1) Assume that Y is a subspace of a strongly κ-Fréchet space X. Let $\{A_F : F \in \kappa^{<\omega}\}$ be a directed decreasing family of non-closed subsets of Y and let $p \in \cap\{\overline{A_F}^Y : F \in \kappa^{<\omega}\} \setminus \cup\{A_F : F \in \kappa^{<\omega}\}$. Then $p \in \overline{A_F}^X \cap Y$ for each $F \in \kappa^{<\omega}$. Since X is strongly κ-Fréchet, there exists a point $x_F \in A_F$ for each $F \in \kappa^{<\omega}$ such that the κ-net $\langle x_F \rangle$ converges to p in X. Let V be an open neighborhood of p in Y. Take an open neighborhood U of p in X such that $V = U \cap Y$. Then there exists $F \in \kappa^{<\omega}$ such that $x_G \in U$ for all $G \in \kappa^{<\omega}$ with $F \subseteq G$. It follows from $x_G \in Y$ that $x_G \in V$. Therefore Y is strongly κ-Fréchet.

(2) Assume that Y is a closed subspace of a strongly κ-sequential space X. Let $\{A_F : F \in \kappa^{<\omega}\}$ be a directed decreasing family of non-closed subsets of Y. Then each A_F is a non-closed subset of X because Y is closed in X. Since X is strongly κ-sequential, there exist a point $p \in \cap\{\overline{A_F}^X : F \in \kappa^{<\omega}\} \setminus \cup\{A_F : F \in \kappa^{<\omega}\}$ and a point $x_F \in A_F$ for each $F \in \kappa^{<\omega}$ such that the κ-net $\langle x_F \rangle$ converges to p. Since $\overline{A_F}^Y = \overline{A_F}^X$ for each $F \in \kappa^{<\omega}$, Y is strongly κ-sequential.

(3) Let Y be an open subspace of a strongly κ-sequential space X. Suppose that Y is not strongly κ-sequential. Let $\{A_F : F \in \kappa^{<\omega}\}$ be a directed decreasing family of non-closed subsets of Y. Since each A_F is a non-closed subset of X and since X is strongly κ-sequential, there exist a point $p \in \cap\{\overline{A_F}^X : F \in \kappa^{<\omega}\} \setminus \cup\{A_F : F \in \kappa^{<\omega}\}$ and a point $x_F \in A_F$ for each $F \in \kappa^{<\omega}$ such that the κ-net $\langle x_F \rangle$ converges to p in X.

Notice that $\langle x_F \rangle$ is a κ-net in Y. By our assumption, $p \notin \overline{A_F}^Y$ for some $F \in \kappa^{<\omega}$. Since $Y \setminus \overline{A_F}^Y$ is open in Y (and hence in X), it is impossible that $p \in Y \setminus \overline{A_F}^Y$ (for, if $p \in Y \setminus \overline{A_F}^Y$, then there exists $F_0 \in \kappa^{<\omega}$ such that $x_G \in Y \setminus \overline{A_F}^Y$ for all $G \in \kappa^{<\omega}$ with $F_0 \subseteq G$. This is a contraction to $x_G \in A_G \subseteq A_F$ for all $G \in \kappa^{<\omega}$ with $F_0 \cup F \subseteq G$.).

Hence $p \in X \setminus Y$. But $p \in \overline{A_F}^X \cap (X \setminus Y) = \overline{A_F \cap (X \setminus Y)}^X = \emptyset$ for each $F \in \kappa^{<\omega}$. This is a contradiction. Therefore Y is strongly κ-sequential. □

Theorem 3.5.

(1) Every closed continuous image of a strongly κ-Fréchet space is strongly κ-Fréchet;
(2) Every closed continuous image of a strongly κ-sequential space is κ-sequential.

Proof. (1) Assume that X is a strongly κ-Fréchet space and $f : X \to Y$ is a closed continuous onto map. Let $\{A_F : F \in \kappa^{<\omega}\}$ be a directed decreasing family of non-closed subsets of Y and let $p \in \cap\{\overline{A_F} : F \in \kappa^{<\omega}\} \setminus \cup\{A_F : F \in \kappa^{<\omega}\}$. Denote $B_F = f^{-1}(A_F)$ for each $F \in \kappa^{<\omega}$. Notice that $f(\overline{B_F}) = \overline{f(B_F)} = \overline{A_F}$ for each $F \in \kappa^{<\omega}$ (because f is a closed continuous onto map). Hence $\overline{B_F} = f^{-1}(\overline{A_F})$ for each $F \in \kappa^{<\omega}$. So we obtain that

$$f^{-1}(p) \subset f^{-1}(\cap\{\overline{A_F} : F \in \kappa^{<\omega}\}) = \cap\{\overline{B_F} : F \in \kappa^{<\omega}\}$$

and

$$f^{-1}(p) \cap (\cup\{B_F : F \in \kappa^{<\omega}\}) = \emptyset.$$

Pick a point $x \in f^{-1}(p)$. Since X is strongly κ-Fréchet, there exists a point $x_F \in B_F$ for each $F \in \kappa^{<\omega}$ such that the κ-net $\langle x_F \rangle$ converges to x in X. Let $y_F = f(x_F)$ for each $F \in \kappa^{<\omega}$. Then $y_F \in A_F$ for each $F \in \kappa^{<\omega}$ and the κ-net $\langle y_F \rangle$ converges to p in Y. Therefore Y is strongly κ-Fréchet.

It follows from the same argument with (1) that (2) can be proved. □

A continuous map $f : X \to Y$ is called *pseudo-open* if for each $y \in Y$ and every neighborhood U of $f^{-1}(y)$ we have $y \in f(U)^\circ$ where A° is the interior of A in Y.

The following lemma is a characterization of a pseudo-open map.

Lemma 3.6 ([1])**.** Let $f : X \to Y$ be a continuous onto map. Then the following conditions are equivalent:

(1) For each $Y' \subset Y$ the restriction f' of f to $X' = f^{-1}(Y')$, the inverse image of Y', is a quotient map of X' onto Y';
(2) For each $y \in Y$ and every open set U in X containing $f^{-1}(y)$, $y \in f(U)^{\circ}$;
(3) Whenever $B \subset Y$ and $y \in Y$ satisfies $y \in \overline{B}$, we have $f^{-1}(y) \cap \overline{f^{-1}(B)} \neq \emptyset$.

It is well known that every continuous closed (or open) map is pseudo-open.

Theorem 3.7.

(1) Any pseudo-open injective image of a strongly κ-Fréchet space is strongly κ-Fréchet;
(2) Any pseudo-open injective image of a strongly κ-sequential space is κ-sequential.

Proof. (1) Assume that X is a strongly κ-Fréchet space and $f : X \to Y$ is a pseudo-open bijection. Let $\{A_F : F \in \kappa^{<\omega}\}$ be a directed decreasing family of non-closed subsets of Y and let $p \in \cap\{\overline{A_F} : F \in \kappa^{<\omega}\} \setminus \cup\{A_F : F \in \kappa^{<\omega}\}$. Denote $B_F = f^{-1}(A_F)$ for each $F \in \kappa^{<\omega}$. Then, by Lemma 3.6, $f^{-1}(p) \cap \overline{B_F} \neq \emptyset$. Since f is injective, $f^{-1}(p)$ is a singleton (denote $f^{-1}(p) = \{x\}$). Hence $x \in \overline{B_F}$ for each $F \in \kappa^{<\omega}$. Namely, $x \in \cap\{\overline{B_F} : F\kappa^{<\omega}\} \setminus \cup\{B_F : F \in \kappa^{<\omega}\}$ where $\{B_F : F \in \kappa^{<\omega}\}$ is a directed decreasing family of non-closed subsets of X. Since X is strongly κ-Fréchet, there exists a point $x_F \in B_F$ such that the κ-net $\langle x_F \rangle$ converges to x in X. Let $y_F = f(x_F)$ for each $F \in \kappa^{<\omega}$. Then $y_F \in A_F$ for each $F \in \kappa^{<\omega}$ and the κ-net $\langle y_F \rangle$ converges to p in Y. Therefore Y is strongly κ-Fréchet.

(2) The proof follows the same argument with (1). □

A family $\{A_F : F \in \kappa^{<\omega}\}$ of subsets of a set X is called a *strictly directed decreasing family* if $A_F \subseteq A_G$ for all $F, G \in \kappa^{<\omega}$ with $|G| \leq |F|$.

Definition 3.8. A space X is called *strictly κ-Fréchet* if for every strictly directed decreasing family $\{A_F : F \in \kappa^{<\omega}\}$ of subsets of X and for every $p \in \cap\{\overline{A_F} : F \in \kappa^{<\omega}\} \setminus \cup\{A_F : F \in \kappa^{<\omega}\}$, there exists a point $x_F \in A_F$ for each $F \in \kappa^{<\omega}$ such that the κ-net $\langle x_F \rangle$ converges to p.

Definition 3.9. A space X is called *strictly κ-sequential* if for every strictly directed decreasing family $\{A_F : F \in \kappa^{<\omega}\}$ of non-closed subsets of X, there exist a point $p \in \cap\{\overline{A_F} : F \in \kappa^{<\omega}\} \setminus \cup\{A_F : F \in \kappa^{<\omega}\}$ and a point $x_F \in A_F$ for each $F \in \kappa^{<\omega}$ such that the κ-net $\langle x_F \rangle$ converges to p.

The following are true by definitions:

Theorem 3.10.

(1) Every strictly κ-Fréchet space is strongly κ-Fréchet;
(2) Every strictly κ-sequential space is strongly κ-sequential;
(3) Every strictly κ-Fréchet space is strictly κ-sequential.

In the rest of this paper, we compare some properties in Proposition 2.1 with the properties on strong versions of κ-Fréchet and κ-net spaces. As (1) in Proposition 2.1 we have the following theorem.

Theorem 3.11. If X is strictly λ-Fréchet and $\lambda \leq \kappa$, then X is strictly κ-Fréchet.

Proof. Suppose that X is strictly λ-Fréchet and $\lambda \leq \kappa$. Let $\{A_F : F \in \kappa^{<\omega}\}$ be a strictly directed decreasing family of non-closed subsets of X and let $p \in \cap\{\overline{A_F} : F \in \kappa^{<\omega}\} \setminus \cup\{A_F : F \in \kappa^{<\omega}\}$.

Let $\{B_F : F \in \lambda^{<\omega}\}$ be a subfamily of $\{A_F : F \in \kappa^{<\omega}\}$ such that $B_F = A_F$ for each $F \in \lambda^{<\omega}$ (it is possible because $\lambda \leq \kappa$). Then $\{B_F : F \in \lambda^{<\omega}\}$ is a strictly directed decreasing family of non-closed subsets of X and $p \in \cap\{\overline{B_F} : F \in \lambda^{<\omega}\} \setminus \cup\{B_F : F \in \lambda^{<\omega}\}$. Since X is strictly λ-Fréchet, there exists a point $x_F \in B_F$ for each $F \in \lambda^{<\omega}$ such that the λ-net $\langle x_F \rangle$ converges to p.

Now, we shall construct a κ-net $\langle y_F : F \in \kappa^{<\omega} \rangle$ converging to p. Let $N = \{x_F : F \in \lambda^{<\omega}\}$ and let $I_n = \{F \in \lambda^{<\omega} : x_F \in N, |F| = n\}$ for each $n \in \mathbb{N}$. We firstly choose a number $n_0 = \min\{n : I_n \neq \emptyset\}$ and fix a point $x_{F_0} \in \{x_F : F \in I_{n_0}\}$. Then we define $y_F = x_{F_0}$ for each $F \in \kappa^{<\omega}$ such that $|F| \leq |F_0|$. We choose again a number $n_1 = \min\{n : I_n \neq \emptyset, n > n_0\}$ and fix a point $x_{F_1} \in \{x_F : F \in I_{n_1}\}$. Then we define $y_F = x_{F_1}$ for each $F \in \kappa^{<\omega}$ such that $|F_0| < |F| \leq |F_1|$.

Inductively, we can choose two sets $\{n_k : k \in \mathbb{N}\}$, $\{x_{F_k} : k \in \mathbb{N}\}$ and a κ-net $\langle y_F : F \in \kappa^{<\omega} \rangle$ satisfying

- $n_k = \min\{n : I_n \neq \emptyset, n > n_{k-1}\}$;
- $x_{F_k} \in \{x_F : F \in I_{n_k}\}$;
- $y_F = x_{F_k}$ for each $F \in \kappa^{<\omega}$ with $|F_{k-1}| < |F| \leq |F_k|$.

Then for each $F \in \kappa^{<\omega}$, there exists a number $k \in \mathbb{N}$ such that $y_F = x_{F_k}$ and $|F| \leq |F_k|$. Since $x_{F_k} \in B_{F_k}$ and $|F| \leq |F_k|$, we have $y_F \in A_F$. It is obvious, by construction, that the κ-net $\langle y_F \rangle$ converges to p. Therefore X is strictly κ-Fréchet. □

We have known from Theorem 3.3 that every strongly κ-Fréchet space is strongly κ-sequential. However, we do not know yet the other implications as in Proposition 2.1. We pose two questions as below.

Question 3.12. Are the following true?

(1) For all κ, $\chi(X) \leq \kappa \Rightarrow X$ is strongly κ-Fréchet(or even strictly κ-Fréchet);
(2) X is strongly κ-sequential $\Rightarrow t(X) \leq \kappa$;
(3) X is strongly Fréchet $\Leftrightarrow X$ is strongly ω-Fréchet;
(4) X is sequential $\Leftrightarrow X$ is strongly ω-sequential.

Question 3.13. Is every κ-radial space strongly κ-Fréchet (or even strictly κ-Fréchet)?

It seems that there is no implication between radial and strongly κ-sequential.

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