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THE GREATEST EXPANDED NUMBER EXPANDED BY SUMMING OF POWERS OF ITS DIGITS

Kyung Ho Jeong and Ihn Sue Kim*

Abstract. In this paper, we proved some properties of the greatest expanded numbers, and give the method to determine the greatest expanded numbers and find the integer x for which $S_{q,p}(x)-x$ is the largest. Additionally, we provide an algorithm to find the greatest expanded number.

1. Introduction

For two positive integers q, $p(\geq 2)$, let $S_{q,p} : Z^+ \to Z^+$ be the map defined by, to each positive integer $x = \sum_{i=0}^n a_i p^i$, assigning the sum of the qth powers of its p-base digits. In other words, for $x = \sum_{i=0}^n a_i p^i, 0 \leq a_i \leq p-1$,

$$S_{q,p}(x) = S_{q,p}\left(\sum_{i=0}^n a_i p^i\right) = \sum_{i=0}^n a_i^q.$$

On the sum of powers of digits of an integer, Singh identified fixed points and periodic orbits in the dynamical system defined by summing the rth powers of the digits of a positive integer repeatedly [2].

Grundman and Teeple presented a method for determining the fixed points and cycles for $S_{q,p}(x)$ and apply it to $S_{5,p}(x)$ with $2 \leq p \leq 10$ [1].

Grundman and Teeple also proved in [1] that

$$\text{if } x \geq p^{q+1}, \text{ then } S_{q,p}(x) < x. \tag{1.1}$$

Thus there can exist the integers with $x \leq S_{q,p}(x)$ for $x < p^{q+1}$. In fact, such integers exist and are much smaller than p^{q+1}.

In this paper, we proved some properties of such numbers and give the method to determine the greatest such numbers and to find the

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*Corresponding author

integer x for which $S_{q,p}(x) - x$ is the largest. Additionally, we provide an algorithm to find the greatest expanded number.

2. Analysis of the difference $S_{q,p}(x) - x$

Definition 2.1. 1. A positive integer $x = \sum_{i=0}^{n} a_i p^i$, $0 \le a_i \le p-1$, is called *an expanded number in qth powers in p-base* if $x \le S_{q,p}(x)$, where $S_{q,p}(x)$ is the sum of the qth powers of its p-base digits.

2. *The greatest expanded number in qth powers in p-base* is the largest number among the all of expanded numbers in qth powers in p-base, and denoted by $M(S_{q,p})$.
3. A positive integer $x = \sum_{i=0}^{n} a_i p^i$, $0 \le a_i \le p-1$, is called *a circulated number in qth powers in p-base* if $x = S_{q,p}(x)$.

Example 2.1. If $p = 10$, $q = 2$, the expanded numbers in 2th powers(squares) in 10-base are $1, 2, 3, \ldots, 99$ of all 51 integers, and 1 is the only circulated number in 2th powers(squares) in 10-base.

Example 2.2. For all natural numbers $p(\ge 2)$, q, 1 is a circulated number in qth powers in p-base.

From (1.1), we know that if $x \le S_{q,p}(x)$ then $x < p^{q+1}$[1]. But, if $x = p^{q+1} - 1$, then

$$\begin{aligned} x &= ((p-1)+1)^{q+1} - 1 \\ &> (p-1)^{q+1} + (q+1)(p+1)^q - 1 \\ &\ge (q+1)(p+1)^q = S_{q,p}(x). \end{aligned}$$

Thus, in any case, $p^{q+1} - 1$ can't be the expanded number. Therefore, we need to analyze the characteristics of the expanded number.

To do this we observe about $S_{q,p}(x) - x$. Since

$$S_{q,p}(x) - x = \sum_{i=0}^{q} a_i^q - \sum_{i=0}^{q} a_i p^i = \sum_{i=0}^{q} (a_i^q - a_i p^i),$$

for nonnegative integer i, let $h_i(t) = t^q - p^i t (0 \le t \le p-1)$. This function h_i gives an important clue to find the greatest expanded number.

In the next lemma we can obtain the required tool to analyze $S_{q,p}(x) - x$.

Lemma 2.1. *When p, q are the integers with $p \ge 3$, $q \ge 2$, $p \ge q$, about the integer $i(\le q)$, let $h_i(t) = t^q - p^i t (0 \le t \le p-1)$. Then*

1. *if $\kappa = [(q-1)\log_p(p-1)] + 1$, the maximum of $h_i(t)$ is*

   (a) *when* $i < \kappa$, $h_i(p-1)$.
   (b) *when* $i \geq \kappa$, 0.
2. $i > j \Rightarrow h_i(p-1) < h_j(p-1)$.
3. *if* $\lambda = \log_p \dfrac{(p-1)^q - 1}{p-2}$,
   (a) *when* $i < \lambda$, $h_i(p-1) > h_i(1)$.
   (b) *when* $i \geq \lambda$, $h_i(p-1) \leq h_i(1)$.
4. *for* i, $\kappa \leq i < n$, *if the real root less than* $p-1$ *of* $h_i(t) = (p-1)^q - (p-1)p^i$ *is* α_i,
   (a) *if* $m_0 = \sqrt[q-1]{\dfrac{p^i}{q}}$, $m_{n+1} = \dfrac{h_i(p-1)\cdot m_n}{h_i(m_n)} = \dfrac{(p-1)^q - (p-1)p^i}{p^i - m_n^{q-1}}$, *for* n,

$$\alpha_i < m_{n+1} < m_n \text{ and } \lim_{n\to\infty} m_n = \alpha_i.$$

   (b) *if* $f(t) = h_i(t) - h_i(p-1)$, $r_0 = 0$, $r - n + 1 = r_n - \dfrac{f(r_n)}{f'(r_n)}$, $[r_n] = [m_n]$, *then* $[\alpha_i] = [m_n]$.

Proof. 1. $h_i'(t) = 0 \Rightarrow t = \sqrt[q-1]{\dfrac{p^i}{q}}$.

When $i \leq q-1$,

$$\sqrt[q-1]{\frac{p^i}{q}} \leq \sqrt[q-1]{\frac{p^{q-1}}{q}} = p\sqrt[q-1]{\frac{1}{q}} < p,$$

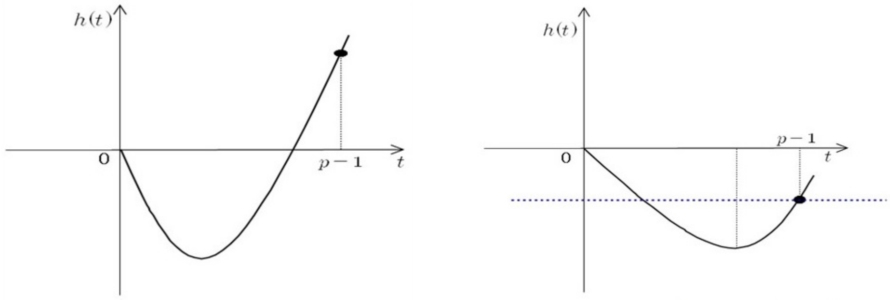
so $h_i(t)$ has a minimum less than one in case of $0 \leq t < p$, and doesn't have a maximum. Therefore the global maximum of $h_i(t)$ is $\max\{h_i(0) = 0, h_i(p-1)\}$. (In case of $t > 0$, $h_i(t)$ has one extremum which is a minimum.)

On the other hand,

$$\begin{aligned} h_i(p-1) < h_i(0) &\Leftrightarrow (p-1)^q - p^i(p-1)) < 0 \\ &\Leftrightarrow p^i > (p-1)^{q-1} \\ &\Leftrightarrow i > (q-1)\log_p(p-1). \end{aligned}$$

Therefore, if $\kappa = [(q-1)\log_p(p-1)] + 1$, the maximum of $h_i(t)$ is when $i < \kappa$, $h_i(p-1)$, when $i \geq \kappa$, 0.

i) $i < \kappa,\ h_i(p-1) \geq 0 = h_i(0)$ ii) $i \geq \kappa,\ h_i(p-1) < 0 = h_i(0)$



Case1 : $i < \kappa,\ h_i(p-1)$ Case2 : $i \geq \kappa,\ h_i(p-1)$

(The graph of $y = h_i(t)$ by the size of i)

2. If $i = j + a$(a is a natural number),

$$\begin{aligned} h_i(p-1) &= (p-1)^q - p^{j+a}(p-1)) \\ &= (p-1)^q - p^j(p-1) - (p^{j+a} - p^j)(p-1) \\ &= h_j(p-1) - (p^{j+a} - p^j)(p-1) \\ &< h_j(p-1). \end{aligned}$$

3.

$$\begin{aligned} h_i(p-1) \leq h_i(1) &\Leftrightarrow (p-1)^q - p^i(p-1)) \leq 1 - p^i \\ &\Leftrightarrow p^i(p-2) \geq (p-1)^q - 1 \\ &\Leftrightarrow i \geq \log_p \frac{(p-1)^q - 1}{p-2} = \lambda. \end{aligned}$$

4. If $m = \sqrt[q-1]{\dfrac{p^i}{q}}$, $h_i(t) = t^q - p^i t(0 \leq t \leq p-1)$ is a minimum at $t = m$, curved down. And if $i \geq \kappa$, $h_i(p-1) < 0$.

In this case, the line connecting the origin and the point $(m, h_i(m))$ meet $y = h_i(p-1)$ and x coordinate of intersection is m_1,

$$m_1 : m = (-h_i(p-1)) : (-h_i(m)).$$

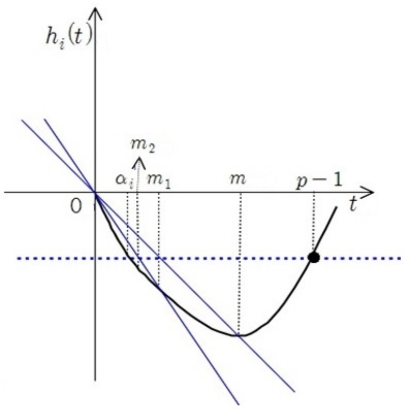
Therefore,

$$m_1 = \frac{-h_i(p-1)\cdot m}{-h_i(m)}$$
$$= \frac{(p-1)p^i-(p-1)^q}{p^i-m^{q-1}}$$
$$= \frac{q(p-1)}{(q-1)}\left(1-\frac{(p-1)^{q-1}}{p^i}\right).$$

In the same way, for a natural number n, if

$$m_{n+1} = \frac{(p-1)p^i-(p-1)^q}{m_n^{q-1}-p^i},$$

$\alpha_i < m_{n+1} < m_n$ and $\lim_{n\to\infty} m_n = \alpha_i$.



Finding approximate value of α_i

On the other hand, if
$f(t) = h_i(t) - h_i(p-1) = t^q - p^i t + (p-1)p^i - (p-1)^q$,
$f'(t) = qt^{q-1} - p^i$.

Let $r_0 = 0$, by applying the method of Newton,

$$r_1 = 0 - \frac{-(p-1)^q+(p-1)p^i}{-p^i} = \frac{-(p-1)^q+(p-1)p^i}{p^i}$$
$$= p-1-\frac{(p-1)^q}{p^i}.$$
$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}$$
$$= r_n - \frac{r_n^q - r_n p^i + (p-1)^i - (p-1)^q}{qr_n^{q-1}-p^i}.$$

At this moment, $\alpha_i > r_{n+1} > r_n$, and $\lim_{n\to\infty} r_n = \alpha_i$. Therefore, by repeating the above two processes, if $[r_n] = [m_n]$, $[\alpha_i] = [m_n]$. □

3. Properties of the greatest expanded number

Now using Lemma 2.1, let us find the useful way to get the greatest expanded number. The next theorem is useful in case of $p \geq q$.

Lemma 3.1. *If $x = \sum_{i=0}^{n} a_i p^i (a \neq 0,\ p > 2,\ q \geq 2$ are natural numbers, $0 \leq a_i \leq p-1$), then*

$$x \leq S_{q,p}(x) \Rightarrow x < (q-1)p^q.$$

Proof. 1. In case of $q > p$, since $(q-1)p^q \geq p \cdot p^q = p^{q+1}$, by (1.1), $x < p^{q+1} \leq (q-1)p^q$.

2. Now let us look into the case of $q \leq p$. By (1.1), it is enough to consider $n \leq q$, let $x = a_q p^q + a_{q-1}p^{q-1} + \cdots + a_1 p + a_0 = \sum_{i=1}^{q} a_i p^i$. Then

$$S_{q,p}(x) - x = \sum_{i=0}^{q} a_i^q - \sum_{i=0}^{q} a_i p^i = \sum_{i=0}^{q} (a_i^q - a_i p^i),$$

$$S_{q,p}(x) - x - (a_q^q - a_q p^q) = \sum_{i=0}^{q-1} (a_i^q - a_i p^i).$$

If $h_i(t) = t^q - p^i t (0 \leq t \leq p-1)$, κ, in Lemma 2.1, is $\kappa = [(q-1)\log_p(p-1)] + q < (q-1) + 1 = q$.

Thus, by Lemma 2.1, the maximum of $S_{q,p}(x) - x - (a_q^q - a_q p^i) = \sum_{i=0}^{q-1}(a_i^q - a_i p^i)$ is

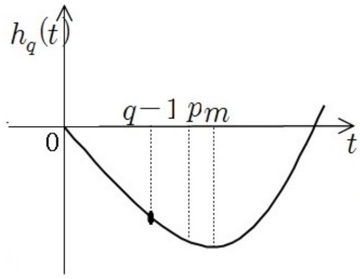
$$\begin{aligned}\sum_{i=0}^{\kappa-1} h_i(p-1) + \sum_{i=0}^{q-1} h_i(0) &= \sum_{i=0}^{\kappa-1}((p-1)^q - (p-1)p^i) \\ &= \kappa(p-1)^q - p^\kappa + 1.\end{aligned}$$

Therefore, when $a_q = q-1$, the maximum of $S_{q,p}(x) - x$ is
$(q-1)^q - (q-1)p^q + \kappa(p-1)^q - p^\kappa + 1$
$< (q-1)^q - (q-1)p^q + (q-1)(p-1)^q - p^\kappa + 1$
$< (q-1)^q - (q-1)\{(p-1)^q + q(p-1)^{q-1}\} + (q-1)(p-1)^q - p^\kappa + 1$
$= (q-1)^q - q(q-1)(p-1)^{q-1} - p^\kappa + 1 < 0$
($\because 2 \leq q \leq p$, so $(q-1)^{q-1} < q(p-1)^{q-1}$).

Also, if $h_q(t) = t^q - p^q t$ has a minimum at $t = m$,

$$m = \sqrt[q-1]{\frac{p^q}{q}} \geq \sqrt[q-1]{\frac{p^q}{p}} = p,$$

so $h_q(t) = t^q - p^q t$ is decreased at $0 \leq t \leq p-1$. Thus, when $a_q \geq q-1$, $S_{q,p}(x) - x < 0$.
Therefore,
$S_{q,p}(x) \geq x \Rightarrow x < (q-1)p^q$.



$h_q(t) = t^q - p^q t.$

□

Example 3.1. If $S_{2,10}(x) \geq x$, $x < (2-1) \cdot 10^2 = 100$.
If $S_{3,10}(x) \geq x$, $x < (3-1) \cdot 10^3 = 2000$.

Actually, $M(S_{2,10}) = 99$, $M(S_{3,10}) = 1999$. Thus, Lemma 3.1 gives an convenient means to find the greatest expanded number in the case of $p \geq q$. Now let us investigate when $(q-1)p^q - 1$ becomes $M(S_{q,p})$.

Corollary 3.2. *For integers p, q such as $p \geq 3$, $q \geq 2$, $p \geq q$, when $h_(p,q) = (q-2)^q - (q-1)p^q + q(p-1)^q + 1$,*

$$h(p,q) \geq 0 \Leftrightarrow (q-1)p^q - 1 = M(S_{q,p}).$$

Proof. In case of $p \geq q$, if $q \geq 2$, since

$$(p-1)p^q - 1 = (q-2)p^q + \sum_{i=0}^{q-1}(p-1)p^i,$$

$$\begin{aligned} S_{q,p}(x) - x &= (q-2)^q - (q-2)p^q + \sum_{i=0}^{q-1}((p-1)^q - (p-1)p^i) \\ &= (q-2)^q - (q-2)p^q + q(p-1)^q - (p^q - 1) \\ &= (q-2)^q - (q-1)p^q + q(p-1)^q + 1 \\ &= h(p,q). \end{aligned}$$

Thus, a necessary and sufficient condition for $x = (q-1)p^q - 1$ to be the greatest expanded number is $h(p,q) \geq 0$. □

Example 3.2. Since

$$\begin{aligned} h(5,2) &= (q-2)^q - (q-1)p^q + q(p-1)^q + 1 \\ &= (2-2)^2 - (2-1) \cdot 5^2 + 2 \cdot (5-1)^2 + 1 \\ &= 8 > 0, \end{aligned}$$

the sum of the squares of digits in 5 base is

$$(q-1)p^q - 1 = (2-1)\cdot 5^2 - 1 = 24 = 44_{(5)}.$$

When q is a constant, $h(p,q)$ is a monic polynomial of p. For given q, if p is big enough, $(q-1)p^q - 1$ is the greatest expanded number.

Here, let us investigate, in case of q is small, according as how big p is, $(q-1)p^q - 1$ comes to be the greatest expanded number.

Corollary 3.3. 1. *If $p \geq 3$, $M(S_{2,p})$ is $p^2 - 1$.*
2. *If $p \geq 8$, $M(S_{3,p})$ is $2p^3 - 1$.*
3. *If $p \geq 15$, $M(S_{4,p})$ is $3p^4 - 1$.*

Proof. 1. $h(p,2) = -p^2 + 2(p-1)^2 + 1 = p^2 - 4p + 3 = (p-2)^2 - 1$.
If $p \geq 3$, $h(p,2) \geq 0$. By Corollary 3.2, $p^2 - 1$ is $M(S_{2,p})$.
2.

$$\begin{aligned} h(p,3) &= (q-2)^q - (q-1)p^q + q(p-1)^q + 1 \\ &= 1 - 2p^3 + 3(p-1)^3 + 1 \\ &= p^3 - 9p^2 + 9p - 1 \\ &= (p-1)((p-4)^2 - 15). \end{aligned}$$

Thus, if $p \geq 8$, then $h(p,3) > 0$.
3. $h(p,4) = 2^2 - 3p^4 + 4(p-1)^4 + 1 = p^4 - 16p^3 + 24p^2 - 16p + 9$,

$$\frac{d}{dp}h(p,4) = 4p^3 - 16\cdot 3p^2 + 24\cdot 2p - 16 = s(p^3 - 12p^2 + 12p - 4).$$

If $p \geq 15$, $\frac{d}{dp}h(p,4) > 0$, and $h(15,4) = 1794 > 0$.
Therefore, if $p \geq 15$, $h(p,4) > 0$.

□

When the greatest expanded number is expressed in their base, at the end part of digits of the greatest expanded number, the same digits, $p-1$, are repeated. For example, $M(S_{2,10})$ is 99, so 2 times 9, $M(S_{3,10})$ is 1999, so 3 times 9, $M(S_{4,5})$ is $10444_{(5)}$, so 3 times 4. This means the greatest expanded number can be denoted like $a\cdot p^b - 1$(a, b are natural numbers.).

Therefore how many times $(p-1)$ are repeated there like this? In other words, when 'the greatest expanded number($M(S_{q,p})$) is denoted like $a\cdot p^b - 1$(a, b are natural numbers.), let us think about the maximum of b (in other words, when a is not multiple of p, the value of b).

Theorem 3.4. *When the greatest expanded number ($M(S_{q,p})$) is denoted in base p, the number of $(p-1)$ at the end part satisfies the next inequality.*

$$\kappa \leq \textit{(the number of } p-1) \leq q,$$

where $\kappa = [(q-1)\log_p(p-1)] + 1$.

Proof. In the proof of Lemma 2.1, if $i < \kappa$, for two natural numbers $x = x_1 p^{i+1} + x_0$, $y = x_1 p^{i+1} + p^{i+1} - 1$, where $x_0 = a_i p^i + a_{i-1} p^{i-1} + \cdots + a_1 p + a_0 < p^{i+1} - 1$, $0 \leq a_m < p$, $m = 0, 1, \ldots, i-1$, we have

$$S_{q,p}(x) - x \geq 0 \Rightarrow S_{q,p}(y) - y \geq 0.$$

$$\begin{aligned} \because 0 \leq S_{q,p}(x) - x &= S_{q,p}(x_1) - x_1 p^{i+1} + \sum_{n=0}^{i} (a_n^q - a_n p^n) \\ &\leq S_{q,p}(x_1) - x_1 p^{i+1} + \sum_{n=0}^{i} ((p-1)^q - (p-1)p^n) \\ &= S_{q,p}(y) - y. \end{aligned}$$

Here, if $x < y$, x can't be $M(S_{p,q})$. In other words, $M(S_{p,q})$ should be the form like y. At this time, the maximum of i is $\kappa - 1$. Therefore, when $M(S_{p,q})$ is denoted in p base, the digits from the position of $1 (= p^0)$ to the position of $p^{\kappa-1}$ are all $p-1$.

Therefore, if $\kappa = [(q-1)\log_p(p-1)] + 1$,

$$\kappa \leq (\text{the number of } p-1) \leq q.$$

□

In case p is not too much less compared q, $\kappa = q-1$.(Table 1.)

Now, let us look into about the digits whose position is more than p^κ of the greatest expanded number.

Theorem 3.5. *For the integers $p \geq 3$, $q \geq 2$, when $\kappa = [(q-1)\log_p(p-1)] + 1$, $x = \sum_{i=0}^{n} a_i p^i (a_n \neq 0,\ 0 \leq a_i \leq p-1)$ and an integer j, $\kappa \leq j \leq n$,*

1. *if x is the expanded number, $y = \sum_{i=j}^{n} a_i p^i + p^\kappa - 1$ is the expanded number.*
2. *if x is $M(S_{q,p})$, a_j is the maximum of b to $\sum_{i=j+1}^{n} a_i p^i + b p^j + p^\kappa - 1$ be the expanded number ($0 \leq b \leq p-1$, if $j = n$, $\sum_{i=j+1}^{n} a_i p^i = \sum_{i=n+1}^{n} a_i p^i = 0$).*

Proof. 1. By Lemma 2.1.(1), if $h_i(t) = t^q - p^i t$, $0 \le t \le p-1$. When $i \ge \kappa$, the maximum of $h_i(t)$ is 0. Therefore, if $i \ge \kappa$,

$$h_i(a_i) = a_i^q - a_i p^i \le 0.$$

Thus, $S_{q,p}(y) - y \ge S_{q,p}(x) - x > 0$.

2. If the maximum of b to make $\sum_{i=j+1}^{n} a_i p^i + bp^j + p^\kappa - 1$ the expanded number is m, when $z = \sum_{i=j+1}^{n} a_i p^i + mp^j + p^\kappa - 1$, x is the greatest expanded number, so $x \ge z$. Therefore $a_j \ge m$.

   By (1), $\sum_{i=j}^{n} a_i p^i + p^\kappa - 1 = \sum_{i=j+1}^{n} a_i p^i + a_j p^j + p^\kappa - 1$ is the expanded number, $a_j \le m$, therefore, $a_j = m$.

□

Theorem 3.5 tells that each digits of the greatest expanded number could be determined one by one.

Now, for the given numbers p, q, and the sum of q-powered in p-base, let us find what number becomes the biggest. In other words, let us find what x is the biggest in $S_{q,p}(x) - x$. This also gives us the information concerning the greatest expanded number. We can find the answer about this in the next theorem.

Theorem 3.6. *When* $\kappa = [(q-1)\log_p(p-1)]+1$, $\lambda = \log_p \dfrac{(p-1)^q - 1}{p-2}$, $p \ge 3$,

1. *a natural number that* $S_{q,p}(x) - x$ *is the maximum is* $p^\kappa - 1$.
2. *when it is denoted in base p, among the natural number* x *of which* $(n+1)$ *is the position number, a natural number that is* $S_{q,p}(x) - x$ *is the maximum*
   (a) *if* $n < \kappa$, $p^n - 1$,
   (b) *if* $\kappa \le n < \lambda$, $(p-1)p^n + p^\kappa - 1$,
   (c) *if* $n = \lambda$*(in case of* λ *is an integer),* $(p-1)p^n + p^\kappa - 1$ *and* $p^n + p^\kappa - 1$,
   (d) *if* $n > \lambda$, $p^n + p^\kappa - 1$.

Proof. 1. By Lemma 2.1, when $h_i(t) = t^q - p^i t$ $(0 \le t \le p-1)$, if $i \ge \kappa$, about t as $1 \le t \le q-1$, $h_i(t) < h_i(0) = 0$, if $i < \kappa$, the maximum of $h_i(t)$ is $h_i(p-1) \ge 0$. Therefore, when $x = \sum_{i=0}^{n} a_i p^i$($a_i$ is nonnegative integer that is less than p), the cases $S_{q,p}(x) - x = \sum_{i=0}^{n}(a_i^q - a_i p^i)$ becomes the maximum are 'for i as $i \ge \kappa$, $a_i = 0$, and for i as $i < \kappa$, $a_i = p-1$'. Therefore, the natural number that $S_{q,p}(x) - x$ is the maximum is $p^\kappa - 1$.

2. In the proof of Lemma 2.1, when $h_i(t) = t^q - p^i t$ $(0 \le t \le p-1)$, if $n < \kappa$, $h_i(p-1) > h_i(0) > h_i(1)$,

if $\kappa \le n \le \lambda$, $h_i(0) > h_i(p-1) > h_i(1)$ (but, in case of $n = \lambda$, $h_i(0) > h_i(p-1) = h_i(1)$),
if $n \ge \lambda$, $h_i(0) > h_i(1) \ge h_i(p-1)$.
From these we get the result.

When $x = \sum_{i=0}^{n} a_i p^i$ (a_i is nonnegative integer that is less than p), if x is the natural number of $(n+1)$ position, it should be $a_n \ge 1$. Therefore, the natural number that $S_{q,p}(x) - x$ is the maximum is like the result above.

□

Lemma 3.1 is very meaningful in case that p is much bigger than given q. Therefore, we need to study the meaningful content to apply this case. About this, let us look into the next theorem.

Lemma 3.7. *About the integers p, q as $p \ge 3$, $q \ge 2$, when $\kappa = [(q-1)\log_p(p-1)] + 1$, $\alpha = \log_p(\kappa(p-1)^q - p^\kappa + 2)$, $\eta = [\alpha]$, if $x = \sum_{i=0}^{n} a_i p^i$ ($a_n \ne 0$, $0 \le a_i \le p-1$) is $M(S_{q,p})$, $n = \eta$.*

Proof. (i)$\alpha \ge \lambda$.
If $p^\kappa \le p(p-1)^{q-1}$ in $\kappa = [(q-1)\log_p(p-1)] + 1$,
$\lambda = \log_p \dfrac{(p-1)^q - 1}{p-2}$, $\alpha = \log_p(\kappa(p - 10^q - p^\kappa + 2)$,

$$\begin{aligned}\frac{p^\alpha - p^\lambda}{p-2} &= (p-2)(\kappa(p-1)^q - p^\kappa + 2) - ((p-1)^q - 1)\\ &\ge (p-2)(\kappa(p-1)^q - p(p-1)^{q-1} + 2) - ((p-1)^q - 1)\\ &= ((p-2)((\kappa-1)(p-1) - 1) - (p-1))(p-1)^{q-1} + 2p - 3\\ &= ((p-2)(p-1)(\kappa-1) - 1)(p-1)^{q-1} + 2p - 3,\end{aligned}$$

if $q \ge 3$, in $\kappa = [2\log_p(p-1)] + 1$ when $p \ge 3$, $\kappa \ge 2$.
Therefore, when $p \ge 3$, $q \ge 3$, $\dfrac{p^\alpha - p^\lambda}{p-2} \ge 0$. Also, if $q = 2$, since $\kappa = 1$, $\dfrac{p^\alpha - p^\lambda}{p-2} \ge -(p-1) + 2(p-2) + 1 = p - 2 \ge 0$. Here, when $p \ge 3$, $q \ge 2$, $\dfrac{p^\alpha - p^\lambda}{p-2} \ge 0$. In other words, $\alpha \ge \lambda$. .... ①
(ii) If there is an integer j as $\lambda \le j \le$, $n = [\alpha] = \eta$.
By Theorem 3.5, if $n \ge \lambda$, $p^n + p^\kappa - 1$ becomes the (n+1) position natural number that $S_{q,p}(y) - y$ is the maximum. In other words,

$$0 \le S_{q,p}(x) - x \le S_{q,p}(y) - y.$$

In

$$y = p^n + (p-1)p^{\kappa-1} + \cdots + (p-1)p + (p-1)$$
$$= p^n + \sum_{i=0}^{\kappa-1}(p-1)p^i,$$

if $S_{q,p}(y) - y = 1 - p^n + \kappa(p-1)^q - p^\kappa + 1$,
$S_{q,p}(y) - y \geq 0$, $p^n \leq \kappa(p-1)^q - p^\kappa + 2$.
In other words, $n \leq \log_p(\kappa(p-1)^q - p^\kappa + 2) = \alpha$. Therefore, if there is an integer j as $\lambda \leq j \leq \alpha$, about such j, when

$$y = p^j + (p-1)p^{\kappa-1} + \cdots + (p-1)p + (p-1)$$
$$= p^j + \sum_{i=0}^{\kappa-1}(p-1)p^i,$$
$$S_{q,p}(y) - y = \kappa(p-1)^q - p^\kappa + 2 - p^j \geq 0.$$

At this time, if $x = \sum_{i=0}^{n} a_i p^i$ ($a_n \neq 0$, $0 \leq a_i \leq p-1$) is the greatest expanded number ($M(S_{q,p})$), n is the maximum of such j.
Therefore, if there is an integer j as $\lambda \leq j \leq \alpha$, $n = [\alpha] = \eta$. .... ②
(iii) And, if there isn't an integer j as $\lambda \leq j \leq \alpha$, when $n > \lambda$, about $x = \sum_{i=0}^{n} a_i p^i$ ($a_n \neq 0$, $0 \leq a_i \leq p-1$), $S_{q,p}(x) - x < 0$.
Therefore, if there is not an integer j as $\lambda \leq j \leq \alpha$, to become $S_{q,p}(x) - x \geq 0$ it should be $n \leq \lambda \leq \alpha$. .... ③
By ①, ②, ③, $n \leq \eta = [\alpha]$. .... ④
(iv) $\alpha \geq \kappa$.

$$p^\alpha - p^\kappa = (\kappa(p-1)^q - p^\kappa + 2) - p^\kappa = \kappa(p-1)^q - 2p^\kappa + 2$$
$$\geq \kappa(p-1)^q - 2p(p-1)^{q-1} + 2$$
$$= ((\kappa-2)(p-1) - 2)(p-1)^{q-1} + 2.$$

Since $p \geq 3$, if $\kappa \geq 3$, $p^\alpha - p^\kappa > 0$,
if $a \geq 4$ (except $p = 3$, $q = 4$), since $\kappa \geq 3$, $p^\alpha - p^\kappa > 0$.
if $p = 3$, $q = 4$, since $\kappa = 2$, $p^\alpha - p^\kappa = 2\cdot(3-1)^4 - 2\cdot 3^2 + 2 = 32 - 18 + 2 > 0$.
if $q = 3$, $\kappa = 2$,

$$p^\alpha - p^\kappa = 2(p-1)^3 - 2p^2 + 2 = 2(p-1)((p-1)^2 - (p+1))$$
$$= 2(p-1)(p^2 - 3p) \geq 0.$$

if $q = 2$, since $\kappa = 1$,
$p^\alpha - p^\kappa = (p-1)^2 - 2p + 2 = p^2 - 4p + 3 = (p-1)(p-3) \geq 0$.
From the above, since $p^\alpha - p^\kappa \geq 0$, $\alpha \geq \kappa$. .... ⑤

(v) By ⑤, since $\eta \geq \kappa$, let us look into

$$x = p^{\eta} + p^{\kappa} - 1 = p^{\eta} + \sum_{i=0}^{\kappa-1}(p-1)p^{i}.$$

Since $p^{\eta} = p^{[\alpha]} \leq p^{\alpha} = \kappa(p-1)^{q} - p^{\kappa} + 2 - p^{\eta} \geq 0$.
Here, $n \geq \eta$ .... ⑥
By ④and ⑥, $n = \eta$. □

The above lemma decides which digit is the maximum of $x \leq S_{q,p}(x)$. Compared with the given q, when p is big enough, $q = \eta$, in that case it is more meaningful to be explained by Lemma 3.1. If not, this theorem is more meaningful.

And, in the proof of Lemma 3.7, in case $\alpha = \log_p(\kappa(p-1)^q - p^\kappa + 2)$ becomes $n = \alpha$, in (ii) since $S_{q,p}(y) - y = 0$, $(n+1)$digit which is not x is the circulated number and the greatest expanded number. When (q,p) is $(3,2)$, $(3,3)$, $(6,3)$, it comes to be.

In the above proof of Lemma 3.7, instead of (iv), it can be explained more simply 'though there is the case of $\eta = \kappa - 1$, when $x = p - 1$ through $S_{q,p}(x) - x > 0$, it should be $n \geq \eta$'. However, '$\alpha \geq \kappa$' is also valuable to know, so it is proved like the above.

Here let us denote the range of η with p, κ.
Since ${}^{\kappa-1} \leq (p-1)^{q-1} < p^{\kappa}$,
$\kappa(p-1)p^{\kappa-1} - p^{\kappa} + 2 \leq \kappa(p-1)^{q} - p^{\kappa} + 2 < \kappa(p-1)p^{\kappa} - p^{\kappa} + 2,$
$\kappa p^{\kappa} - \kappa p^{\kappa-1} - p^{\kappa} + 2 \leq \kappa(p-1)^{q} - p^{\kappa} + 2 < \kappa p^{\kappa+1}(\kappa+1)p^{\kappa} + 2,$
$p^{\kappa-1}(-\kappa - p) < \kappa(p-1)^{q} - p\kappa + 2 < \kappa p^{\kappa+1},$
$\log_p(\kappa p - \kappa - p) + \kappa - 1 < \log_p \kappa + \kappa + 1,$
$[\log_p(\kappa p - \kappa - p)] + \kappa - 1 \leq \eta \leq [\log_p \kappa] + \kappa + 1.$

In other words, the digit of the greatest expanded number is less than or equal to $[\log_p \kappa] + \kappa + 2$, $\eta - \kappa$ is less than or equal to $[\log_p \kappa] + 1$.

Example 3.3. In case of $p = 3$, $q = 10$,
$\kappa = [(q-1)\log_p(p-1)] + 1 = [9\log_3 2] + 1 = [5.678] + 1 = 6,$
$\eta = [\log_q(\kappa(p-1)^q - p^\kappa + 2)] = [7.83] = 7.$
Thus, $M(S_{10,3})$ is $\eta + 1 = 8$ digit number (Actually, $M(S_{10,3})$ is $3^8 - 1 = 22222222_{(3)}$).

Similarly to Corollary 3.2, let us find in what case $p^{\eta+1} - 1$ becomes the greatest expanded number.

Since $S_{q,p}(p^{\eta+1}-1)=(\eta+1)(p-1)^q$,

$$S_{q,p}(p^{\eta+1}-1)-(p^{\eta+1}-1)\geq 0 \Leftrightarrow (\eta+1)(p-1)^q-(p^{\eta+1}-1)\geq 0$$
$$\Leftrightarrow p^{\eta+1}\geq(\eta+1)(p-1)^q+1.$$

However, the case that the above inequality is satisfied is not common, so it is useful only in some cases.

And, when in the above the value of κ and the greatest expanded number are denoted in base p, the number of $(p-1)$ and the $(\eta+1)$ digit number of the greatest expanded number at the end are shown like the next.

q / part / p	2			3			4			5			6			7			8			9			10		
	κ	amount	digit	κ	amount	digit	κ	amount	digit	κ	amount	digit	κ	amount	digit	κ	amount	digit	κ	amount	digit	κ	amount	digit	κ	amount	digit
3	1	2	2	2	2	3	2	3	3	3	4	4	4	5	5	4	6	6	5	6	7	6	6	8	6	8	8
4	1	2	2	2	3	3	3	4	4	4	5	5	4	6	6	5	6	7	6	6	8	7	8	9	8	9	10
5	1	2	2	2	3	3	3	3	5	4	4	6	5	5	7	6	6	8	7	8	8	7	9	9	8	10	10
6	1	2	2	2	2	4	3	3	5	4	5	6	5	6	7	6	6	8	7	7	9	8	8	10	9	9	11
7	1	2	2	2	2	4	3	4	5	4	5	6	5	6	7	6	7	8	7	8	9	8	9	10	9	10	11
8	1	2	2	2	3	4	3	4	5	4	5	6	5	5	7	6	6	8	7	7	9	8	8	10	9	9	11
9	1	2	2	2	3	4	3	4	5	4	4	6	5	5	7	6	7	8	7	8	9	8	9	10	9	10	11
10	1	2	2	2	3	4	3	4	5	4	4	6	5	6	7	6	7	8	7	8	9	8	9	10	9	9	11

Table 1. The value of κ according to p, q and the amount and digit of $(p-1)$ at the end of the greatest equally expanded number.

And, when we clarify the above Lemma 2.1, Lemma 3.1, Lemma 3.7 the next theorem is completed.

Theorem 3.8. *About the integer p, q as $p\geq 3$, $q\geq 2$, when $\kappa=[(q-1)\log_p(p-1)]+1$, $\alpha=\log_p(\kappa(p-1)^q-p^\kappa+2)$, $\eta=[\alpha]$, if $x=\sum_{i=0}^n a_ip^i(a_n\neq 0,\ 0\leq a_i\leq p-1)$ is the greatest expanded number, $\kappa\leq n=\eta\leq q$, and the next makes sense.*

1. *About the integer i as $0\leq i<\kappa$, $a_i=p-1$.*
2. *About the integer i as $\kappa\leq i\leq n$, if the real root that is less than $p-1$ of $t^q-p^it=(p-1)^q-(p-1)p^i$ is α_i, $a_i=p-1$ or $a_i\leq\alpha_i$.*
3. *In case of $n=q$, $a_n<q-1$.*

Proof. 1. By Lemma 3.1, Lemma 3.7, $\kappa\leq n=\eta\leq q$.

2. In case of $i\geq\kappa$, $h_i(t)=t^q-p^it(0\leq t\leq p-1)$ is the minimum at $t=[q-1]\dfrac{p^i}{q}$. Thus, if the real root that is less than $p-1$ of $t^q-p^it=(p-1)^q-(p-1)p^i$ is α_i, when $\alpha_i<t<p-1$, since

$h_i(t) = t^q - p^i t < h_i(p-1)$,
$x = A \cdot p^{i+1} + a_i p^i + B$, $y = A \cdot p^{i+1} + (p-1)p^i + B$ (A is the natural number, B is the nonnegative integer less than p^i, when a_i is the integer as $\alpha_i < a_i < p-1$, if y is not the expanded number,

$$S_{q,p}(x) - x = S_{q,p}(y) - y + (a_i^q - p^i a_i) - ((p-1)^q - p^i(p-1)) < 0).$$

In other words, if y is not the expanded number, x is also not the expanded number.

3. By Lemma 3.1, $a_n < q-1$.

□

The above Theorem 3.8 explains if x is the greatest expanded number,

$$p^\eta + \sum_{i=0}^{\kappa-1}(p-1)p^i \le x < \min\{(q-1)p^q, p^{\eta+1}\}.$$

4. Finding algorithm of the greatest expanded number

To find $M(S_{q,p})$, we can check the natural number less than $x = (q-1)p^q - 1$, or $x = p^{\eta+1} - 1$, but in case either p or q is big, actually it is not easy to calculate them, therefore, we need to find the way to find the greatest expanded number using the above characteristics.

First, by Corollary 3.2, in case of $p \ge q$ and $h(p,q) > 0$, $(q-1)p^q - 1$ becomes just the greatest expanded number. And, by Lemma 3.7 the digit of the greatest equally expanded number can be calculated. Of course, when $\eta = q$ and $p > q$ the highest digit value is less than $(q-1)$. Also, by Theorem 3.4, the κ digit at the end can be decided by $(p-1)$. And, by using Theorem 3.5, from the value of p^κ digit to p^η digit, the value of $(\eta - \kappa + q)$ digit only have to be decided in turn.

Finding algorithm of the greatest expanded number

1. Calculate $\kappa = [(q-1)\log_p(p-1)]+1$, $\alpha = \log_p(\kappa(p-1)^q - p^\kappa + 2)$, $\eta = [\alpha]$.
2. By Theorem 3.8 check the natural number x as $x \leq p^{\eta+1} - 1$. In case of $\eta = q$ and $p \geq q$, check the natural number x as $x \leq (q-1)p^q - 1$.
3. If $x = \sum_{i=\kappa}^{\eta} a_i p^i + bp^j + p^\kappa - 1$, and it is decided by the order of a_η, $a_{\eta-1}$, $a_{\eta-2}$, $\ldots a_\kappa$.
   - (a) in case it is decided from a_η to a_{j+1}, the maximum of b to make $\sum_{i=j+1}^{\eta} a_i p^i + bp^j + p^\kappa - 1$ become the expanded number is decided by a_j.
   - (b) We can substitute one by one by one and check from $p-1$ of b, but in case of $(p-1)$ if it does not exist, calculate α_i in Lemma 2.1, and check from $[\alpha_i]$.

The above way is good to apply by using computer, and when $p \geq q$, in case of calculating it directly, like Example 3.2, it is useful to check $h(p,q)$ first.

In case that α_i is not used in the above way, since the value of p^η digit is more than 1, the case that the checking time is the maximum is that the greatest expanded number is $p^\eta + p^\kappa - 1$, and the number of times is $\eta = q$ and when $p \geq q$, the number of times is $(q-2)+p(q-\kappa)$, in the other case the number of times is $(p-1)(\eta-\kappa+1)$. Actually we can find the greatest expanded number by much less checking.

Example 4.1. Find $M(S_{10,20})$.
Here, we may not apply α_i;

First, we can calculate,
$\kappa = [(q-1)\log_p(p-1)] + 1 = [9 \cdot \log_{20} 19] + 1 = 9$.
$\eta = [10.56] = 10$. Therefore, $\eta = q$.
Thus, the greatest expanded number to solve is $x = a_{10} \cdot 20^{10} + a_9 \cdot 20^9 + 20^9 - 1$. Also, $a_{10} \leq q-2 = 8$. Now, when $s_1 = b_1^{10} - 20^{10} b_1 + 9 \times 19^{10} - (20^9 - 1)$ is denoted, substitute 8, 7, $\ldots$ to b_1 in turns and check whether '$s_1 > 0$' exists or not. Then, when $b_1 = 5$, it exists first. Therefore, $a_{10} = 5$.

When $s_2 = b_2^9 - 20^9 b_2 + 5^{10} - 5 \times 20^{10} + 9 \times 19^{10} - (20^9 - 1)$ is denoted, substitute 19, 18, $\ldots$ to b_2 in turns and check whether '$s_2 > 0$' exists or not. Then, when $b_2 = 6$, it exists first. Therefore, $a_9 = 6$. Therefore $M(S_{10,20}) = 5 \cdot 20^{10} + 6 \ldots 20^9 + 20^9 - 1 = 5 \cdot 20^{10} + 7 \cdot 20^9 - 1 = 5.7 \times 20^{10} - 1_{(20)}$. (We can find $(8-5+1)+(19-6+1) = 18$ the number of times by checking.)

When we apply α_i above, when $b_2 = 19$, '$s_2 > 0$' doesn't exist.

If $h_9(t) = t^{10} - 20^9 t$, $h_9(t)$ is the minimum at $m = \sqrt[9]{\dfrac{20^9}{10}}$. $m_1 = \dfrac{h_i(p-1)\cdot m}{h(m)} = \dfrac{q(p-1)}{q-1}\left(1 - \dfrac{(p-1)^{q-1}}{p^i}\right) \approx 7.806$

$r_1 = p - 1 - \dfrac{(p-1)^q}{p^i} \approx 7.025.$

Since $[r_1] = [m_1] = 7$, $[\alpha_9 = 7]$.

Instead of finding r_1, by checking $t = [m_1] = 7$, you can find $[\alpha_9]$. Therefore you can start checking from the case of $b_2 = 7$. Including the calculation of m_1, r_1, you can find the greatest expanded number by checking the number of times $(8-5+1)+2+2 = 9$.

The table shows the greatest expanded numbers obtained from the algorithm.(Each number is denoted in its base.)

p \ q	2	3	4	5	6	7	8	9	10	20
3	22	122	222	2222	22222	222222	1222222	10222222	22222222	$3^{15}-1$
4	33	333	3333	33333	333333	2333333	20333333	133333333	1333333333	$3.1\times 4^{17}-1$
5	44	444	10444	104444	1044444	10444444	44444444	444444444	4444444444	$5^{19}-1$
6	55	1055	11555	155555	1555555	13555555	125555555	1255555555	11555555555	$2.4\times 6^{19}-1$
7	66	1066	16666	166666	1666666	16666666	166666666	1666666666	16666666666	$6\times 7^{19}-1$
8	77	1777	17777	177777	2077777	21777777	227777777	2277777777	21777777777	$1.2\times 8^{20}-1$
9	88	1888	18888	208888	2388888	28888888	288888888	2888888888	28888888888	$1.7\times 9^{20}-1$
10	99 $=10^2-1$	1999 $=2\times 10^3-1$	19999 $=2\times 10^4-1$	229999 $=2.3\times 10^5-1$	2999999 $=3\times 10^6-1$	29999999 $=3\times 10^7-1$	299999999 $=3\times 10^8-1$	2999999999 $=3\times 10^9-1$	30999999999 $=3.1\times 10^{10}-1$	$2.3\times 10^{20}-1$
11	(10)(10)	$2\times 11^3-1$	$2\times 11^4-1$	$3\times 11^5-1$	$3\times 11^6-1$	$3\times 11^7-1$	$3.(02)\times 11^8-1$	$3.(04)\times 11^9-1$	$3.(05)\times 11^{10}-1$	$2.(09)\times 11^{20}-1$
12	(11)(11)	$2\times 12^3-1$	$2.(01)\times 12^4-1$	$3\times 12^5-1$	$3\times 12^6-1$	$3.(03)\times 12^7-1$	$3.(05)\times 12^8-1$	$4\times 12^9-1$	$4\times 12^{10}-1$	$3.(04)\times 12^{20}-1$
13	(12)(12)	$2\times 13^3-1$	$2.(02)\times 13^4-1$	$3\times 13^5-1$	$3.(01)\times 13^6-1$	$3.(05)\times 13^7-1$	$4\times 13^8-1$	$4\times 13^9-1$	$4\times 13^{10}-1$	$4\times 13^{20}-1$
14	(13)(13)	$2\times 14^3-1$	$2.(03)\times 14^4-1$	$3\times 14^5-1$	$3.(02)\times 14^6-1$	$4\times 14^7-1$	$4\times 14^8-1$	$4.(01)\times 14^9-1$	$4.(04)\times 14^{10}-1$	$4.(04)\times 14^{20}-1$
15	(14)(14)	$2\times 15^3-1$	$3\times 15^4-1$	$3\times 15^5-1$	$3.(04)\times 15^6-1$	$4\times 15^7-1$	$4\times 15^8-1$	$4.(04)\times 15^9-1$	$5\times 15^{10}-1$	$5\times 15^{20}-1$
16	(15)(15)	$2\times 16^3-1$	$3\times 16^4-1$	$3\times 16^5-1$	$4\times 16^6-1$	$4\times 16^7-1$	$4.(02)\times 16^8-1$	$5\times 16^9-1$	$5\times 16^{10}-1$	$5.(03)\times 16^{20}-1$
20	(19)(19)	$2\times 20^3-1$	$3\times 20^4-1$	$3.(01)\times 20^5-1$	$4\times 20^6-1$	$4.(03)\times 20^7-1$	$5\times 20^8-1$	$5\times 20^9-1$	$5.(07)\times 20^{10}-1$	$7\times 20^{20}-1$
30	(29)(29)	$2\times 30^3-1$	$3\times 30^4-1$	$4\times 30^5-1$	$4.(02)\times 30^6-1$	$5\times 30^7-1$	$6\times 30^8-1$	$6\times 30^9-1$	$7\times 30^{10}-1$	$(10)\times 30^{20}-1$
40	(39)(39)	$2\times 40^3-1$	$3\times 40^4-1$	$4\times 40^5-1$	$5\times 40^6-1$	$5.(01)\times 40^7-1$	$6\times 40^8-1$	$7\times 40^9-1$	$7\times 40^{10}-1$	$(12)\times 40^{20}-1$
50	(49)(49)	$2\times 50^3-1$	$3\times 50^4-1$	$4\times 50^5-1$	$5\times 50^6-1$	$6\times 50^7-1$	$6\times 50^8-1$	$7\times 50^9-1$	$8\times 50^{10}-1$	$(13)\times 50^{20}-1$
60	(59)(59)	$2\times 60^3-1$	$3\times 60^4-1$	$4\times 60^5-1$	$5\times 60^6-1$	$6\times 60^7-1$	$6.(07)\times 60^8-1$	$7\times 60^9-1$	$8\times 60^{10}-1$	$(14)\times 60^{20}-1$

Table 2. The greatest expanded number according to p, q.

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Kyung Ho Jeong
Jangdeok Middle School,
Gwangju 500-, Korea.
E-mail: jkh0702@empas.com

Ihn Sue Kim
Department of Mathematics Education, Chonnam National University,
Gwangju 500-757, Korea.
E-mail: ikim@jnu.ac.kr