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$(k_1, \cdots, k_n)$-CONVEXITY IN $\mathbb{R}^n$

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Abstract. In this paper, we first introduce and study new concepts of $(k_1, \cdots, k_n)$-convexity and k-segment. Secondly, we shall discuss some properties of nonisotropically starlike domains in $\mathbb{R}^n$ with respect to the origin.

1. Introduction

In this paper, we introduce new notions of convexity, derived from the notion of "generalized balanced domain", can be found in [1] and [3]. We also study some basic properties of these new notions. Next, we propose a modified line segment "k-segment" and give its elementary properties. Finally, we recall the definition of a nonisotropically starlike domain in $\mathbb{R}^n$ with respect to the origin and present some related elementary results.

We refer to, e.g. [2], for general information about convexity in the classical sense used throughout this paper.

2. $(k_1, \cdots, k_n)$-convex sets in $\mathbb{R}^n$

Throughout in this paper we fix a natural number $n \in \mathbb{N}$ and we always let $\mathbf{k} := (k_1, \cdots, k_n) \in \mathbb{N}^n$. A set $S \subset \mathbb{R}^n$ is called $\mathbf{k}$-*convex* if either $S = \varnothing$ or, whenever $(1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y} \in S$ for any $\mathbf{x}, \mathbf{y} \in S$ and any $t \in [0, 1]$, where we denote $t^{\mathbf{k}}\mathbf{x} := (t^{k_1}x_1, \cdots, t^{k_n}x_n)$ for $\mathbf{x} := (x_1, \cdots, x_n)$ and $t \geq 0$.

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Example 2.1. *The followings are some examples of* $\mathbf{k}$*-convex sets:*

(a) For $\mathbf{x}^0 := (x_1^0, \cdots, x_n^0) \in \mathbb{R}^n$ *the set* $\{\mathbf{x} \in \mathbb{R}^n : \operatorname{sgn}(x_j) = \operatorname{sgn}(x_j^0),\ j = 1, \cdots, n\}$ *is* $\mathbf{k}$*-convex.*

(b) If $k_1 \leq \cdots \leq k_n$*, then the set* $\{\mathbf{x} \in \mathbb{R}^n : x_1 \geq \cdots \geq x_n \geq 0\}$ *is* $\mathbf{k}$*-convex.*

(c) For $r_1, \cdots, r_n > 0$ *the open (resp. closed) n-polydisk* $\prod_{j=1}^n (-r_j, r_j)$, $\prod_{j=1}^n [-r_j, r_j]$ *are* $\mathbf{k}$*-convex.*

(d) It follows from the Minkowski inequality that for every $r > 0$ *the open (resp. closed) n-ball* $\{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n x_j^2 < r^2\}$, $\{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n x_j^2 \leq r^2\}$ *are* $\mathbf{k}$*-convex.*

Example 2.2. *Let* $\mathbf{1} := (1, \cdots, 1) \in \mathbb{R}^n$ *and let* $k \in \mathbb{N}$.

(a) Obviously, any $\mathbf{1}$*-convex set is convex in the classical sense.*

(b) A closed half-space or an open half-space is a $k\mathbf{1}$*-convex set.*

(c) A set that is the intersection of a finite number of close half-spaces is called a convex polytope*. Since a half-space is a* $k\mathbf{1}$*-convex set, any convex polytope is a* $k\mathbf{1}$*-convex set.*

We can easily check the following results

Proposition 2.3. *(a) If a set* $S \subset \mathbb{R}^n$ *is* $\mathbf{k}$*-convex, so is* $\overline{S}$.

(b) If two sets $S, T \subset \mathbb{R}^n$ *are* $\mathbf{k}$*-convex, so is* $S + T := \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in S,\ \mathbf{y} \in T\}$.

(c) The intersection of any number of $\mathbf{k}$*-convex sets is* $\mathbf{k}$*-convex.*

Remark 2.4. *The empty set is* $\mathbf{k}$*-convex, but every nonempty finite subset of* $\mathbb{R}^n$ *is not* $\mathbf{k}$*-convex. For example, if* $x_j \neq 0$, $k_j \neq 1$ *for some* j*, then the singleton set* $\{\mathbf{x}\}$ *is not* $\mathbf{k}$*-convex, because*

$$\left(\frac{1}{2}\right)^{\mathbf{k}} \mathbf{x} + \left(\frac{1}{2}\right)^{\mathbf{k}} \mathbf{x} = \left(\frac{1}{2}\right)^{\mathbf{k-1}} \mathbf{x} \neq \mathbf{x}.$$

In particular, in contrast to Proposition 2.3 (c), a union of $\mathbf{k}$*-convex sets is not* $\mathbf{k}$*-convex in general.*

Remark 2.5. *Since a translated set* $\mathbf{x} + S := \{\mathbf{x} + \mathbf{y} : \mathbf{y} \in S\}$ *of a* $\mathbf{k}$*-convex set* S *with* $\mathbf{x} \neq \mathbf{0}$ *is not* $\mathbf{k}$*-convex in general, e.g., consider the set*

$$(1, 3) \times \cdots \times (1, 3) = 2\mathbf{1} + (-1, 1) \times \cdots \times (-1, 1)$$

and see Example 5.2. Furthermore, any open n-ball with center $\mathbf{x} \neq \mathbf{0}$ is not $\mathbf{k}$-convex whenever $\mathbf{k} \neq \mathbf{1}$. In particular, the $\mathbf{k}$-convexity of sets is in general not translation invariant.

3. $(k_1, \cdots, k_n)$-convex functions

Now we introduce the notion of a $\mathbf{k}$-convex function.

Definition 3.1. *Let $S \subset \mathbb{R}^n$ be a $\mathbf{k}$-convex set. A function $f : S \longrightarrow \mathbb{R}$ is called $\mathbf{k}$-convex whenever*

$$f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) \leq (1-t)f(\mathbf{x}) + tf(\mathbf{y})$$

for $\mathbf{x}, \mathbf{y} \in S$ and $t \in [0, 1]$. If strict inequality holds in the previous inequality whenever $\mathbf{x} \neq \mathbf{y}$ and $0 < t < 1$, we say that f is strictly $\mathbf{k}$-convex on S. The epigraph of f is defined as

$$\mathrm{Epi}(f) := \{(\mathbf{x}, a) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{x} \in S,\ f(\mathbf{x}) \leq a\}.$$

Remark 3.2. *(a) Any linear combination with positive coefficients of $\mathbf{k}$-convex functions is $\mathbf{k}$-convex.*

(b) If f is a $\mathbf{k}$-convex function on a $\mathbf{k}$-convex subset S of $\mathbb{R}^n$ and a function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is non-decreasing and convex, then $\phi \circ f : S \longrightarrow \mathbb{R}$ is $\mathbf{k}$-convex.

Example 3.3. *The function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, defined by*

$$f(\mathbf{x}) := \sum_{j=1}^{n} |x_j|^{1/k_j} \qquad (\mathbf{x} \in \mathbb{R}^n),$$

is $\mathbf{k}$-convex, because

$$|(1-t)^{k_j} x_j + t^{k_j} y_j|^{1/k_j} \leq (1-t)|x_j|^{1/k_j} + t|y_j|^{1/k_j} \quad (j = 1, \cdots, n)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $t \in [0, 1]$. In particular, in view of Remark 3.2 (b), there are lots of other $\mathbf{k}$-convex functions from $\mathbb{R}^n$ to $\mathbb{R}$.

We now present a characterization of $\mathbf{k}$-convexity for functions

Theorem 3.4. *Let $S \subset \mathbb{R}^n$ be a $\mathbf{k}$-convex set. For a function $f : S \longrightarrow \mathbb{R}$, the following statements are equivalent:*

(a) *f is $\mathbf{k}$-convex.*
(b) $\mathrm{Epi}(f)$ *is $(\mathbf{k}, 1)$-convex, where* $(\mathbf{k}, 1) := (k_1, \cdots, k_n, 1)$.

(c) (Jensen's Inequality) *For every* $x^1, \cdots, x^m \in S$ *and nonnegative real numbers* $t_1, \cdots, t_m$ *with* $1 = \sum_{i=1}^m t_i$, *we have*

$$f\left(\sum_{i=1}^m t_i^{\mathbf{k}} \mathbf{x}^i\right) \le \sum_{i=1}^m t_i f(\mathbf{x}^i) \tag{1}$$

Proof. (a)$\Longrightarrow$(b): Let $(\mathbf{x}, a), (\mathbf{y}, b) \in \mathrm{Epi}(f)$ and let $0 \le t \le 1$. Then $f(\mathbf{x}) \le a$, $f(\mathbf{y}) \le b$, and also the $(\mathbf{k}, 1)$-convexity of f implies that

$$f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) \le (1-t)f(\mathbf{x}) + tf(\mathbf{y}) \le (1-t)a + tb.$$

But since $(1-t)^{(\mathbf{k},1)}(\mathbf{x}, a) + t^{(\mathbf{k},1)}(\mathbf{y}, b) = ((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}, (1-t)a + tb)$, the set $\mathrm{Epi}(f)$ is $(\mathbf{k}, 1)$-convex.

(b)$\Longrightarrow$(a): Let $\mathbf{x}, \mathbf{y} \in S$ and let $0 \le t \le 1$. It is clear that $(\mathbf{x}, f(\mathbf{x}))$, $(\mathbf{y}, f(\mathbf{y})) \in \mathrm{Epi}(f)$, and by the $(\mathbf{k}, 1)$-convexity of $\mathrm{Epi}(f)$ one has

$$\begin{aligned}&((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}, (1-t)f(\mathbf{x}) + tf(\mathbf{y}))\\ &\qquad = (1-t)^{(\mathbf{k},1)}(\mathbf{x}, f(\mathbf{x})) + t^{(\mathbf{k},1)}(\mathbf{y}, f(\mathbf{y})) \in \mathrm{Epi}(f),\end{aligned}$$

that is, $f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) \le (1-t)f(\mathbf{x}) + tf(\mathbf{y})$. Thus f is $\mathbf{k}$-convex.

(c)$\Longrightarrow$(a): It is trivial.

(a)$\Longrightarrow$(c): Let us use mathematical induction on m. The inequality (1) is true for $m = 1$. Assume the inequality (1) holds for the integer $m \ge 1$, let $\mathbf{x}^1, \cdots, \mathbf{x}^{m+1} \in S$ and let $t_i \ge 0\,(i = 1, \cdots, m+1)$ with $1 = \sum_{i=1}^{m+1} t_i$. At least one of $t_1, \cdots, t_{m+1}$ must be less than 1 (otherwise the inequality (1) is trivial). Now we may assume that $t_{m+1} < 1$. Let $t := 1 - t_{m+1}$ and $s_i := t_i/t\,(i = 1, \cdots, m)$. Then $0 \le t \le 1$, $\sum_{i=1}^m s_i = 1$, $\mathbf{y} := \sum_{i=1}^m s_i^{\mathbf{k}}\mathbf{x}^i \in S$, and

$$\mathbf{x} := \sum_{i=1}^{m+1} t_i^{\mathbf{k}}\mathbf{x}^i = t^{\mathbf{k}}\mathbf{y} + (1-t)^{\mathbf{k}}\mathbf{x}^{m+1}.$$

But since f is $\mathbf{k}$-convex on S, $f(\mathbf{x}) \le tf(\mathbf{y}) + (1-t)f(\mathbf{x}^{m+1})$ and, by our induction hypothesis, $f(\mathbf{y}) \le \sum_{i=1}^m s_i f(\mathbf{x}^i)$. Hence, combining the above two inequalities, we get that

$$\begin{aligned} f\left(\sum_{i=1}^{m+1} t_i^{\mathbf{k}} \mathbf{x}^i\right) &\le t \sum_{i=1}^{m} s_i f(\mathbf{x}^i) + (1-t) f(\mathbf{x}^{m+1}) \\ &\le \sum_{i=1}^{m} t_i f(\mathbf{x}^i) + t_{m+1} f(\mathbf{x}^{m+1}) = \sum_{i=1}^{m+1} t_i f(\mathbf{x}^i). \end{aligned}$$

Thus, the inequality (1) is established for $m \rightsquigarrow m+1$, and therefore, by mathematical induction, it holds for any natural number m. □

Notice that the infimum taken over the empty set is, by convention, assumed to be $+\infty$.

Theorem 3.5. *Let $E \subset \mathbb{R}^{n+1}$ be a $(\mathbf{k}, 1)$-convex set. If a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $f(\mathbf{x}) := \inf\{c \in \mathbb{R} : (\mathbf{x}, c) \in E\}$ is $\mathbf{k}$-convex.*

Proof. In view of Theorem 3.4, it suffices to show that the set $\mathrm{Epi}(f)$ is $(\mathbf{k}, 1)$-convex. For this, let $(\mathbf{x}, a), (\mathbf{y}, b) \in \mathrm{Epi}(f)$ and let $\epsilon > 0$. Since $f(\mathbf{x}) < a + \epsilon$ and $f(\mathbf{y}) < b + \epsilon$, it follows from the definition of f that there exist $c, d \in \mathbb{R}$ such that $c < a + \epsilon$, $d < b + \epsilon$, and $(\mathbf{x}, c), (\mathbf{y}, d) \in E$. The $(\mathbf{k}, 1)$-convexity of E implies that for any $t \in [0, 1]$ one has

$$((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}, (1-t)c + td) = (1-t)^{(\mathbf{k},1)}(\mathbf{x}, c) + t^{(\mathbf{k},1)}(\mathbf{y}, d) \in E$$

and so $f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) \le (1-t)c + td < (1-t)a + tb + \epsilon$. But since ϵ was arbitrary, $f((1-t)^{\mathbf{k}}\mathbf{x}+t^{\mathbf{k}}\mathbf{y}) \le (1-t)a+tb$, that is, $(1-t)^{(\mathbf{k},1)}(\mathbf{x}, a)+ t^{(\mathbf{k},1)}(\mathbf{y}, b) \in \mathrm{Epi}(f)$, as desired. □

The next result can be easily checked.

Proposition 3.6. *Let $S \subset \mathbb{R}^n$ be a $\mathbf{k}$-convex set. If $f : S \longrightarrow \mathbb{R}$ is $\mathbf{k}$-convex, then, to every $c \in \mathbb{R}$ two sets $S_c(f) := \{\mathbf{x} \in S : f(\mathbf{x}) < c\}$ and $\overline{S}_c(f) := \{\mathbf{x} \in S : f(\mathbf{x}) \le c\}$ are $\mathbf{k}$-convex.*

Remark 3.7. *The fact that the converse of the above proposition does not hold in general can be easily seen from the function $x \longmapsto x^{1/(2k)}$ from $[0, \infty)$ to $\mathbb{R}$, where $k \in \mathbb{N}$.*

Example 3.8. *(cf. [2], p.109) Let S be nonempty and closed. Define*

$$d_S(\mathbf{x}) := \min\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in S\} \qquad \text{for any } \mathbf{x} \in \mathbb{R}^n.$$

Then the function d_S is $\mathbf{k}$-convex on $\mathbb{R}^n$ if and only if S is a $\mathbf{k}$-convex set.

Proposition 3.9. *If $\{f_\alpha\}_{\alpha\in\Lambda}$ is a family of* **k***-convex functions on a* **k***-convex set $S \subset \mathbb{R}^n$, its upper envelope $f : S \longrightarrow \mathbb{R}$, defined by $f(\mathbf{x}) := \sup_{\alpha\in\Lambda} f_\alpha(\mathbf{x})$ for $\mathbf{x} \in S$ is also* **k***-convex.*

Proof. Since each f_α is **k**-convex, in virtue of Theorem 3.4, each $\mathrm{Epi}(f_\alpha)$ is $(\mathbf{k}, 1)$-convex. Hence the $(\mathbf{k}, 1)$-convexity of $\mathrm{Epi}(f)$ follows immediately from the fact that $\mathrm{Epi}(f) = \bigcap_{\alpha\in\Lambda} \mathrm{Epi}(f_\alpha)$ and, by Theorem 3.4, the function f is **k**-convex. □

We finish this section by introducing the notion of quasi-**k**-convexity.

Definition 3.10. *A function f defined on a* **k***-convex set $S \subset \mathbb{R}^n$ is called quasi-***k***-convex whenever $f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\}$ for any $\mathbf{x}, \mathbf{y} \in S$ and $t \in [0, 1]$.*

Proposition 3.11. *A function f defined on a* **k***-convex set $S \subset \mathbb{R}^n$ is quasi-***k***-convex if and only if the sub-level set $\overline{S}_c(f)$ is* **k***-convex for every $c \in \mathbb{R}$.*

Proof. ($\Longrightarrow$) Fix $c \in \mathbb{R}$. Let $\mathbf{x}, \mathbf{y} \in \overline{S}_c(f)$ and let $t \in [0, 1]$. Then the **k**-convexity of S(resp. f) implies that $(1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y} \in S$ and

$$f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) \le (1-t)f(\mathbf{x}) + tf(\mathbf{y}) \le (1-t)c + tc = c,$$

so $(1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y} \in \overline{S}_c(f)$.

($\Longleftarrow$) Let $\mathbf{x}, \mathbf{y} \in S$ and let $t \in [0, 1]$. If we put $c := \max\{f(\mathbf{x}), f(\mathbf{y})\}$, it is clear that $\mathbf{x}, \mathbf{y} \in \overline{S}_c(f)$, and so by the **k**-convexity of $\overline{S}_c(f)$ one has $(1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y} \in \overline{S}_c(f)$, i.e., $f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\}$. □

Corollary 3.12. *(i) If $f_1, \cdots, f_m$ are quasi-***k***-convex on a* **k***-convex subset S of $\mathbb{R}^n$ and $c_1, \cdots, c_m \ge 0$, then $f := \max\{c_j f_j : j = 1, \cdots, m\}$ is a quasi-***k***-convex function on S.*

*(ii) If f is a quasi-***k***-convex function on a* **k***-convex subset S of $\mathbb{R}^n$ and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing function, then $\phi \circ f : S \longrightarrow \mathbb{R}$ is quasi-***k***-convex*

The proofs of the above results are trivial and we omit it.

4. Extrema of $(k_1, \cdots, k_n)$-convex functions

Proposition 4.1. *Let $S \subset \mathbb{R}^n$ be compact and* **k***-convex. If a function $f : S \longrightarrow \mathbb{R}$ is continuous and strictly* **k***-convex, then there exists a unique $\mathbf{x}^0 \in S$ such that $f(\mathbf{x}^0) \le f(\mathbf{x})$ for any $\mathbf{x} \in S$.*

Proof. The existence of $\mathbf{x}^0 \in S$ is guaranteed by the fact that S is compact and f is continuous on S. To show the uniqueness of the point $\mathbf{x}^0$, we assume that there is a $\mathbf{x}^1 \in S$ with $\mathbf{x}^1 \neq \mathbf{x}^0$ such that $f(\mathbf{x}^1) \leq f(\mathbf{x})$ for any $\mathbf{x} \in S$. For any $t \in (0,1)$, using the strict $\mathbf{k}$-convexity of f,

$$f((1-t)^{\mathbf{k}}\mathbf{x}^0 + (1-t)^{\mathbf{k}}\mathbf{x}^1) < (1-t)f(\mathbf{x}^0) + tf(\mathbf{x}^1);$$

moreover, using the fact that $\mathbf{x}^0$ and $\mathbf{x}^1$ are minimums, one has

$$\begin{aligned} &f((1-t)^{\mathbf{k}}\mathbf{x}^0 + (1-t)^{\mathbf{k}}\mathbf{x}^1) \\ &\quad < (1-t)f((1-t)^{\mathbf{k}}\mathbf{x}^0 + (1-t)^{\mathbf{k}}\mathbf{x}^1) + tf((1-t)^{\mathbf{k}}\mathbf{x}^0 + (1-t)^{\mathbf{k}}\mathbf{x}^1) \\ &\quad = f((1-t)^{\mathbf{k}}\mathbf{x}^0 + (1-t)^{\mathbf{k}}\mathbf{x}^1) \end{aligned}$$

which is a contradiction. □

We now present a maximum principle for $\mathbf{k}$-convex functions as follows.

Theorem 4.2. *If f is a $\mathbf{k}$-convex function on a $\mathbf{k}$-convex subset S of $\mathbb{R}^n$ and attains a global maximum at an interior point of S, then f is constant on S.*

Proof. Assume that f is not constant and attains a global maximum at the point $\mathbf{p} \in \mathrm{int}S$. Choose $r > 0$ with $\mathbb{B}_n(\mathbf{p}; r) := \{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u} - \mathbf{p}\| < r\} \subset S$ and $\mathbf{x} \in S$ with $f(\mathbf{x}) < f(\mathbf{p})$. For any $0 < \epsilon < 1$ we put $\mathbf{y}_\epsilon := (y_1, \cdots, y_n) \in \mathbb{R}^n$, where

$$y_j := (1+\epsilon)^{k_j} p_j - \epsilon^{k_j} x_j \qquad \text{for} \quad j = 1, \cdots, n.$$

Observe that

$$\begin{aligned} \mathbf{y}_\epsilon - \mathbf{p} &= \sum_{j=1}^{n} \left\{ \left[(1+\epsilon)^{k_j} - 1 \right] p_j - \epsilon^{k_j} x_j \right\} \mathbf{e}_j \\ &= \sum_{j=1}^{n} \left[\left\{ \sum_{i=1}^{k_j - 1} \binom{k_j}{i} \epsilon^i p_j \right\} + \epsilon^{k_j} (p_j - x_j) \right] \mathbf{e}_j \end{aligned}$$

where $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$, and also

$$\begin{aligned} \|\mathbf{y}_\epsilon - \mathbf{p}\| &\leq \sqrt{n} \max_{j=1,\cdots,n} \left[\left\{ \sum_{i=1}^{k_j - 1} \binom{k_j}{i} \epsilon^i |p_j| \right\} + \epsilon^{k_j} |p_j - x_j| \right] \\ &\leq \sqrt{n} M \epsilon \|\mathbf{p}\|_\infty + \epsilon \|\mathbf{p} - \mathbf{x}\|_\infty \leq \epsilon \left(\sqrt{n} M \|\mathbf{p}\| + \|\mathbf{p} - \mathbf{x}\| \right) \end{aligned}$$

where $M := \max_{j=1,\cdots,n} \sum_{i=1}^{k_j-1} \binom{k_j}{i}$, $\|\mathbf{p}\|_\infty := \max_{j=1,\cdots,n} |p_j|$. Hence, we can choose a sufficiently small $\epsilon \in (0,1)$ so that $\mathbf{y}_\epsilon \in \mathbb{B}_n(\mathbf{p}; r)$. On the other hand, for every $j = 1, \cdots, n$ one has

$$p_j = \frac{1}{(1+\epsilon)^{k_j}} \left(y_j + \epsilon^{k_j} x_j \right) = \left(\frac{1}{1+\epsilon} \right)^{k_j} y_j + \left(\frac{\epsilon}{1+\epsilon} \right)^{k_j} x_j,$$

that is, $\mathbf{p} = (1/(1+\epsilon))^k \, \mathbf{y}_\epsilon + (\epsilon/(1+\epsilon))^k \, \mathbf{x}$, which yields a contradiction since

$$f(\mathbf{p}) \le \frac{1}{1+\epsilon} f(\mathbf{y}_\epsilon) + \frac{\epsilon}{1+\epsilon} f(\mathbf{x}) < \frac{1}{1+\epsilon} f(\mathbf{p}) + \frac{\epsilon}{1+\epsilon} f(\mathbf{p}) = f(\mathbf{p})$$

by the $\mathbf{k}$-convexity of f. □

Proposition 4.3. *Let f be a $\mathbf{k}$-convex function on a $\mathbf{k}$-convex subset S of $\mathbb{R}^n$. Then the set of points, $A_{\min}(f)$, at which f attains its minimum is $\mathbf{k}$-convex.*

Proof. Assume that $A_{\min}(f) \neq \varnothing$. Let m be the minimal value attained by f on S. For any $\mathbf{x}, \mathbf{y} \in A_{\min}(f)$ and any $t \in [0,1]$, one has $(1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y} \in S$ and

$$m \le f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) \le (1-t)f(\mathbf{x}) + tf(\mathbf{y}) = (1-t)m + tm = m$$

and so $(1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y} \in A_{\min}(f)$. Hence the set $A_{\min}(f)$ is $\mathbf{k}$-convex. □

Proposition 4.4. *Let f be a $\mathbf{k}$-convex function on an open $\mathbf{k}$-convex subset S of $\mathbb{R}^n$. Then every local minimum of f is a global minimum of f on S.*

Proof. Suppose that f attains a local minimum at a point $\mathbf{x}^0 \in S$. Then $f(\mathbf{x}) \ge f(\mathbf{x}^0)$ for all $\mathbf{x}$ in a sufficiently small neighborhood $\mathbb{B}_n(\mathbf{x}^0; \delta) \subset S$. Let $\mathbf{x}$ be any point in S. Since $t \longmapsto (1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{x}^0$ is a continuous function passing through the point $\mathbf{x}^0$, we can find some $t \in (0,1)$ suffficiently close to 1 such that $(1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{x}^0 \in \mathbb{B}_n(\mathbf{x}^0; \delta)$, and also $f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{x}^0) \ge f(\mathbf{x}^0)$. But since f is $\mathbf{k}$-convex on S, one has $(1-t)f(\mathbf{x}) + tf(\mathbf{x}^0) \ge f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{x}^0)$. Combining the last two inequalities, we get that $f(\mathbf{x}) \ge f(\mathbf{x}^0)$. Hence f attains a global minimum at the point $\mathbf{x}^0$. □

Proposition 4.5. *Let f be a strictly $\mathbf{k}$-convex function on an open $\mathbf{k}$-convex subset S of $\mathbb{R}^n$. If f attains its minimum on S, it is attained at a unique point of S.*

Proof. Suppose that $A_{\min}(f)$ contains two distinct points $\mathbf{x}, \mathbf{y} \in S$ and put $m := f(\mathbf{x}) = f(\mathbf{y})$. By the strictly $\mathbf{k}$-convexity of f, one has

$$m \le f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) < (1-t)f(\mathbf{x}) + tf(\mathbf{y}) = m \qquad \text{for} \quad 0 < t < 1$$

which is a contradiction. □

5. k-segments in $\mathbb{R}$

Let $k \in \mathbb{N}$. For $x, y \in \mathbb{R}$ with $x \le y$, *the k-segment (determined by x and y)* is defined by

$$[x, y]_k := \{(1-t)^k x + t^k y : 0 \le t \le 1\}.$$

First we will give some elementary properties of this k-segment.

Lemma 5.1. *Let $k \in \mathbb{N}$. Then any k-segment is a closed interval in $\mathbb{R}$, and $[x, y] \subseteq [x, y]_k$, where the equality holds iff $k = 1$ or $xy \le 0$. More explicitly, there exists a point $\Phi_k(x, y) \in \mathbb{R}$ such that*

$$[x, y] \subset [x, y]_k = \begin{cases} [\Phi_k(x, y), y] \subset (0, y] & (y > x > 0) \\ [x, y] & (x < 0 < y \text{ or } xy = 0) \\ [x, \Phi_k(x, y)] \subset [x, 0) & (x < y < 0) \end{cases}$$

where

$$\Phi_k(x, y) = \left(\frac{y^{1/(k-1)}}{x^{1/(k-1)} + y^{1/(k-1)}} \right)^k x + \left(\frac{x^{1/(k-1)}}{x^{1/(k-1)} + y^{1/(k-1)}} \right)^k y$$

for $y > x > 0$ and $\Phi_k(x, y) = -\Phi_k(-y, -x)$ for $x < y < 0$. In particular,

$$0 < \Phi_k(x, y) < x \qquad (y > x > 0), \tag{2}$$

$$y < \Phi_k(x, y) < 0 \qquad (x < y < 0). \tag{3}$$

Proof. Define $\phi : [0, 1] \longrightarrow \mathbb{R}$ by $\phi(t) := (1-t)^k x + t^k y$, $t \in [0, 1]$. Then ϕ is a (continuous) polynomial of one variable t connecting $x = \phi(0)$ with $y = \phi(1)$, which gives $[x, y] \subset \phi([0, 1])$, and also

$$\phi'(t) = k\{t^{k-1}y - (1-t)^{k-1}x\} \qquad (0 \le t \le 1).$$

Moreover, if $k > 1$ then we have:

(i) in case $xy \ge 0$, $x \ne 0$: one has

$$\phi'(t) = 0 \iff t = \frac{1}{1 + \sqrt[k-1]{y/x}} =: t_0$$

and in particular,

- in case $x > 0$: $y > 0$ and ϕ is strictly decreasing and increasing on $[0, t_0]$ and $[t_0, 1]$, respectively, and ϕ has the minimum at t_0. In particular, $[x, y] \subset [x, y]_k = [\phi(t_0), y] \subset (0, y]$;
- in case $x < 0$: $y < 0$ and ϕ is strictly increasing and decreasing on $[0, t_0]$ and $[t_0, 1]$, respectively, and ϕ has the maximum at t_0. In particular, $[x, y] \subset [x, y]_k = [x, \phi(t_0)] \subset [x, 0)$.

(ii) in case $xy \leq 0$ and $x \neq 0$: $x < 0$, $y > 0$, and $\phi'(t) \geq 0$, $t \in [0, 1]$, with equality holds iff $t = 1, y = 0$, which implies that ϕ is strictly increasing on $[0, 1]$. In particular, $[x, y] = [x, y]_k$.

(iii) in case $xy = 0$: $\phi'(t) \geq 0$, $t \in [0, 1]$, with equality holds iff $x = t = 0$ or $y = t - 1 = 0$, which implies that ϕ is strictly increasing on $[0, 1]$. In particular, $[0, y] = [0, y]_k$ and $[x, 0] = [x, 0]_k$,

as desired. □

Example 5.2. *Note that $[0, 0]_k = [0, 0]$ and $([0, 0]_k)_k = [0, 0]$. For any $x \geq 0$ one has $[x, x]_k = \left[\frac{x}{2^{k-1}}, x\right]$, since by Lemma 5.1*

$$\Phi_k(x, x) = \left(\frac{1}{2}\right)^k x + \left(\frac{1}{2}\right)^k x = \frac{x}{2^{k-1}} \qquad (x > 0).$$

In particular, any singleton set with a nonzero real number is never k-convex for $k \in \mathbb{N} \setminus \{1\}$.

Lemma 5.3. *Let $k \in \mathbb{N} \setminus \{1\}$. For any $a, b, x, y \in \mathbb{R}$ with $0 \leq a < b$ and $0 \leq x < y$, one has*

$$\Phi_k(a, b) \leq \Phi_k(x, y) \iff BY(X - A) - AX(B - Y) \geq 0 \tag{4}$$

where Φ_k is as in Lemma 5.1 and $A := a^{1/(k-1)}$, $B := b^{1/(k-1)}, X := x^{1/(k-1)}$, $Y := y^{1/(k-1)}$.

Proof. Let $p, q \in \mathbb{R}$ with $0 \leq p < q$. Observe that $\Phi_k(0, q) = 0$, and in case $p > 0$ one has

$$\begin{aligned}\Phi_k(p, q) &= \left(\frac{Q}{P + Q}\right)^k P^{k-1} + \left(\frac{P}{P + Q}\right)^k Q^{k-1} \\ &= \left(\frac{PQ}{P + Q}\right)^k \left(\frac{1}{P} + \frac{1}{Q}\right) \\ &= \left(\frac{1}{P} + \frac{1}{Q}\right)^{1-k}\end{aligned}$$

where $P := p^{1/(k-1)}$, $Q := q^{1/(k-1)}$. Hence,

$$\Phi_k(a, b) \leq \Phi_k(x, y) \iff \frac{1}{X} + \frac{1}{Y} \leq \frac{1}{A} + \frac{1}{B}$$

which implies (4) as desired. □

Proposition 5.4. *Let $k \in \mathbb{N}\backslash\{1\}$. For $x, y, b \in \mathbb{R}$ with $0 \le x \le y \le b$, we have $[y, b]_k \subset [x, b]_k$, and $[y, b]_k \subset ([y, b]_k)_k$.*

Proof. If we put $X := x^{1/(k-1)}$, $Y := y^{1/(k-1)}$, $B := b^{1/(k-1)}$, then $BB(Y - X) - XY(B - B) = B^2(Y - X) \ge 0$, and so the first assertion is true by Lemma 5.3. And the second assertion is also true, because

$$\Phi_k(\Phi_k(y, b), b) < \Phi_k(y, b) \qquad (0 \le y < b)$$

by (2) of Lemma 5.1. □

In contrast to the result of Proposition 5.4, the inclusion $[x, y]_k \subset [a, b]_k$ does not hold for $0 < a \le x < y \le b$ in general, as follows:

Example 5.5. *In the case: $k \ge 2$, $a := 1$, $x := (3/2)^{k-1}$, $y := 2^{k-1}$, $b > 0$. Note that*

$$BY(X - A) - AX(B - Y) = B \cdot 2 \cdot \frac{1}{2} - 1 \cdot \frac{3}{2} \cdot (B - 2) = \frac{6 - B}{2}$$

which implies that in that case $\Phi_k(a, b) \le \Phi_k(x, y)$ iff $0 < b \le 6^{k-1}$. More generally, if $a := 1$, $x := (m + 1)^{k-1}$, $y < (1 + \frac{1}{m})^{k-1}$ with $0 < m < 1$, one has $a < x < y < b$, $Ym < m + 1$, and

$$BY(X - A) - AX(B - Y) = B(Ym - m - 1) + (m + 1)Y,$$

which implies that in that case

$$\Phi_k(a, b) \le \Phi_k(x, y) \iff 0 < b \le \left\{ \frac{(m + 1)y^{1/(k-1)}}{m + 1 - my^{1/(k-1)}} \right\}^{k-1}.$$

The following result gives us the concrete form for iterated k-segment.

Proposition 5.6. *Let $k \in \mathbb{N}\backslash\{1\}$ and let $x, y \in \mathbb{R}$, $0 \le x \le y$, where x, y are not all zero. Then we have for any $n \in \mathbb{N}$,*

$$(5) \qquad [x_n, y] := \underbrace{((([x, y]_k)_k) \cdots)_k}_{n-times} = \left[\frac{xy}{\left\{ nx^{1/(k-1)} + y^{1/(k-1)} \right\}^{k-1}}, y \right].$$

In particular, if $x > 0$ then $0 < x_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. We can easily verify that

$$(6) \qquad x_n = \Phi_k(x_{n-1}, y) = \frac{x_{n-1}y}{\left\{ x_{n-1}^{1/(k-1)} + y^{1/(k-1)} \right\}^{k-1}}$$

where $x_0 := x$, using mathematical induction and Lemma 5.1. Note that

$$x_1 = \frac{xy}{\left\{x^{1/(k-1)} + y^{1/(k-1)}\right\}^{k-1}} = \left(\frac{x^{1/(k-1)}y^{1/(k-1)}}{x^{1/(k-1)} + y^{1/(k-1)}}\right)^{k-1}.$$

Assume that (5) is true for any positive integer n. Observe that

$$x_n = \frac{xy}{\left\{nx^{1/(k-1)} + y^{1/(k-1)}\right\}^{k-1}} = \left(\frac{x^{1/(k-1)}y^{1/(k-1)}}{nx^{1/(k-1)} + y^{1/(k-1)}}\right)^{k-1}$$

and so, by (6),

$$\begin{aligned} x_{n+1} &= \frac{x_n y}{\left\{x_n^{1/(k-1)} + y^{1/(k-1)}\right\}^{k-1}} \\ &= \frac{\frac{xy}{\left\{nx^{1/(k-1)}+y^{1/(k-1)}\right\}^{k-1}} y}{\left\{\frac{x^{1/(k-1)}y^{1/(k-1)}}{nx^{1/(k-1)}+y^{1/(k-1)}} + y^{1/(k-1)}\right\}^{k-1}} \\ &= \frac{xy^2}{\left\{(n+1)x^{1/(k-1)} + y^{1/(k-1)}\right\}^{k-1} y} \\ &= \frac{xy}{\left\{(n+1)x^{1/(k-1)} + y^{1/(k-1)}\right\}^{k-1}} \end{aligned}$$

as desired. Hence by mathematical induction (5) is correct for all positive integers n. □

6. Nonisotropically Starlike Sets in $\mathbb{R}^n$

Definition 6.1. *Let $n \in \mathbb{N}$ with $n \geq 2$. A set $S \subset \mathbb{R}^n$ is said to be* $\mathbf{k}$*-nonisotropically starlike with respect to the origin* whenever $t^{\mathbf{k}}\mathbf{x} \in S$ *for any $\mathbf{x} \in S$ and $t \in [0,1]$.*

Remark 6.2. *(a) A set $S \subset \mathbb{R}^n$ is stralike with respect to the origin if and only if it is* $\mathbf{1}$*-nonisotropically starlike with respect to the origin.*

(b) Any $\mathbf{k}$*-convex subset of $\mathbb{R}^n$ containing the origin is* $\mathbf{k}$*-nonisotropically starlike with respect to the origin.*

(c) If f is a $\mathbf{k}$*-convex function on a* $\mathbf{k}$*-nonisotropically starlike set $S \subset \mathbb{R}^n$ with respect to the origin satisfying $f(\mathbf{0}) \leq 0$, then*

$$f(t^{\mathbf{k}}\mathbf{x}) = f(t^{\mathbf{k}}\mathbf{x} + (1-t)^{\mathbf{k}}\mathbf{0}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{0}) \leq tf(\mathbf{x})$$

for any $\mathbf{x} \in S$ and $t \in [0,1]$.

Definition 6.3. *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called positively* **k**-*homogeneous whenever*

$$f(t^{\mathbf{k}}\mathbf{x}) = tf(\mathbf{x}) \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^n \quad \text{and} \quad t \geq 0.$$

Lemma 6.4. *A positively* **k**-*homogeneous function $f : \mathbb{R}^n \longrightarrow [0, \infty)$ is* **k**-*convex if and only if*

$$f(\mathbf{x}+\mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y}) \qquad \text{for any} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{7}$$

Proof. ($\Longleftarrow$) Let $(\mathbf{x}, a), (\mathbf{y}, b) \in \mathrm{Epi}(f)$. Since $f(\mathbf{x}) \leq a$ and $f(\mathbf{y}) \leq b$, it follows from the subadditivity (7) and the positive **k**-homogeneity of f that

$$\begin{aligned} f((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) &\leq f((1-t)^{\mathbf{k}}\mathbf{x}) + f(t^{\mathbf{k}}\mathbf{y}) \\ &= (1-t)f(\mathbf{x}) + tf(\mathbf{y}) \leq (1-t)a + tb, \end{aligned}$$

for every $t \in [0, 1]$, so $(1-t)^{(\mathbf{k},1)}(\mathbf{x}, a) + t^{(\mathbf{k},1)}(\mathbf{y}, b) = ((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}, (1-t)a + tb) \in \mathrm{Epi}(f)$. Hence, $\mathrm{Epi}(f)$ is $(\mathbf{k}, 1)$-convex and we conclude that f is a **k**-convex function according to Theorem 3.4.

($\Longrightarrow$) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and let $\epsilon > 0$. Put

$$\mathbf{x}_\epsilon := \left(\frac{1}{f(\mathbf{x}) + \frac{\epsilon}{2}}\right)^{\mathbf{k}} \mathbf{x}, \qquad \mathbf{y}_\epsilon := \left(\frac{1}{f(\mathbf{y}) + \frac{\epsilon}{2}}\right)^{\mathbf{k}} \mathbf{y},$$

$$\alpha := \frac{f(\mathbf{x}) + \frac{\epsilon}{2}}{f(\mathbf{x}) + f(\mathbf{y}) + \epsilon}, \qquad \beta := \frac{f(\mathbf{y}) + \frac{\epsilon}{2}}{f(\mathbf{x}) + f(\mathbf{y}) + \epsilon}.$$

Clearly, $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Observe that

$$\alpha^{\mathbf{k}}\mathbf{x}_\epsilon + \beta^{\mathbf{k}}\mathbf{y}_\epsilon = \left(\frac{1}{f(\mathbf{x}) + f(\mathbf{y}) + \epsilon}\right)^{\mathbf{k}} (\mathbf{x} + \mathbf{y}).$$

By the positive **k**-homogeneity and by the **k**-convexity of f we get that

$$\frac{f(\mathbf{x}+\mathbf{y})}{f(\mathbf{x}) + f(\mathbf{y}) + \epsilon} \leq \alpha f(\mathbf{x}_\epsilon) + \beta f(\mathbf{y}_\epsilon) \leq \alpha \frac{f(\mathbf{x})}{f(\mathbf{x}) + \frac{\epsilon}{2}} + \beta \frac{f(\mathbf{y})}{f(\mathbf{y}) + \frac{\epsilon}{2}} < 1$$

and so $f(\mathbf{x}+\mathbf{y}) < f(\mathbf{x}) + f(\mathbf{y}) + \epsilon$. But since ϵ was arbitrary, we obtain the required subadditive inequality (7), and the proof is complete. $\square$

From now on we assume that $S \subset \mathbb{R}^n$ is a **k**-nonisotropically starlike domain with respect to the origin.

Definition 6.5. *We define a functional $h_{\mathbf{k},S} : \mathbb{R}^n \longrightarrow [0, +\infty)$ by*

$$h_{\mathbf{k},S}(\mathbf{x}) := \inf\{t > 0 : (1/t)^{\mathbf{k}}\mathbf{x} \in S\} \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^n,$$

which is called the **k**-*Minkowski functional of S.*

We now recall the following elementary properties of Minkowski functionals, which can be found in [1] and [3].

Proposition 6.6. *The following properties hold:*

(a) $h_{\mathbf{k},S}(t^{\mathbf{k}}\mathbf{x}) = th_{\mathbf{k},S}(\mathbf{x})$ *for any* $t \geq 0$ *and* $\mathbf{x} \in \mathbb{R}^n$.

(b) $S = \{\mathbf{x} \in \mathbb{R}^n : h_{\mathbf{k},S}(\mathbf{x}) < 1\}$.

(c) $h_{\mathbf{k},S}$ *is uniquely determined by* (a) *and* (b)

(d) $h_{\mathbf{k},S}$ *is upper semicontinuous on* $\mathbb{R}^n$.

Finally, we characterize **k**-convex sets in terms of its **k**-Minkowski functionals:

Theorem 6.7. *The following three properties are equivalent:*

(a) S *is a* **k**-*convex set*

(b) $h_{\mathbf{k},S}$ *is subadditive, i.e, it satisfies the triangle inequality.*

(c) $h_{\mathbf{k},S}$ *is a* **k**-*convex function*

Proof. Since $h_{\mathbf{k},S}$ is nonnegative and positively **k**-homogeneous by Proposition 6.6 (a), the equivalence of (b) and (c) is a consequence of Lemma 6.4.

(b)$\Longrightarrow$(a): Let $\mathbf{x}, \mathbf{y} \in S$ and $0 \leq t \leq 1$. Then by (b) and Proposition 6.6, one has

$$\begin{aligned} h_{\mathbf{k},S}((1-t)^{\mathbf{k}}\mathbf{x} + t^{\mathbf{k}}\mathbf{y}) &\leq h_{\mathbf{k},S}((1-t)^{\mathbf{k}}\mathbf{x}) + h_{\mathbf{k},S}(t^{\mathbf{k}}\mathbf{y}) \\ &= (1-t)h_{\mathbf{k},S}(\mathbf{x}) + th_{\mathbf{k},S}(\mathbf{y}) < (1-t) + t = 1. \end{aligned}$$

Hence making use of Proposition 6.6 once more, we have $(1-t)^{\mathbf{k}}\mathbf{x}+t^{\mathbf{k}}\mathbf{y} \in S$.

(a)$\Longrightarrow$(b): Let $\mathbf{x}, \mathbf{y} \in S$ and let $\epsilon > 0$. Put $\mathbf{x}_\epsilon, \mathbf{y}_\epsilon, \alpha, \beta$ be as in the proof of Lemma 6.4. Clearly, $\mathbf{x}_\epsilon, \mathbf{y}_\epsilon \in S$ by Proposition 6.6 (a) and (b). Note that $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. So the **k**-convexity of S implies that

$$\left(\frac{1}{h_{\mathbf{k},S}(\mathbf{x}) + h_{\mathbf{k},S}(\mathbf{y}) + \epsilon}\right)^{\mathbf{k}} (\mathbf{x} + \mathbf{y}) = \alpha^{\mathbf{k}}\mathbf{x}_\epsilon + \beta^{\mathbf{k}}\mathbf{y}_\epsilon \in S.$$

By using again Proposition 6.6 (a) and (b), one has $h_{\mathbf{k},S}(\mathbf{x} + \mathbf{y}) < h_{\mathbf{k},S}(\mathbf{x})+h_{\mathbf{k},S}(\mathbf{y})+\epsilon$, and the desired result follows since ϵ was arbitrary. □

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