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A NOTE ON ANALOGUE OF WIENER SPACE WITH VALUES IN ORLICZ SPACE

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Abstract. In this note we find the upper bound for $\rho(u^n, M) = \int_0^T \int_0^{|u(t)|^n} p(s)dsdt$ and show that $F(y) = y^n$ is m_ϕ^M-Bochner integrable on $C(\mathcal{O}_M)$ for $0 \le t \le T$ when $\int_{\mathcal{O}_M} \|u_0\|_M^n d\phi(u_0)$ is finite.

1. Introduction

In this section, we present some notations, definitions, Theorems and Remarks from [4][9].

(1) A real valued continuous function $M(u)$ is called an N-function if it is even and satisfies $\lim_{u\to 0} \frac{M(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$, equivalent to it admits of the representation $M(u) = \int_0^{|u|} p(t)dt$ when the function $p(t)$ is right continuous for $t \ge 0$, positive for $t > 0$ and nondecreasing which satisfies the condition $p(0) = 0$ and $\lim_{t\to\infty} p(t) = \infty$.

(2) Let $p(t)$ be a function which is positive for $t > 0$, right continuous for $t \ge 0$, nondecreasing, and satisfying conditions $p(0) = 0$ and $\lim_{t\to\infty} p(t) = \infty$. We defined the function $q(s)(s \ge 0)$ by the equality $q(s) = \sup_{p(t)\le s} t$. Then $q(s)$ is positive for $s > 0$, right continuous for $s \ge 0$, nondecreasing and satisfies $q(0) = 0$, $\lim_{s\to\infty} q(s) = \infty$. The functions $M(u) = \int_0^{|u|} p(t)dt$, $N(v) = \int_0^{|v|} q(s)ds$ are called mutually complementary N-functions.

(3) We say that the N-function $M(u)$ satisfies the Δ_2-condition for large values of u if there exist constants $k > 0, u_0 \ge 0$ such that

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$M(2u) \leq kM(u)$,$(u \geq u_0)$ and we say that the N-function $M(u)$ satisfies the Δ_a-condition if $\overline{\lim}_{u\to\infty}\frac{M(u^2)}{M(u)} < \infty$.

Remark 1.1. *(a) The* Δ_2*-condition is equivalent to* $M(lu) \leq k(l)M(u)$, $l > 1$ *for* $u \geq u_0$.

(b) The N*-function* $M(u)$ *satisfies the* Δ_2 *-condition is equivalent* $\overline{\lim}_{u\to\infty}\frac{M(2u)}{M(u)} < \infty$.

(c) If N*-function* $M(u)$ *satisfies the* Δ_2*-condition, then there are two constants* $\alpha > 1$, $c > 0$ *such that* $M(u) \leq c|u|^\alpha$ *for large value of* u.

(d) If N*-function* $M(u)$ *satisfies the* Δ_a*-condition,* $M(u)$ *satisfies the* Δ_2*-condition.*

Proof. Since $\overline{\lim}_{u\to\infty}\frac{M(u^2)}{M(u)} < \infty$, there exist $u_0 > 2$ and $k \geq 1$ such that $M(u^2) \leq kM(u)$ for $u \geq u_0$. If $u > 2$, then $2u < u^2$ and $M(2u) \leq M(u^2) \leq kM(u)$ for $u \geq u_0$. So $M(u)$ satisfies the Δ_2-condition. □

(4) For a N-function M and a measurable function $u : [0, T] \to R$, let $\rho(u, M) = \int_0^T M(u(t))dt = \int_0^T \int_0^{|u(t)|} p(s)dsdt$. The space $\mathcal{K}_M = \{u|u : [0, T] \to R, \rho(u, M) < \infty\}$ is called Orlicz class and let $\mathbf{K}_M$ be the space of all equivalence classes of functions in $\mathcal{K}_M$ which are equal almost everywhere with respect to the Lebesgue measure.

Remark 1.2. $\mathbf{K}_M$ *is linear iff* M *satisfies the* Δ_2*-condition.*

Lemma 1.3. *If* N*-function* $M(u)$ *satisfies the* Δ_a*-condition, then for* $k \geq 1$,

$$\int_0^{a^n} p(s)ds \leq k^n \int_0^a p(s)ds.$$

Proof. We show by mathematical induction. Let $n = 1$. Then $\int_0^a p(s)ds \leq k \int_0^a p(s)ds$, $k \geq 1$. Since $M(u)$ satisfies the Δ_a-condition, $\overline{\lim}_{u\to\infty}\frac{M(2u)}{M(u)} < \infty$. So there exist $k > 1, u_0 > 2$ such that $M(u^2) \leq kM(u)$ for $u \geq u_0$. Therefore $\int_0^{a^2} p(s)ds = M(a^2) \leq kM(a) = k\int_0^a p(s)ds \leq k^2 \int_0^a p(s)ds$ for $k \geq 1$. Assume that $n = m$. Then $\int_0^{a^m} p(s)ds \leq k^m \int_0^a p(s)ds$. If $n \geq m + 1$ n=2m,

$$\begin{aligned}\int_0^{a^n} p(s)ds = \int_0^{a^{2m}} p(s)ds = \int_0^{(a^m)^2} p(s)ds \leq k \int_0^{a^m} p(s)ds \\ \leq kk^m \int_0^a p(s)ds = k^{m+1} \int_0^a p(s)ds \leq k^{2m} \int_0^a p(s)ds.\end{aligned}$$

If $n = 2m+1$,

$$\int_0^{a^n} p(s)ds = \int_0^{a^{2m+1}} p(s)ds \le \int_0^{a^{2m+2}} p(s)ds = \int_0^{(a^{m+1})^2} p(s)ds$$

$$\le k \int_0^{a^{m+1}} p(s)ds \le k \int_0^{a^{2m}} p(s)ds \le kk^{2m} \int_0^{a} p(s)ds = k^{2m+1} \int_0^{a} p(s)ds.$$

Hence for all n, $\int_0^{a^n} p(s)ds \le k^n \int_0^a p(s)ds$. □

Lemma 1.4. *Let N-function $M(u)$ satisfies the Δ_a-condition. For a N-function $M(u)$ and a measurable function $u : [0,T] \to R$, let $\rho(u^n, M) = \int_0^T \int_0^{|u(t)|^n} p(s)dsdt$. Then*

$$\rho(u^n, M) \le k^n \int_0^T \max\{\int_0^{|u(t)|} p(s)ds, \int_0^a p(s)ds\}dt.$$

Proof. Let $A = \{t \in [0,T] | u(t) \ge a\}$ and let $B = [0,T] - A = \{t \in [0,T] | u(t) < a\}$.

$$\begin{aligned}
\rho(u^n, M) &= \int_0^T \int_0^{|u(t)|^n} p(s)dsdt \\
&= \int_A (\int_0^{|u(t)|^n} p(s)ds)dt + \int_B (\int_0^{|u(t)|^n} p(s)ds)dt \\
&\le \int_A (\int_0^{|u(t)|^n} p(s)ds)dt + \int_B (\int_0^{a^n} p(s)ds)dt \\
&\le \int_A k^n (\int_0^{|u(t)|} p(s)ds)dt + \int_B k^n (\int_0^{a} p(s)ds)dt \\
&\le k^n \int_0^T \max\{\int_0^{|u(t)|} p(s)ds, \int_0^a p(s)ds\}dt.
\end{aligned}$$

□

(5) Let M and N be mutually complementary N-functions. We let $\mathcal{D}_M = \{u \in \mathbf{K}_M | u : [0,T] \to R$ is measurable such that for all v in $\mathbf{K}_N$, $(u,v) = \int_{[0,T]} u(t)v(t)dt < \infty\}$. Let $\mathcal{O}_M$ be the space of all equivalence classes of functions in $\mathcal{D}_M$ which are equal almost everywhere with respect to the Lebegue measure. From Young's inequality, we have $\mathbf{K}_M \subset \mathcal{O}_M$. For u in $\mathcal{O}_M$, $\|u\|_M = \sup_{\rho(v,N)\le 1} |(u,v)|$ is called the Orlicz norm of u and $\|u\|_{(M)} = \inf k$, where the infimum is taken over all $k > 0$ such that $\rho(u/k, M) \le 1$, is called Luxemberg norm of u.

Remark 1.5. *(a) For u in $\mathcal{O}_M$,$\|u\|_M = 1 + \rho(u, M)$.*

(b) If M satisfies the Δ_2-condition, then $(\mathcal{O}_M\ ,\|u\|_M\ )$ is a separable Banach space and $\mathbf{K}_M = \mathcal{O}_M$.

(c) For u in $\mathcal{O}_M$,$\|u\|_{(M)} \le \|u\|_M \le 2\|u\|_{(M)}$.

(d) Let M and N be mutually complementary N-functions. Let E_M be the closure of L_∞ with respect to the topology generated by the norm $\|\cdot\|_M$ and V^* be the dual space of the normed vector space V. Then $(E_M, \|\cdot\|_{(M)})^* = (\mathcal{O}_N, \|\cdot\|_N)$ and $(E_M, \|\cdot\|_M)^* = (\mathcal{O}_N, \|\cdot\|_{(N)})$.

(e) If M satisfies the Δ_2-condition, then $E_M = \mathcal{O}_M = \mathbf{K}_M$. So if M satisfies the Δ_2-condition, then $(\mathcal{O}_M, \|\cdot\|_M)$ is reflexive.

(f) Since $L_\infty \subset L_2 \subset \mathcal{O}_M$, the closure of L_2 with respect to the topology generated by the $\|\cdot\|_M$ is $\mathcal{O}_M$.

(g) Let M and N be mutually complementary N-functions. For u in $\mathcal{O}_M$ and v in $\mathcal{O}_N$, $|(u,v)| \le \rho(u,M) + \rho(v,N)$, $|(u,v)| \le \|u\|_M\|v\|_{(N)}$, and $|(u,v)| \le \|u\|_{(M)}\|v\|_N$. Hence for u in L_2, $\|u\|_2 \le \|u\|_M \le 2\|u\|_{(M)}$.

(h) If M satisfies the Δ_a-condition, then for u in $\mathcal{O}_M$, u^2 belongs to $\mathcal{O}_M$.

(i) If M satisfies the Δ_a-condition, then for u, v in $\mathcal{O}_M$, there exists a constant c such that $\|uv\|_M \le c\|u\|_M\|v\|_M$.

(6) A subset I of L_2 of the form $I = \{u \in L_2 | P(u) \in F\}$ is called a cylinder set where P is a finite dimensional orthogonal projection of L_2 and F is a Borel subset of $P(L_2)$. The Gaussian measure on L_2 is a set function of all cylinder sets defined as follows: If $I = \{u \in L_2 | P(u) \in F\}$ then $\mu(I) = (2\pi)^{-n/2} \int_F e^{-\|t\|^2/2} dt$ where n is the dimension of $P(L_2)$. Then μ is not σ-finite. Suppose $\{e_n | n \in N\}$ be an orthonormal basis of L_2. Let $\mu_{e_1,\cdots,e_n}(F) = \mu\{u \in L_2 | ((u,e_1),(u,e_2),\cdots,(u,e_n)) \in F\}$. Then $\{\mu_{e_1,\cdots,e_n}\}$ is a consistence family of probability measure. By Kolomogorov's theorem, there exists a probability measure space (Ω, ω) and random variables $\xi_n : \Omega \to R$ $(n \in N)$ such that $\omega(\{z \in \Omega | (\xi_1(z), \xi_2(z), \cdots, \xi_n(z)) \in F\}) = \mu_{e_1,\cdots,e_n}(F)$. Without loss of generality, we can put $\Omega = \mathcal{O}_M$ because $\mathcal{O}_M \subset L_0$, the space of all measurable functions on $[0,T]$ with the topology of convergence in measure.

Remark 1.6. *(a) $\mathcal{O}_M$ is a closed subset of L_0.*

(b) For nonzero v in $\mathcal{O}_N$ and for a real number a,

$$\omega(\{u \in \mathcal{O}_M | (u,v) < a\}) = \frac{1}{\sqrt{2\pi\|v\|_{(N)}}} \int_{-\infty}^{a} e^{-t^2/(2\|v\|_{(N)})} dt.$$

(7) For two Borel measures m_1 and m_2, we let $m_1 * m_2(E+F) = m_1 \times m_2(E \times F)$ for E, F in $\mathcal{B}(\mathcal{O}_M)$, the set of all Borel subsets of $\mathcal{O}_M$. For $\lambda > 0$ and for B in $\mathcal{B}(\mathcal{O}_M)$, let $\omega_\lambda(B) = \omega(\lambda^{-1/2}B)$. Then for two positive real numbers s and t, $\omega_\lambda * \omega = \omega_{\sqrt{s^2+t^2}}$ and $\omega_\lambda * \delta_0 = \omega_\lambda$, where δ_0 is the Dirac measure centered at 0.

2. The analogue of Wiener space with value in Orlicz Space

In this section, let M be an N-function which satisfies the Δ_a-condition. Let M and N be mutually complementary N-functions. Let $C(\mathcal{O}_M)$ be the space of all continuous functions defined on the interval $[0,T]$ with values in $\mathcal{O}_M$ in the norm $\|y\|_{C(\mathcal{O}_M)} = \sup_{0\le t\le T}\|y(t)\|_M$ and ϕ be a probability Borel measure on $\mathcal{O}_M$. Let $\vec{t} = (t_0, t_1, \cdots, t_n)$ be given with $0 = t_0 < t_1 < \cdots < t_n \le T$ and let $T_{\vec{t}} : \mathcal{O}_M^{n+1} \to \mathcal{O}_M^{n+1}$ be a function given by

$$T_{\vec{t}}(x_0, x_1, \cdots, x_n) = (x_0, x_0 + \sqrt{t_1}x_1, \cdots, x_0 + \sum_{j=1}^{n}\sqrt{t_j - t_{j-1}}x_j).$$

We define a set function $v_{\vec{t}}^{\phi}$ on $\mathcal{B}(\mathcal{O}_M^{n+1})$ given by

$$v_{\vec{t}}^{\phi}(B) = \int_{\mathcal{O}_M}[\int_{\mathcal{O}_M^n}(\chi_B \circ T_{\vec{t}})((x_0, x_1, \cdots, x_n))d(\prod_{j=1}^{n}\omega)(x_1, \cdots, x_n)]d\phi(x_0),$$

where χ_B is a characteristic function associated with B. Then $v_{\vec{t}}^{\phi}$ is a Borel measure on $(\mathcal{O}_M^{n+1}, \mathcal{B}(\mathcal{O}_M^{n+1}))$. Let $J_{\vec{t}} : C(\mathcal{O}_M) \to \mathcal{O}_M^{n+1}$ be a function with $J_{\vec{t}}(y) = (y(t_0), y(t_1), \cdots, y(t_n))$. For Borel subsets $B_0, B_1, \cdots, B_n$ in $\mathcal{B}(\mathcal{O}_M)$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n}B_j)$ of $C(\mathcal{O}_M)$ is called an interval. Let $\mathcal{J}$ be the set all such intervals. Then from [7], $\mathcal{J}$ is a semi algebra. We define a set function M_ϕ on $\mathcal{J}$ by $M_\phi(J_{\vec{t}}^{-1}(\prod_{j=0}^{n}B_j)) = v_{\vec{t}}^{\phi}(\prod_{j=0}^{n}B_j)$. Then from [7], M_ϕ is well defined on $\mathcal{J}$, $\mathcal{B}(C(\mathcal{O}_M))$ coincides with the smallest σ-algebra generated by $\mathcal{J}$ and there exists a unique measure m_ϕ^M on $(C(\mathcal{O}_M), \mathcal{B}(C(\mathcal{O}_M)))$ such that $m_\phi^M(I) = M_\phi(I)$ for all I in $\mathcal{J}$. This measure space $(C(\mathcal{O}_M), \mathcal{B}(C(\mathcal{O}_M)), m_\phi^M)$ is called the analogue of Wiener measure space with values in Orlicz Space.

From the change of variable theorem, we have the following two theorems from [9].

Theorem 2.1. *If $f : \mathcal{O}_M^{n+1} \to R$ is Borel measurable and $F : C(\mathcal{O}_M) \to R$ is a function with $F(y) = f(y(t_0), y(t_1), \cdots, y(t_n))$ then the following equality holds*

$$\int_{C(\mathcal{O}_M)} F(y) dm_\phi^M(y) = \int_{C(\mathcal{O}_M)} f(y(t_0), y(t_1), \cdots, y(t_n)) dm_\phi^M(y)$$
$$\doteq \int_{\mathcal{O}_M} [\int_{\mathcal{O}_M^n} (f \circ T_{\vec{t}})((x_0, x_1, \cdots, x_n)) d(\prod_{j=1}^n \omega)(x_1, \cdots, x_n)] d\phi(x_0)$$

where $\doteq$ means that if one side exists then both sides exist and the two values are equal.

Theorem 2.2. *If $f : R^{n+1} \to R$ is Borel measurable and v is a nonzero element in $\mathcal{O}_N$.*

$$\int_{C(\mathcal{O}_M)} f((v, y(t_0)), (v, y(t_1)), \cdots, (v, y(t_n))) dm_\phi^M(y)$$
$$\doteq \{(2\pi)^n \|v\|_{(N)} \prod_{j=1}^n \sqrt{t_j - t_{j-1}}\}^{-1/2} \int_R [\int_{R^n} f(s_0, s_1, \cdots, s_n)$$
$$\exp\{-\frac{1}{2\|v\|_{(N)}} \sum_{j=1}^n \frac{(s_j - s_{j-1})^2}{t_j - t_{j-1}}\} ds_n ds_{n-1} \cdot ds_1] d\phi(s_0)$$

where $\doteq$ means that if one side exists, then both sides exist and the two values are equal.

Lemma 2.3. *Let M satisfies the Δ_a-condition. For $u \in \mathcal{O}_M$,$\|u^n\|_M \le \rho(u^n, M) + 1$.*

Proof. By Remark 1.5 (a) and (h), $\|u\|_M = 1 + \rho(u, M)$. Since $\mathcal{O}_M$ is linear, $u^n \in \mathcal{O}_M$. So we replace u with u^n. Hence $\|u^n\|_M \le \rho(u^n, M) + 1$. □

Theorem 2.4. *Suppose $\int_{\mathcal{O}_M} \|u_0\|_M^n d\phi(u_0)$ is finite. Then for $0 \le t \le T$, $F(y) = y^n$ is m_ϕ^M-Bochner integrable on $C(\mathcal{O}_M)$.*

Proof. Let D be the set of all rational numbers in $[0, T]$. Then we can write $D = \{t_n|$ n is a natural numbers $\}$. For a natural number m, let $D_m = \{t_1, t_2, \cdots, t_m\}$. Then by the monotone convergence theorem and by Theorem 2.1, and let $A = \max\{\rho(u_1^k, M) + 1 | k = 0, 1, 2, \cdots, n\}$.

$$\begin{aligned}
&\int_{C(\mathcal{O}_M)} \|y^n\|_{C(\mathcal{O}_M)} dm_\phi^M(y) = \int_{C(\mathcal{O}_M)} \sup_{t\in D} \|y(t)^n\|_M dm_\phi^M(y) \\
&= \int_{C(\mathcal{O}_M)} \lim_{m\to\infty} \sup_{t\in D_m} \|y(t)^n\|_M dm_\phi^M(y) \\
&= \lim_{m\to\infty} \int_{C(\mathcal{O}_M)} \sup_{t\in D_m} \|y(t)^n\|_M dm_\phi^M(y) \\
&= \lim_{m\to\infty} \int_{\mathcal{O}_M} \int_{\mathcal{O}_M} \sup_{t\in D_m} \|(u_0 + \sqrt{t}u_1)^n\|_M d\omega(u_1) d\phi(u_0) \\
&\le \lim_{m\to\infty} \int_{\mathcal{O}_M} \int_{\mathcal{O}_M} \sum_{k=0}^{n} \binom{n}{k} \|(\sqrt{t}u_1)^k\|_M \|u_0^{n-k}\|_M d\omega(u_1) d\phi(u_0) \\
&\le \sum_{k=0}^{n} \binom{n}{k} (\sqrt{T})^k \int_{\mathcal{O}_M} \int_{\mathcal{O}_M} \|u_1^k\|_M \|u_0^{n-k}\|_M d\omega(u_1) d\phi(u_0) \\
&\le \sum_{k=0}^{n} \binom{n}{k} (\sqrt{T})^k \int_{\mathcal{O}_M} [\rho(u_1^k, M) + 1] d\omega(u_1) \int_{\mathcal{O}_M} \|u_0\|_M^{n-k} d\phi(u_0) \\
&\le \sum_{k=0}^{n} \binom{n}{k} (\sqrt{T})^k \cdot A\omega(\mathcal{O}_M) \int_{\mathcal{O}_M} \|u_0\|_M^{n-k} d\phi(u_0) < \infty.
\end{aligned}$$

Since F is weakly measurable and $C(\mathcal{O}_M)$ is separable, from [2], $F(y) = y^n$ is m_ϕ^M-Bochner integrable on $C(\mathcal{O}_M)$. □

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