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STABILITY OF A 3-DIMENSIONAL QUADRATIC-ADDITIVE TYPE FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate a stability problem for a functional equation

$$f(-x-y-z)-f(x+y)-f(y+z)-f(x+z)$$
$$+2f(x)+2f(y)+2f(z)-f(-x)-f(-y)-f(-z)=0$$

by applying the direct method.

1. Introduction

In 1940, Ulam [9] proposed the stability problem of of the additive functional equation

$$f(x+y)-f(x)-f(y)=0. \tag{1}$$

In 1941, Hyers [3] gave an affirmative answer to this problem between Banach spaces. Subsequently many mathematicians came to deal with this problem (cf. [1, 2, 8]).

A solution of the functional equation (1) is called an additive mapping and a solution of the functional equation

$$f(x+y)-f(x-y)-2f(x)-2f(y)=0, \tag{2}$$

is called a quadratic mapping. A mapping f is called a quadratic-additive mapping if f is represented by sum of a quadratic mapping and an additive mapping [6]. A functional equation is called a quadratic-additive type functional equation provided that each solution of that

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equation is a quadratic-additive mapping, and vice versa [6]. Now we consider the following functional equation

$$\begin{aligned} &f(-x-y-z)-f(x+y)-f(y+z)-f(x+z) \\ &\quad +2f(x)+2f(y)+2f(z)-f(-x)-f(-y)-f(-z)=0. \end{aligned} \tag{3}$$

The mapping $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x):=ax^2+bx$ is a solution of this functional equation, where a,b are real constants.

In this paper, we will show that every solution of functional equation (3) is a quadratic-additive mapping and we will prove the stability of functional equation (3) by using the direct method presented by Hyers in [3]. Namely, starting from the given mapping f that approximately satisfies the functional equation (3), a solution F of the functional equation (3) is explicitly constructed as either

$$\lim_{n\to\infty}\left(\frac{f(2^nx)+f(-2^nx)}{2\cdot 2^{2n}}+\frac{f(2^nx)-f(-2^nx)}{2\cdot 2^n}\right),$$

or

$$\lim_{n\to\infty}\left(\frac{2^{2n}}{2}\left(f\left(\frac{x}{2^n}\right)+f\left(\frac{-x}{2^n}\right)\right)+\frac{2^n}{2}\left(f\left(\frac{x}{2^n}\right)-f\left(\frac{-x}{2^n}\right)\right)\right),$$

or

$$\lim_{n\to\infty}\left(\frac{f(2^nx)+f(-2^nx)}{2\cdot 2^{2n}}+\frac{2^n}{2}\left(f\left(\frac{x}{2^n}\right)-f\left(\frac{-x}{2^n}\right)\right)\right)$$

which approximate the mapping f.

2. Preliminaries

Throughout this paper, let V and W be real vector spaces, let X be a normed space, and let Y be a Banach space. For a given mapping $f:V\to W$, we use the following abbreviations

$$\begin{aligned} f_o(x) &:= \frac{f(x)-f(-x)}{2}, \\ f_e(x) &:= \frac{f(x)+f(-x)}{2}, \\ Af(x,y) &:= f(x+y)-f(x)-f(y), \\ Qf(x,y) &:= f(x+y)+f(x-y)-2f(x)-2f(y), \\ Df(x,y,z) &:= f(-x-y-z)-f(x+y)-f(y+z)-f(x+z) \\ &\quad +2f(x)+2f(y)+2f(z)-f(-x)-f(-y)-f(-z) \end{aligned}$$

for all $x, y, z \in V$. Now we will show that the functional equation (3) is quadratic-additive type functional equation.

Lemma 2.1. *A mapping $f : V \to W$ satisfies the functional equation $Df(x, y, z) = 0$ if and only if f is a quadratic-additive mapping.*

Proof. If $f : V \to W$ is a solution of the functional equation $Df(x, y, z) = 0$, then $f(0) = Df(0, 0, 0) = 0$. Since $f(0) = 0$, $f_e(-x) = f_e(x)$, and $f_o(-x) = -f_o(x)$ hold for all $x \in V$, we get the desired equalities

$$\begin{aligned} Qf_e(x, y) &= -Df_e(x, y, -y) = 0, \\ Af_o(x, y) &= Df_o\left(\frac{x+y}{2}, \frac{x-y}{2}, \frac{y-x}{2}\right) - Df_o\left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{-x-y}{2}\right) = 0 \end{aligned}$$

for all $x, y \in V$.

Conversely, let $f : V \to W$ be a quadratic-additive mapping which is represented by $f = g + h$, where g is a quadratic mapping and h is an additive mapping. Because $g : V \to W$ is a quadratic mapping and $h : V \to W$ is an additive mapping, the equalities $g(-x) = g(x)$, $g(x) = 4g\left(\frac{x}{2}\right)$, and $h(-x) = -h(x)$ hold for all $x \in V$. So we obtain the equalities

$$\begin{aligned}
&Dg(\ x,y,z)\\
&\quad = g(x+y+z) - g(x+y) - g(x+z) - g(y+z) + g(x) + g(y) + g(z)\\
&\quad = g(x+y+z) + g(x-y) - 2g\left(x+\frac{z}{2}\right) - 2g\left(y+\frac{z}{2}\right)\\
&\qquad - g(x+z) - g(x) + 2g\left(x+\frac{z}{2}\right) + 2g\left(\frac{z}{2}\right)\\
&\qquad - g(y+z) - g(y) + 2g\left(y+\frac{z}{2}\right) + 2g\left(\frac{z}{2}\right)\\
&\qquad - g(x+y) - g(x-y) + 2g(x) + 2g(y)\\
&\quad = Qg\left(x+\frac{z}{2}, y+\frac{z}{2}\right) - Qg\left(x+\frac{z}{2}, \frac{z}{2}\right) - Qg\left(y+\frac{z}{2}, \frac{z}{2}\right) - Qg(x,y)\\
&\quad = 0,\\
&Dh(\ x,y,z)\\
&\quad = \ - h(x+y+z) - h(x+y) - h(x+z) - h(y+z)\\
&\qquad + 3h(x) + 3h(y) + 3h(z)\\
&\quad = \ - \big(h(x+y+z) - h(x+y) - h(z)\big) - 2\big(h(x+y) - h(x) - h(y)\big)\\
&\qquad - \big(h(x+z) - h(x) - h(z)\big) - \big(h(y+z) - h(y) - h(z)\big)\\
&\quad = \ - Ah(x+y,z) - 2Ah(x,y) - Ah(x,z) - Ah(y,z)\\
&\quad = 0
\end{aligned}$$

for all $x, y, z \in V$ for all $x, y, z \in V$, which imply that

$$Df(x,y,z) = Dg(x,y,z) + Dh(x,y,z) = 0$$

for all $x, y, z \in V$. □

Lemma 2.2. *If $f : V \to W$ is a mapping such that $Df(x,y,z) = 0$ for all $x, y, z \in V \backslash \{0\}$, then*

$$Df(x,y,z) = 0$$

for all $x, y, z \in V$.

Proof. For any $x \in V\backslash\{0\}$, we get

$$f(0) = \frac{Df(2x,-x,-x) + Df(-2x,x,x)}{2} = 0,$$

which implies that

$$f(0) = 0,$$

$$Af_o(x,y) = \begin{cases} Df_o\left(\frac{x+y}{2}, \frac{x-y}{2}, \frac{y-x}{2}\right) \\ -Df_o\left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{-x-y}{2}\right) = 0 & \text{if } y \notin \{x, -x\}, \\ \frac{1}{4}Df_o(2x, -x, -x) = 0 & \text{if } y = x, \\ 0 & \text{if } y = -x \end{cases}$$

for all $x, y \in V\backslash\{0\}$. Because $f_e(-x) = f_e(x)$, $f_o(-x) = -f_o(x)$, and $Af_o(x,y) = 0$ for all $x, y \in V\backslash\{0\}$, we easily know that $Df_e(x,y,0) = 0$, $Df_e(x,0,z) = 0$, $Df_e(0,y,z) = 0$, $Df_e(x,0,0) = 0$, $Df_e(0,0,z) = 0$, $Df_e(0,y,0) = 0$, $Df_e(0,0,0) = 0$, $Df_o(x,y,0) = 0$, $Df_o(x,0,z) = 0$, $Df_o(0,y,z) = 0$, $Df_o(x,0,0) = 0$, $Df_o(0,0,z) = 0$, $Df_o(0,y,0) = 0$, $Df_o(0,0,0) = 0$ for all $x, y, z \in V\backslash\{0\}$. Since $Df(x,y,z) = Df_e(x,y,z) + Df_o(x,y,z)$ for all $x, y, z \in V$, the equality $Df(x,y,z) = 0$ holds for all $x, y, z \in V$ as we desired. □

The following lemmas are the same as [7, Corollary 4, Corollary 5].

Lemma 2.3. *Let $a > 1$ be a rational number, let $\phi : V\backslash\{0\} \to [0,\infty)$ be a function satisfying either*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{a^i}\phi(a^i x) < \infty, \tag{4}$$

for all $x \in V\backslash\{0\}$ or

$$\Phi(x) := \sum_{i=0}^{\infty} a^{2i}\phi\left(\frac{x}{a^i}\right) < \infty \tag{5}$$

for all $x \in V\backslash\{0\}$, and let $f : V \to Y$ be an arbitrarily given mapping. If there exists a quadratic-additive mapping $F : V \to Y$ satisfying

$$\|f(x) - F(x)\| \leq \Phi(x) \tag{6}$$

for all $x \in V\backslash\{0\}$, then F is a unique quadratic-additive mapping satisfying (6).

Lemma 2.4. *Let $a > 1$ be a rational number, let $\phi, \psi : V\backslash\{0\} \to [0, \infty)$ be functions satisfying each of the following conditions*

$$\begin{aligned}&\sum_{i=0}^{\infty} a^i \psi\left(\frac{x}{a^i}\right) < \infty, &&\sum_{i=0}^{\infty} \frac{1}{a^{2i}} \phi(a^i x) < \infty,\\ &\tilde{\Phi}(x) := \sum_{i=0}^{\infty} a^i \phi\left(\frac{x}{a^i}\right) < \infty, &&\tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{a^{2i}} \psi(a^i x) < \infty\end{aligned} \tag{7}$$

for all $x \in V\backslash\{0\}$, and let $f : V \to Y$ be an arbitrarily given mapping. If there exists a quadratic-additive mapping $F : V \to Y$ satisfying the inequality

$$\|f(x) - F(x)\| \leq \tilde{\Phi}(x) + \tilde{\Psi}(x) \tag{8}$$

for all $x \in V\backslash\{0\}$, then F is a unique quadratic-additive mapping satisfying (8).

3. Main results

Theorem 3.1. *Let V be a real vector space, let Y be a real Banach space, and let $\varphi : (V\backslash\{0\})^3 \to [0, \infty)$ be a function satisfying the condition*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 2^i z)}{2^i} < \infty \tag{9}$$

for all $x, y, z \in V\backslash\{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and

$$\|Df(x, y, z)\| \leq \varphi(x, y, z) \tag{10}$$

for all $x, y, z \in V\backslash\{0\}$, then there exists a unique quadratic-additive mapping $F : V \to Y$ satisfying

$$\begin{aligned}\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \bigg(&\frac{\varphi(2^i x, 2^i x, -2^i x) + \varphi(2^i x, -2^i x, -2^i x)}{2 \cdot 2^{2i+2}}\\ &+ \frac{\varphi(2^i x, 2^i x, -2^i x) + \varphi(2^i x, -2^i x, -2^i x)}{2 \cdot 2^{i+1}} \bigg)\end{aligned} \tag{11}$$

for all $x \in V\backslash\{0\}$.

Proof. It follows from (10) that

$$\begin{aligned}
&\left\| \frac{f(2^n x) + f(-2^n x)}{2 \cdot 2^{2n}} + \frac{f(2^n x) - f(-2^n x)}{2 \cdot 2^n} \right. \\
&\qquad \left. - \frac{f(2^{n+m} x) + f(-2^{n+m} x)}{2 \cdot 2^{2n+2m}} - \frac{f(2^{n+m} x) - f(-2^{n+m} x)}{2 \cdot 2^{n+m}} \right\| \\
&\qquad \le \sum_{i=n}^{n+m-1} \left\| \frac{f(2^i x) + f(-2^i x)}{2 \cdot 2^{2i}} + \frac{f(2^i x) - f(-2^i x)}{2 \cdot 2^i} \right. \\
&\qquad \left. - \frac{f(2^{i+1} x) + f(-2^{i+1} x)}{2 \cdot 2^{2i+2}} - \frac{f(2^{i+1} x) - f(-2^{i+1} x)}{2 \cdot 2^{i+1}} \right\| \\
&\qquad = \sum_{i=n}^{n+m-1} \left\| \frac{Df(2^i x, 2^i x, -2^i x) + Df(2^i x, -2^i x, -2^i x)}{2 \cdot 2^{2i+2}} \right. \\
&\qquad \left. + \frac{Df(2^i x, 2^i x, -2^i x) - Df(2^i x, -2^i x, -2^i x)}{2 \cdot 2^{i+1}} \right\| \\
&\qquad \le \sum_{i=n}^{n+m-1} \left(\frac{\varphi(2^i x, 2^i x, -2^i x) + \varphi(2^i x, -2^i x, -2^i x)}{2 \cdot 2^{2i+2}} \right. \\
&\qquad \left. + \frac{\varphi(2^i x, 2^i x, -2^i x) + \varphi(2^i x, -2^i x, -2^i x)}{2 \cdot 2^{i+1}} \right) \qquad (12)
\end{aligned}$$

for all $x \in V \backslash \{0\}$ and $m, n \in \mathbb{N} \cup \{0\}$. So, it is easy to show that the sequence $\{\frac{f(2^n x)+f(-2^n x)}{2 \cdot 2^{2n}} + \frac{f(2^n x)-f(-2^n x)}{2 \cdot 2^n}\}$ is a Cauchy sequence for all $x \in V$. Since Y is complete and $f(0) = 0$, the sequence $\{\frac{f(2^n x)+f(-2^n x)}{2 \cdot 2^{2n}} + \frac{f(2^n x)-f(-2^n x)}{2 \cdot 2^n}\}$ converges for all $x \in V \backslash \{0\}$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n \to \infty} \left(\frac{f(2^n x) + f(-2^n x)}{2 \cdot 2^{2n}} + \frac{f(2^n x) - f(-2^n x)}{2 \cdot 2^n} \right)$$

for all $x \in V$. Moreover, if we put $n = 0$ and let $m \to \infty$ in (12), we obtain the inequality (11). From the definition of F, we get

$$\begin{aligned}
\|DF(x,y,z)\| &= \lim_{n\to\infty} \Bigg\| \frac{Df\big(2^n x, 2^n y, 2^n z\big) + Df\big(-2^n x, -2^n y, -2^n z\big)}{2\cdot 2^{2n}} \\
&\qquad + \frac{Df\big(2^n x, 2^n y, 2^n z\big) - Df\big(-2^n x, -2^n y, -2^n z\big)}{2\cdot 2^{n}} \Bigg\| \\
&\le \lim_{n\to\infty} \frac{\varphi\big(2^n x, 2^n y, 2^n z\big) + \varphi\big(-2^n x, -2^n y, -2^n z\big)}{2\cdot 2^{2n}} \\
&\qquad + \frac{\varphi\big(2^n x, 2^n y, 2^n z\big) + \varphi\big(-2^n x, -2^n y, -2^n z\big)}{2\cdot 2^{n}} \\
&= 0
\end{aligned}$$

for all $x, y, z \in V\backslash\{0\}$ i.e, $DF(x,y,z) = 0$ holds for all $x, y, z \in V$ by Lemma 2.2. By Lemma 2.1, F is a quadratic-additive mapping. In view of Lemma 2.3, there exists a unique quadratic-additive mapping $F : V \to Y$ satisfying the inequality (11), since the inequality

$$\begin{aligned}
\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \Bigg(\frac{\varphi(2^i x, 2^i x, -2^i x) + \varphi(2^i x, -2^i x, -2^i x)}{2\cdot 2^{2i+2}} \\
&\qquad + \frac{\varphi(2^i x, 2^i x, -2^i x) + \varphi(2^i x, -2^i x, -2^i x)}{2\cdot 2^{i+1}} \Bigg) \\
&\le \sum_{i=0}^{\infty} \frac{\phi(2^i x)}{2^i}
\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $a = 2$ and $\phi(x) = \frac{\varphi(x,x,-x)+\varphi(x,-x,-x)}{2}$. □

Theorem 3.2. *Let $\varphi : (V\backslash\{0\})^3 \to [0, \infty)$ be a function satisfying the condition*

$$\sum_{i=0}^{\infty} 2^{2i} \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}\right) < \infty \tag{13}$$

for all $x, y, z \in V\backslash\{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and the inequality (10) *for all $x, y, z \in V\backslash\{0\}$, then there exists a unique quadratic-additive mapping $F : V \to Y$ satisfying*

$$\begin{aligned}
\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \Bigg(&\frac{2^{2i}}{2} \left(\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{-x}{2^{i+1}}\right) + \varphi\left(\frac{x}{2^{i+1}}, \frac{-x}{2^{i+1}}, \frac{-x}{2^{i+1}}\right) \right) \\
&+ \frac{2^{i}}{2} \left(\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{-x}{2^{i+1}}\right) + \varphi\left(\frac{x}{2^{i+1}}, \frac{-x}{2^{i+1}}, \frac{-x}{2^{i+1}}\right) \right) \Bigg)
\end{aligned} \tag{14}$$

for all $x \in V\backslash\{0\}$.

Proof. It follows from (10) that

$$\begin{aligned}
&\Big\| \frac{2^{2n}}{2}\Big(f\Big(\frac{x}{2^n}\Big)+f\Big(\frac{-x}{2^n}\Big)\Big)+\frac{2^n}{2}\Big(f\Big(\frac{x}{2^n}\Big)-f\Big(\frac{-x}{2^n}\Big)\Big) \\
&\qquad -\frac{2^{2n+2m}}{2}\Big(f\Big(\frac{x}{2^{n+m}}\Big)+f\Big(\frac{-x}{2^{n+m}}\Big)\Big) \\
&\qquad -\frac{2^{n+m}}{2}\Big(f\Big(\frac{x}{2^{n+m}}\Big)-f\Big(\frac{-x}{2^{n+m}}\Big)\Big)\Big\| \\
&\le \sum_{i=n}^{n+m-1}\Big\| \frac{2^{2i}}{2}\Big(f\Big(\frac{x}{2^i}\Big)+f\Big(\frac{-x}{2^i}\Big)\Big)+\frac{2^i}{2}\Big(f\Big(\frac{x}{2^i}\Big)-f\Big(\frac{-x}{2^i}\Big)\Big) \\
&\quad -\frac{2^{2i+2}}{2}\Big(f\Big(\frac{x}{2^{i+1}}\Big)+f\Big(\frac{-x}{2^{i+1}}\Big)\Big)-\frac{2^{i+1}}{2}\Big(f\Big(\frac{x}{2^{i+1}}\Big)-f\Big(\frac{-x}{2^{i+1}}\Big)\Big)\Big\| \\
&= \sum_{i=n}^{n+m-1}\Big\| \frac{2^{2i}}{2}\Big(-Df\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)-Df\Big(\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)\Big) \\
&\quad -\frac{2^i}{2}\Big(Df\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)-Df\Big(\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)\Big)\Big\| \\
&\le \sum_{i=n}^{n+m-1}\Big(\frac{2^{2i}}{2}\Big(\varphi\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)+\varphi\Big(\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)\Big) \\
&\quad +\frac{2^i}{2}\Big(\varphi\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)+\varphi\Big(\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)\Big)\Big)
\end{aligned} \tag{15}$$

for all $x \in V\backslash\{0\}$ and $m,n \in \mathbb{N}\cup\{0\}$. So, it is easy to show that the sequence $\{\frac{2^{2n}}{2}(f(\frac{x}{2^n})+f(\frac{-x}{2^n}))+\frac{2^n}{2}(f(\frac{x}{2^n})-f(\frac{-x}{2^n}))\}$ is a Cauchy sequence for all $x \in V$. Since Y is complete and $f(0)=0$, the sequence $\{\frac{2^{2n}}{2}(f(\frac{x}{2^n})+f(\frac{-x}{2^n}))+\frac{2^n}{2}(f(\frac{x}{2^n})-f(\frac{-x}{2^n}))\}$ converges for all $x \in V\backslash\{0\}$. Hence, we can define a mapping $F: V \to Y$ by

$$F(x) := \lim_{n\to\infty}\Big[\frac{2^{2n}}{2}\Big(f\Big(\frac{x}{2^n}\Big)+f\Big(\frac{-x}{2^n}\Big)\Big)+\frac{2^n}{2}\Big(f\Big(\frac{x}{2^n}\Big)-f\Big(\frac{-x}{2^n}\Big)\Big)\Big]$$

for all $x \in V$. Moreover, if we put $n = 0$ and let $m \to \infty$ in (15), we obtain the inequality (14). From the definition of F, we get

$$\begin{aligned}
\|DF(x,y,z)\| &= \lim_{n\to\infty} \bigg\| \frac{2^{2n}}{2}\bigg(Df\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) + Df\Big(\frac{-x}{2^n},\frac{-y}{2^n},\frac{-z}{2^n}\Big)\bigg) \\
&\quad + \frac{2^n}{2}\bigg(Df\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) - Df\Big(\frac{-x}{2^n},\frac{-y}{2^n},\frac{-z}{2^n}\Big)\bigg)\bigg\| \\
&\le \lim_{n\to\infty} \bigg[\frac{2^{2n}}{2}\bigg(\varphi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) + \varphi\Big(\frac{-x}{2^n},\frac{-y}{2^n},\frac{-z}{2^n}\Big)\bigg) \\
&\quad + \frac{2^n}{2}\bigg(\varphi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) + \varphi\Big(\frac{-x}{2^n},\frac{-y}{2^n},\frac{-z}{2^n}\Big)\bigg)\bigg] \\
&= 0
\end{aligned}$$

for all $x, y, z \in V\backslash\{0\}$, i.e, $DF(x, y, z) = 0$ for all $x, y, z \in V$ by Lemma 2.2. By Lemma 2.1, F is a quadratic-additive mapping. In view of Lemma 2.3, there exists a unique quadratic-additive mapping $F : V \to Y$ satisfying the inequality (14), since the inequality

$$\begin{aligned}
\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \bigg[\frac{2^{2i}}{2}\bigg(\varphi\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big) + \varphi\Big(\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)\bigg) \\
&\quad + \frac{2^i}{2}\bigg(\varphi\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big) + \varphi\Big(\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)\bigg)\bigg] \\
&\le \sum_{i=0}^{\infty} 2^{2i}\phi\Big(\frac{x}{2^i}\Big)
\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $a = 2$ and $\phi(x) = \varphi(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}) + \varphi(\frac{x}{2}, -\frac{x}{2}, -\frac{x}{2})$. □

Theorem 3.3. *Let $\varphi : (V\backslash\{0\})^3 \to [0,\infty)$ be a function satisfying the conditions*

$$(16) \qquad \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 2^i z)}{2^{2i}} < \infty, \qquad \sum_{i=0}^{\infty} 2^i \varphi\Big(\frac{x}{2^i},\frac{y}{2^i},\frac{z}{2^i}\Big) < \infty$$

for all $x, y, z \in V\backslash\{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and the inequality (10) for all $x, y, z \in V\backslash\{0\}$, then there exists a unique *quadratic-additive mapping* $F : V \to Y$ *satisfying*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty}\Bigg[\frac{\varphi(2^i x, -2^i x, -2^i x) + \varphi(-2^i x, 2^i x, 2^i x)}{2\cdot 2^{2i+2}} + \frac{2^i}{2}\left(\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{-x}{2^{i+1}}\right) + \varphi\left(\frac{x}{2^{i+1}}, \frac{-x}{2^{i+1}}, \frac{-x}{2^{i+1}}\right)\right)\Bigg] \tag{17}$$

for all $x \in V\backslash\{0\}$.

Proof. It follows from (10) that

$$\begin{aligned}
&\Bigg\|\frac{f(2^n x) + f(-2^n x)}{2\cdot 2^{2n}} + \frac{2^n}{2}\left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right)\right) \\
&\quad - \frac{f(2^{n+m} x) + f(-2^{n+m} x)}{2\cdot 2^{2n+2m}} - \frac{2^{n+m}}{2}\left(f\left(\frac{x}{2^{n+m}}\right) - f\left(\frac{-x}{2^{n+m}}\right)\right)\Bigg\| \\
&\le \sum_{i=n}^{n+m-1}\Bigg\|\frac{f(2^i x) + f(-2^i x)}{2\cdot 2^{2i}} + \frac{2^i}{2}\left(f\left(\frac{x}{2^i}\right) - f\left(\frac{-x}{2^i}\right)\right) \\
&\quad - \frac{f(2^{i+1} x) + f(-2^{i+1} x)}{2\cdot 2^{2i+2}} - \frac{2^{i+1}}{2}\left(f\left(\frac{x}{2^{i+1}}\right) - f\left(\frac{-x}{2^{i+1}}\right)\right)\Bigg\| \\
&= \sum_{i=n}^{n+m-1}\Bigg\|\frac{Df(2^i x, -2^i x, -2^i x) + Df(-2^i x, 2^i x, 2^i x)}{2\cdot 2^{2i+2}} \\
&\quad - \frac{2^i}{2}\left(Df\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{-x}{2^{i+1}}\right) - Df\left(\frac{x}{2^{i+1}}, \frac{-x}{2^{i+1}}, \frac{-x}{2^{i+1}}\right)\right)\Bigg\| \\
&\le \sum_{i=n}^{n+m-1}\Bigg[\frac{\varphi(2^i x, -2^i x, -2^i x) + \varphi(-2^i x, 2^i x, 2^i x)}{2\cdot 2^{2i+2}} \\
&\quad + \frac{2^i}{2}\left(\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{-x}{2^{i+1}}\right) + \varphi\left(\frac{x}{2^{i+1}}, \frac{-x}{2^{i+1}}, \frac{-x}{2^{i+1}}\right)\right)\Bigg]
\end{aligned} \tag{18}$$

for all $x \in V\backslash\{0\}$ and $m, n \in \mathbb{N}\cup\{0\}$. So, it is easy to show that the sequence $\{\frac{f(2^n x)+f(-2^n x)}{2\cdot 2^{2n}} + \frac{2^n}{2}(f(\frac{x}{2^n}) - f(\frac{-x}{2^n}))\}$ is a Cauchy sequence for all $x \in V$. Since Y is complete and $f(0) = 0$, the sequence $\{\frac{f(2^n x)+f(-2^n x)}{2\cdot 2^{2n}} + \frac{2^n}{2}(f(\frac{x}{2^n}) - f(\frac{-x}{2^n}))\}$ converges for all $x \in V\backslash\{0\}$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n\to\infty}\left[\frac{f(2^n x) + f(-2^n x)}{2\cdot 2^{2n}} + \frac{2^n}{2}\left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right)\right)\right]$$

for all $x \in V$. Moreover, if we put $n = 0$ and let $m \to \infty$ in (18), we obtain the first inequality in (17). From the definition of F, we get

$$\begin{aligned}\|DF(x,y,z)\| &= \lim_{n\to\infty}\bigg\|\frac{Df\big(2^n x, 2^n y, 2^n z\big) + Df\big(-2^n x, -2^n y, -2^n z\big)}{2\cdot 2^{2n}} \\ &\quad + \frac{2^n}{2}\bigg(Df\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) - Df\Big(\frac{-x}{2^n},\frac{-y}{2^n},\frac{-z}{2^n}\Big)\bigg)\bigg\| \\ &\le \lim_{n\to\infty}\bigg(\frac{\varphi\big(2^n x, 2^n y, 2^n z\big) + \varphi\big(-2^n x, -2^n y, -2^n z\big)}{2\cdot 2^{2n}} \\ &\quad + \frac{2^n}{2}\bigg(\varphi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) + \varphi\Big(\frac{-x}{2^n},\frac{-y}{2^n},\frac{-z}{2^n}\Big)\bigg)\bigg) \\ &= 0\end{aligned}$$

for all $x, y, z \in V\backslash\{0\}$ i.e, $DF(x,y,z) = 0$ for all $x, y, z \in V$ by Lemma 2.2. By Lemma 2.1, F is a quadratic-additive mapping. In view of Lemma 2.4, there exists a unique quadratic-additive mapping $F : V \to Y$ satisfying the inequality (14), since the inequality

$$\begin{aligned}\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty}\bigg(\frac{\varphi(2^i x, -2^i x, -2^i x) + \varphi(-2^i x, 2^i x, 2^i x)}{2\cdot 2^{2i+2}} \\ &\quad + \frac{2^i}{2}\bigg(\varphi\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big) + \varphi\Big(\frac{x}{2^{i+1}},\frac{-x}{2^{i+1}},\frac{-x}{2^{i+1}}\Big)\bigg)\bigg) \\ &\le \sum_{i=0}^{\infty} 2^i \phi\Big(\frac{x}{2^i}\Big) + \sum_{i=0}^{\infty}\frac{1}{2^{2i}}\psi(2^i x)\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $\phi(x) = \frac{1}{2}[\varphi(\frac{x}{2},\frac{x}{2},-\frac{x}{2}) + \varphi(\frac{x}{2},-\frac{x}{2},-\frac{x}{2})]$ and $\psi(x) = \frac{\varphi(x,x,-x)+\varphi(x,-x,-x)}{8}$.

□

Corollary 3.4. *Let X be a normed space and let p, θ be real constants such that $p \notin \{1,2\}$ and $\theta > 0$. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{19}$$

for all $x, y, z \in X\backslash\{0\}$ with $f(0) = 0$, then there exists a unique quadratic-additve mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \left(\frac{3}{|4-2^p|} + \frac{3}{|2-2^p|}\right)\theta\|x\|^p \tag{20}$$

for all $x \in X\backslash\{0\}$. In particular, if $p < 0$, then f itself is a quadratic-additive mapping.

Proof. Set $\varphi(x,y,z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x,y,z \in X\backslash\{0\}$. Then φ satisfies (9) when $p < 1$, φ satisfies (13) when $p > 2$, and φ satisfies (16) when $1 < p < 2$. Therefore, by Theorems 3.1, 3.2, and 3.3, there exists a unique quadratic-additive mapping F satisfying the inequality (20) for all $x \in X\backslash\{0\}$. If $p < 0$, then it follows from (19), $DF(x,y,z) \equiv 0$, and (20) that

$$\begin{aligned}
2\|f(x) - F(x)\| &\le \|(Df - DF)((k+1)x, -kx, -kx)\| \\
&\quad + \|(F-f)((k-1)x)\| + \|(F-f)(-2kx)\| \\
&\quad + 2\,\|(f-F)((k+1)x)\| + 4\,\|(f-F)(-kx)\| \\
&\quad + \|(f-F)(-(k+1)x)\| + 2\,\|(f-F)(kx)\| \\
&\le \Bigg(|k+1|^p + 2\cdot|k|^p + \left(\frac{3}{|4-2^p|} + \frac{3}{|2-2^p|}\right) \\
&\quad \times (3|k+1|^p + 6\cdot|k|^p + |k-1|^p + |2k|^p) \Bigg)\theta\|x\|^p \\
&\to 0, \text{ as } k \to \infty,
\end{aligned}$$

for all $x \in X\backslash\{0\}$. Since $f(0) = 0 = F(0)$, we have the equality $f(x) = F(x)$ for all $x \in X$ as desired. □

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