Honam Mathematical J. **37** (2015), No. 4, pp. 431–440
http://dx.doi.org/10.5831/HMJ.2015.37.4.431

SLICE THEOREM FOR SEMIALGEBRAICALLY PROPER ACTIONS

Sangwook Kim and Dae Heui Park*

Abstract. Let G be a semialgebraic group which is not necessarily compact. Let X be a semialgebraically proper G-set such that the orbit space has a semialgebraic structure. In this paper we prove the existence of semialgebraic slices of X. Moreover X can be covered by finitely many semialgebraic G-tubes.

1. Introduction

A semialgebraic set is a subset of some $\mathbb{R}^n$ defined by finite number of polynomial equations and inequalities. Throughout this paper we consider the semialgebraic sets in $\mathbb{R}^n$ equipped with the subspace topology induced by the usual topology of $\mathbb{R}^n$. A continuous map $f\colon X \to Y$ between semialgebraic sets $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ is called *semialgebraic* if its graph is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$. Note that all semialgebraic maps are assumed to be continuous.

In this paper we discuss topological properties of semialgebraic sets with semialgebraic actions of semialgebraic groups. Let X be a semialgebraic set and let G be a semialgebraic group which is not necessarily compact. We say X is a semialgebraic G-set if the action $\theta\colon G \times X \to X$ is semialgebraic. A semialgebraic G-set X is called *semialgebraically proper* if the associated action

$$\vartheta_*\colon G \times X \to X \times X, \quad (g, x) \mapsto (\theta(g, x), x)$$

is semialgebraically proper, i.e., $\vartheta_*^{-1}(C)$ is compact for every compact semialgebraic subset C of $X \times X$. Since C should be semialgebraic in the definition, this notion is weaker than the condition that ϑ_* is

Received July 6, 2015. Accepted September 22, 2015.
2010 Mathematics Subject Classification. 57Sxx, 14P10, 54B15.
Key words and phrases. noncompact, proper actions, semialgebraic, slice.
This study was financially supported by Chonnam National University, 2013.
*Corresponding author

topologically proper. But we know that ϑ_* is semialgebraically proper if and only if it is topologically proper. However, when dealing with semialgebraic objects, we only need the semialgebraical properness. For this reason, we will only treat semialgebraically proper maps in this paper.

The existence of a slice plays an important role on further developments in the theory of topological and smooth transformation groups. When G is a compact Lie group and X is a completely regular G-space, the existence of a slice was studied by Gleason [9], Montgomery and Yang [12], and finally proved in the most general form by Mostow [13].

In this paper we prove the existence of slices in the semialgebraic category. Namely we have the following theorem which is restated in Theorem 4.3.

Slice Theorem. *Let G be a semialgebraic group, and let X be a semialgebraically proper G-set such that the orbit space X/G has a semialgebraic structure. Then there exists a semialgebraic slice at every point of X. Moreover X can be covered by finitely many semialgebraic G-tubes.*

When G is compact, this can be found in [4].

2. Semialgebraic sets and maps

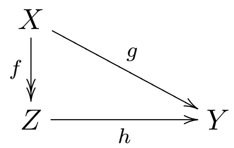
In this section we review some background materials from the semialgebraic category. For the general theory of semialgebraic sets and semialgebraic maps, we refer the reader to [1, 7].

The class of semialgebraic sets in $\mathbb{R}^n$ is the smallest collection of subsets containing all subsets of the form $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ for a real valued polynomial $p(x) = p(x_1, \ldots, x_n)$, which is stable under finite unions, finite intersections and complements.

Proposition 2.1. *Let X and Y be semialgebraic sets.*

(1) *Every semialgebraic set has a finite number of path connected components, which are also semialgebraic.*
(2) *If A is a semialgebraic subset of X, then then the closure $\overline{A}$, the interior $\mathring{A}$ and the complement A^c in X are all semialgebraic.*
(3) *Composition of two semialgebraic maps is semialgebraic.*
(4) *Let $f\colon X \to Y$ be semialgebraic. If A is a semialgebraic subset of X, then its image $f(A)$ is semialgebraic. If B is a semialgebraic subset of Y, then its preimage $f^{-1}(B)$ is semialgebraic.*

(5) *Let $f\colon X \to Z$ and $g\colon X \to Y$ be semialgebraic. Assume f is surjective. If $h\colon Z \to Y$ is a continuous map such that $h \circ f = g$, then h is semialgebraic.*



(6) *If $f\colon X \to Y$ is a semialgebraic homeomorphism, then the inverse f^{-1} is automatically semialgebraic.*

We now deal with the semialgebraic triangulation of semialgebraic sets. Let $a_0, \dots, a_n$ be generically independent points of $\mathbb{R}^m$. The *n-simplex* $\langle a_0, \dots, a_n\rangle$ spanned by $a_0, \dots, a_n$ is defined by

$$\langle a_0, \dots, a_n\rangle = \Big\{\sum_{i=0}^{n} t_i a_i \in \mathbb{R}^m \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\Big\}.$$

The *open n-simplex* $(a_0, \dots, a_n)$ spanned by $a_0, \dots, a_n$ is defined by

$$(a_0, \dots, a_n) = \Big\{\sum_{i=0}^{n} t_i a_i \in \mathbb{R}^m \mid \sum_{i=0}^{n} t_i = 1,\ t_i > 0\Big\}.$$

Note that the open 0-simplex (a) is equal to $\langle a\rangle$ from the definition. Clearly both $\langle a_0, \dots, a_n\rangle$ and $(a_0, \dots, a_n)$ are semialgebraic sets in $\mathbb{R}^m$.

A *finite open simplicial complex* $\big(K, (\sigma_i \mid i \in I)\big)$ is defined as a subset of some $\mathbb{R}^m$ equipped with a partition $(\sigma_i \mid i \in I)$ composed of a finite number of open simplices σ_i in $\mathbb{R}^m$, such that the intersection $\bar{\sigma}_i \cap \bar{\sigma}_j$ of the closures of any two open simplices σ_i and σ_j is either empty or a common face of $\bar{\sigma}_i$ and $\bar{\sigma}_j$. Thus a finite open simplicial complex K is obtained by deleting some open simplices from a "usual" finite simplicial complex. The following theorem can be seen in many places, for instance, [6, Section 2], [1, Chapter 9] and as classical references see [11] for semi-analytic sets and [10] for algebraic sets.

Theorem 2.2. *Let X be a semialgebraic set, and let $X_1, \dots, X_k$ be semialgebraic subsets of X. Then there exist a finite open simplicial complex K and a semialgebraic homeomorphism $\tau\colon |K| \to X$ such that each X_i is a finite union of some of $\tau(\sigma)$, where σ is an open simplex of K.*

We call such K a *finite open triangulation* of X compatible with $X_1, \dots, X_k$. Concerning semialgebraic maps, we have the following useful result. See [1, Theorem 9.3.2] for the proof.

Theorem 2.3 (Semialgebraic triviality)**.** *Let X and Y be two semialgebraic sets and $f\colon X \to Y$ a semialgebraic map. Then there exits a finite decomposition of Y into semialgebraic subsets $\{B_i\}$ such that for each B_i there exists a semialgebraic homeomorphism $\varphi_i\colon B_i \times f^{-1}(b_i) \to f^{-1}(B_i)$ such that $f|_{f^{-1}(B_i)} = p_i \circ \varphi_i^{-1}$, where $b_i \in B_i$ and $p_i\colon B_i \times f^{-1}(b_i) \to B_i$ is the projection to the first factor.*

$$\begin{array}{ccc} B_i \times f^{-1}(b_i) & \xrightarrow{\ \varphi_i\ } & f^{-1}(B_i) \\ & {\scriptstyle p_i}\searrow \quad \swarrow{\scriptstyle f} & \\ & B_i & \end{array}$$

3. Semialgebraically proper actions

In this section we summarize some results about semialgebraically proper actions. For more details, see [14, 15, 16].

The definition of a semialgebraic group is given obviously, i.e., a semialgebraic set $G \subset \mathbb{R}^n$ is called a *semialgebraic group* if it is a topological group such that the group multiplication and the inversion are semialgebraic. A semialgebraic homomorphism between two semialgebraic groups is a semialgebraic map that is a group homomorphism. If H is a subgroup and semialgebraic subset of a semialgebraic group G, then H is called a *semialgebraic subgroup* of G.

Proposition 3.1. (1) *Every semialgebraic group has a Lie group structure, and hence locally compact.*
(2) *Every semialgebraic subgroup of a semialgebraic group is closed.*
(3) *The normalizer $N(H)$ of a semialgebraic subgroup H of a semialgebraic group G is also a semialgebraic subgroup of G.*
(4) *Let G be a semialgebraic group and H a semialgebraic subgroup of G. If $gHg^{-1} \subset H$ for some $g \in G$, then $gHg^{-1} = H$.*

For (topological) proper actions the following proposition appears in [8, Section 1.3] whose proofs are straightforward.

Proposition 3.2. *Let X be a semialgebraically proper G-set and let $x \in X$, then*
(1) *the isotropy subgroup $G_x = \{g \in G \mid g(x) = x\}$ is compact and semialgebraic,*
(2) *the orbit $G(x) = \{gx \in X \mid g \in G\}$ is a closed semialgebraic subset of X,*

(3) *the evaluation map* $\theta_x\colon G \to X$, $g \mapsto gx$, *is semialgebraically proper,*
(4) *the fixed point set* $X^G = \{x \in X \mid gx = x \text{ for all } g \in G\}$ *is a closed semialgebraic subset of* X.

Let X be a semialgebraically proper G-set. If H is a semialgebraic subgroup of G, then the restriction $\vartheta_*|\colon H \times X \to X \times X$ of ϑ_* is semialgebraically proper; hence X is a semialgebraically proper H-set. Moreover, if A is a G-invariant semialgebraic subset of X, then the restriction $\vartheta_*|\colon G \times A \to A \times A$ is semialgebraically proper; hence A is also a semialgebraically proper G-set.

Here is a natural question. Does there exist a semialgebraic structure of the orbit space X/G in the natural sense? A semialgebraic structure (N, f) of X/G is a semialgebraic set $N \subset \mathbb{R}^k$ together with a semialgebraic map $f\colon X \to N$ which is a topological quotient map of X by G. In this case we can substitute X/G and the orbit map $\pi\colon X \to X/G$ with N and f respectively. Scheiderer [17] gave us a partially positive answer of this question as follows.

Theorem 3.3 ([17]). *Let X be a semialgebraically proper G-set which is locally compact. Then the orbit space X/G has a semialgebraic structure such that the orbit map $\pi\colon X \to X/G$ is semialgebraic.*

When G is compact, this was proved by Brumfiel [3]. We remark that when G is compact, every semialgebraic G-set is semialgebraically proper.

As a specific example of a semialgebraically proper action, we consider the following situation; let G be a semialgebraic group and H a semialgebraic subgroup of G. Then G can be seen as a semialgebraically proper H-set where H acts by the right multiplication on G. Note that every semialgebraic group is locally compact, hence that the quotient space G/H has a semialgebraic structure.

We say two homogeneous semialgebraic G-sets are equivalent if they are semialgebraically G-homeomorphic. Let (G/H) denote the equivalence class of G/H. Moreover, the set of all equivalence classes of homogeneous semialgebraic G-sets has the natural partial ordering defined as $(G/K) \leq (G/H)$ if there exists a semialgebraic G-map $G/H \to G/K$. Then $(G/K) \leq (G/H)$ if and only if H is conjugate to a subgroup of K.

When a homogeneous space X of G is equivalent to G/H, the conjugacy class (H) of H in G is called the isotropy type of X. For semialgebraic subgroups H, K of G, a partial ordering $\leq$ is given by $(H) \leq (K)$ if and only if $(G/H) \geq (G/K)$. For more details, see [2, 8].

Let X be a semialgebraic G-set. As the theory of Lie group actions, the natural map

$$\alpha_x \colon G/G_x \to G(x), \quad gG_x \mapsto gx$$

is a semialgebraic G-homeomorphism([14]). We define an *orbit type map* from X to the set of equivalence classes of homogeneous semialgebraic G-sets by $\operatorname{type}(x) = (G/G_x)$. We call $\operatorname{type}(x)$ the *orbit type* of x. The following theorem is one of the main results of [15].

Theorem 3.4 ([15]). *Every semialgebraically proper G-set has only finitely many orbit types.*

For a semialgebraic G-set X and a semialgebraic subgroup H of G, we set

$$\begin{aligned} X_{(H)} &= \{x \in X \mid (G_x) = (H)\} \\ &= \{x \in X \mid G_x = gHg^{-1} \text{ for some } g \in G\}. \end{aligned}$$

Then $X_{(H)}$ is the set of points on orbits of type (G/H).

Corollary 3.5. *Let G be a semialgebraic group and H a semialgebraic subgroup of G, and X a semialgebraically proper G-set. Then $X_{(H)}$ is a semialgebraic G-subset of X.*

Proof. In fact, it is proved in [14, Proposition 2.8] when X has finitely many orbit types. So it follows from Theorem 3.4. □

Let X be a semialgebraically proper G-set and H a semialgebraic subgroup of G. Then X^H is a semialgebraically proper $N(H)$-set by Propositions 3.1 and 3.2 where $N(H)$ is the normalizer of H in G. As in the theory of Lie groups, we have the following proposition.

Proposition 3.6. *Let X be a semialgebraically proper G-set with only one orbit type (G/H). Suppose that X/G has a semialgebraic structure. Then $X^H/N(H)$ has a semialgebraic structure such that the inclusion $j \colon X^H \hookrightarrow X$ induces a semialgebraic homeomorphism $\tilde{j} \colon X^H/N(H) \to X/G$.*

Remark 3.7. *For a semialgebraic subgroup H of G and a semialgebraically proper G-set X, the set*

$$\begin{aligned} X_{(\geq H)} &= \{x \in X \mid (G_x) \geq (H)\} \\ &= \{x \in X \mid G_x \supset gHg^{-1} \text{ for some } g \in G\} \end{aligned}$$

is a closed semialgebraic G-subset of X because $X_{(\geq H)} = GX^H$. Since (G/H) is the largest orbit type occurring in GX^H, $X^H/N(H)$ has a *semialgebraic structure such that the inclusion* $j\colon X^H \hookrightarrow GX^H$ *induces a semialgebraic homeomorphism* $\tilde{j}\colon X^H/N(H) \to GX^H/G$.

4. Semialgebraic slices

In this section, we prove the existence of semialgebraic slices for semialgebraically proper G-sets.

Definition 4.1. *Let* G *be a semialgebraic group. Let* X *be a semialgebraic* G*-set and* H *a semialgebraic subgroup of* G. *A semialgebraic subset* S *of* X *will be called* a semialgebraic H-slice *if* GS *is an open semialgebraic subset of* X *and there exists a semialgebraic* G*-map* $f\colon GS \to G/H$ *such that* $f^{-1}(eH) = S$. *For* $x \in X$ a semialgebraic slice *at* x *means a semialgebraic* G_x*-slice* S *in* X *such that* $x \in S$. *We call* GS a semialgebraic G-tube *about* $G(x)$.

Lemma 4.2. *Let* X *be a semialgebraic set and* A *a closed semialgebraic subset of* X. *Suppose that* A *is a semialgebraic strong deformation retract of* X. *Then for a given semialgebraic neighborhood* U *of* A *there is a closed semialgebraic neighborhood* N *of* A *contained in* U *with a semialgebraic map* $\rho\colon X \to U$ *such that* $\rho(x) = x$ *for* $x \in N$ *and* $\rho(X - N) \subset U - N$.

Proof. See [5, Lemma 3.2]. □

We now prove the slice theorem for semialgebraically proper actions.

Theorem 4.3 (Slice Theorem)**.** *Let* G *be a semialgebraic group, and let* X *be a semialgebraically proper* G*-set such that the orbit space* X/G *has a semialgebraic structure. Then there exists a semialgebraic slice at every point of* X. *Moreover* X *can be covered by finitely many semialgebraic* G*-tubes.*

Proof. Let x_0 be a given point in X. By Theorem 3.4 X has finitely many orbit types. Let $\{(G/H_i) \mid i = 1, \dots, n\}$ be the set of orbit types occurring in X. Each X^{H_i} is viewed as a semialgebraic $N(H_i)$-set. The orbit map $\pi\colon X \to X/G$ restricts to $\pi_i\colon X^{H_i} \to X^{H_i}/N(H_i)$. Note that $X = \cup_{i=1}^n GX^{H_i}$ and $X/G = \cup_{i=1}^n X^{H_i}/N(H_i)$. We apply Theorem 2.3 to each π_i, we get finitely many B_{i_j} such that $\cup_j B_{i_j} = X^{H_i}/N(H_i)$ and semialgebraic cross section $s_{i_j}\colon B_{i_j} \to X^{H_i}$. We simply write the index i_j just as i. Now consider finite semialgebraic subsets $B_1, \dots, B_r$, $b_0 = \pi(x_0)$, and apply Theorem 2.2 to get a finite open semialgebraic triangulation K of X/G compatible with the listed semialgebraic subsets.

Then b_0 is a 0-simplex. We can replace K by its barycentric subdivision. Then the following three properties follow easily.

(1) For each open simplex $\sigma \in K$ the closure $\bar{\sigma}$ contains a 0-simplex,
(2) for each open simplex $\sigma \in K$ there exists a semialgebraic cross section $s\colon \sigma \to X$ of $\pi\colon X \to X/G = K$, and
(3) every point in $s(\sigma)$ has the same isotropy subgroup.

Let $St(b_0)$ denote the (open) star neighborhood of $b_0 = \pi(x_0)$ in $X/G = K$, and let $St^{(k)}(b_0)$ be the k-th skeleton of $St(b_0)$. Let $H_0 = G_{x_0}$. We shall construct a continuous semialgebraic G-map $\psi\colon \pi^{-1}(St(b_0)) \to G/H_0$, so that $\psi^{-1}(eH_0)$ is a semialgebraic slice at x_0 and $\pi^{-1}(St(b_0))$ is a semialgebraic G-tube about $G(x_0)$. The construction is done by inductive construction of semialgebraic G-maps

$$\psi^{(k)}\colon \pi^{-1}(St^{(k)}(b_0)) \to G/H_0.$$

For $k = 0$, because $\pi^{-1}(St^{(0)}(b_0)) = G(x_0)$, we can take $\psi^{(0)}\colon G(x_0) \to G/H_0$ to be the canonical semialgebraic G-homeomorphism. Assume we have constructed a semialgebraic G-map $\psi^{(k-1)}$ that extends $\psi^{(k-2)}$. To extend $\psi^{(k-1)}$ to $\psi^{(k)}$ semialgebraically (and continuously), it is enough to consider a k-dimensional open simplex $\sigma \subset St^{(k)}(b_0)$ where $\psi^{(k-1)}$ is defined on $\pi^{-1}(\bar{\sigma} \cap St^{(k-1)}(b_0))$ and extend $\psi^{(k-1)}$ to $\psi^{(k)}$ on $\bar{\sigma} \cap St^{(k)}(b_0)$. For notational convenience we simply denote δ and $\partial\delta$ for $\bar{\sigma} \cap St^{(k)}(b_0)$ and $\bar{\sigma} \cap St^{(k-1)}(b_0)$ respectively. Note that $\overset{\circ}{\delta} = \delta - \partial\delta = \sigma$. By (2), (3) there is a semialgebraic cross section $s\colon \overset{\circ}{\delta} \to X^{H_k} \subset X$ where $(H_k) \in \{(H_i) \mid i = 1, \ldots, n\}$. Let $\tilde{\delta}$ denote the closure of $s(\overset{\circ}{\delta})$ in X^{H_k}. Then $\pi|_{\tilde{\delta}}\colon \tilde{\delta} \to \delta$ is also semialgebraically proper, hence $\pi|_{\tilde{\delta}}$ is surjective.

We now claim that there exists a semialgebraic retraction

$$\tilde{r}\colon \tilde{\delta} \to \tilde{\delta} \cap \pi^{-1}(\partial\delta).$$

First triangulate $\tilde{\delta}$ and take the open regular neighborhood $\widetilde{U}$ of $\tilde{\delta} \cap \pi^{-1}(\partial\delta)$ in $\tilde{\delta}$. Since $\widetilde{U}$ is open and π is an open map, $U = \pi(\widetilde{U})$ is an open semialgebraic neighborhood of $\partial\delta$ in δ.

Since it is immediate that $\partial\delta$ is a semialgebraic strong deformation retract of δ, we can apply Lemma 4.2 to find a closed semialgebraic neighborhood N of $\partial\delta$ in U and a semialgebraic map $\rho\colon \delta \to U$ such that $\rho(x) = x$ for $x \in N$ and $\rho(\delta - N) \subset U - N$. Now define $r'\colon \tilde{\delta} \to \pi^{-1}(U) \cap \tilde{\delta} = \widetilde{U}$ by

$$r'(x) = \begin{cases} s \circ \rho \circ \pi(x), & x \in \tilde{\delta} - \pi^{-1}(\partial\delta) \\ x, & x \in \tilde{\delta} \cap \pi^{-1}(\partial\delta). \end{cases}$$

Since $\rho|_N = \mathrm{id}$, $s \circ \rho \circ \pi(x) = s \circ \pi(x) = x$ for $x \in \pi^{-1}(N - \partial\delta) \cap \tilde{\tilde{\delta}}$. Therefore the map γ' is continuous. Since the regular neighborhood $\widetilde{U}$ has a semialgebraic retraction to $\tilde{\tilde{\delta}} \cap \pi^{-1}(\partial\delta)$ the composition of r' followed by this retraction gives a semialgebraic retraction $\tilde{r}\colon \tilde{\tilde{\delta}} \to \tilde{\tilde{\delta}} \cap \pi^{-1}(\partial\delta)$.

Now consider the composition

$$r = \psi^{(k-1)} \circ \tilde{r}\colon \tilde{\tilde{\delta}} \to G/H_0.$$

Since any element in $\pi^{-1}(\delta)$ is of the form gx for some $g \in G$ and $x \in \tilde{\tilde{\delta}}$, we can extend r to a semialgebraic G-map

$$r_G\colon \pi^{-1}(\delta) \to G/H_0, \quad gx \mapsto gr(x).$$

This completes the inductive construction of a semialgebraic G-map

$$\psi\colon \pi^{-1}(St(x_0)) \to G/H_0.$$

This completes the existence of semialgebraic G-tube and slice at x_0. Since the triangulation K of X is finite, K has finitely many vertices, say $v_1, \ldots, v_l$. By (1) it is obvious that X can be covered by finite number of G-tubes $\pi^{-1}(St(v_1)), \ldots, \pi^{-1}(St(v_l))$. □

We conclude this paper with the following observation for semialgebraically proper G-sets.

Proposition 4.4. *Let X be a semialgebraically proper G-set and S a semialgebraic slice at x. Then:*

(1) *The map $\varphi\colon G \times_{G_x} S \to GS$ defined by $[g, s] \mapsto gs$ is a semialgebraic G-homeomorphism.*
(2) *The map $\kappa\colon S/G_x \to GS/G$ defined by $[s] \mapsto [s]$ is a semialgebraic homeomorphism.*

Proof. The maps are well-known to be (G-)homeomorphisms, see e.g. [2, Chapter II]. That these maps are semialgebraic follows easily from Proposition 2.1(5). □

References

[1] J. Bochnak, M. Coste and M.-F. Roy, *Real Agebraic Geometry*, Erg. der Math. und ihrer Grenzg., vol. 36, Springer-Verlag, Berlin Heidelberg, 1998.
[2] G. E. Bredon, *Introduction to Compact Transformation Groups*, Pure and Applied Mathematics, vol. 46, Academic Press, New York, 1972.
[3] G. W. Brumfiel, *Quotient space for semialgebraic equivalence relation*, Math. Z. **195** (1987), 69–78.
[4] M.-J. Choi, D. H. Park, and D. Y. Suh, *The existence of semialgebraic slices and its applications*, J. Korean Math. Soc. **41** (2004), 629–646.

[5] M.-J. Choi, D. H. Park, and D. Y. Suh, *Proof of semialgebraic covering mapping cylinder conjecture with semialgebraic covering homotopy theorem*, Top. Appl. **154** (2007), 69–89.
[6] H. Delfs and M. Knebusch, *On the homology of algebraic varieties over real closed fields*, J. Reine Angew. Math. **335** (1981), 122–163.
[7] H. Delfs and M. Knebusch, *Locally Semialgebraic Spaces*, Lecture Notes in Math. **1173**, Springer, Berlin, 1985.
[8] T. tom Dieck, *Transformation Groups*, Walter de Gruyter, New York, 1987.
[9] A. M. Gleason, *Space with a compact Lie group of transformations*, Proc. Amer. Math. Soc. **1** (1950), 35–43.
[10] H. Hironaka, *Triangulations of algebraic varieties*, Proc. Sympos. Pure Math. **29** (1975), 165–185.
[11] S. Łojasiewicz, *Triangulation of semi-analytic sets*, Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat. **18**(3) (1964), 449–474.
[12] D. Montgomery and C. T. Yang, *The existence of a slice*, Ann. of Math. **63** (1957), 108–116.
[13] G. D. Mostow, *Equivariant embeddings in euclidean space*, Ann. of Math. **65**(3) (1957), 432–446.
[14] D. H. Park and D. Y. Suh, *Linear embeddings of semialgebraic G-spaces*, Math. Z. **242** (2002), 725–742.
[15] D. H. Park, *On orbit types of semialgebraically proper actions* Arch. Math. **101** (2013), 33–41.
[16] D. H. Park, *A note on semialgebraically proper maps*, Ukrainian Math. J. **66**(9) (2015), 1414–1422.
[17] C. Scheiderer, *Quotients of semi-algebraic spaces*, Math. Z. **201** (1989), 249–271.

Sangwook Kim
Department of Mathematics, Chonnam National University,
Gwangju 500-757, Korea.
E-mail: swkim.math@jnu.ac.kr

Dae Heui Park
Department of Mathematics, Chonnam National University,
Gwangju 500-757, Korea.
E-mail: dhpark3331@jnu.ac.kr