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# ON THE IDENTITIES BETWEEN THE ARITHMETIC FUNCTIONS

Insuk Kim

**Abstract.** Dirichlet series is a Riemann zeta function attached with an arithmetic function. Here, we studied the properties of Dirichlet series and found some identities between arithmetic functions.

## 1. Introduction

In multiplicative analytic number theory, many problems depend on properties of the zeta function, such as zero free region and zero density estimates. Thus a better understanding of the zeta function theory is the simplest of a large class of Dirichlet series which are known as zeta functions attached with arithmetical functions.

A function defined on the set of natural numbers is called an arithmetic function. There exist many other interesting problems involving summatory functions of arithmetical functions, where the generating series of the arithmetical functions in question factors into a product. Since these problems give some information of the properties of zero density estimates and distribution of primes, several arithmetical functions were studied in [2], [3]. In this paper, we study certain relations between arithmetical functions.

## 2. Arithmetical functions

We list some of mutiplicative arithmetic functions which we will study in this paper.

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Möbius function, $\mu(n)$ which is defined as follows.

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes,} \\ 0 & \text{if } n \text{ is divisible by the square of a prime.} \end{cases}$$

The sum of the divisors of $n$ is denoted by $\sigma(n)$,

$$\sigma(n) = \sum_{d|n} d.$$

Generally, $\sigma_a(n) = \sum_{d|n} d^a$.

Euler $\varphi$ function which is defined to be the number of positive integers not exceeding $n$ that are relatively prime to $n$ can be rewrite as:

$$\varphi(n) = \sum_{d|n} \frac{n}{d}\mu(d) = n \prod (1 - p^{-1}).$$

A divisor function, $d_k(n)$, is defined by

$$d_k(n) = \sum_{d_1 \cdot d_2 \cdots d_k = n} 1.$$

**Remark 2.1.** *If $k = 2$, then $d(n) = d_2(n)$ is the number of divisors of $n$.*

Liouville function, $\lambda(n)$,

$$\lambda(n) = \sum_{d^2|n} \mu(n/d^2).$$

Let $\omega(n)$ be the number of distinct prime factors of $n$. Since $\omega(n)$ is not multiplicative, we consider $2^{\omega(n)}$ as a multiplicative arithmetic function.

Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (\mathrm{Re}(s) > 1).$$

Historically, the zeta function arose from the need for an analytic tool capable of dealing with the problems involing prime numbers. Thus to study the properties of Riemann zeta function, we need a Dirichlet series $F(s)$ which is a function generated by an arithmetic function $f(n)$, i.e.

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s},$$

provided that such a series converges for some $s = s_0$.

For examples,

$$\sum_{n=1}^{\infty} \lambda(n) n^{-s} = \frac{\zeta(2s)}{\zeta(s)}, \quad \sum_{n=1}^{\infty} 2^{\omega(n)} n^{-s} = \frac{\zeta^2(s)}{\zeta(2s)}.$$

From the above definitions, we have following lemmas which are well known. Since we will use these lemmas for the main theorems, we state them with short proofs.

**Lemma 2.2.** *We have*

$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \sigma_a(n) n^{-s}.$$

*Proof.* See p.66 in [1].

□

**Lemma 2.3.** *Let the notations be as above. Then*

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \varphi(n) n^{-s}$$

*for $\sigma > 2$.*

*Proof.* Since $\varphi$ is a multiplicative function,

$$\begin{aligned} \sum_{n=1}^{\infty} \varphi(n) n^{-s} &= \prod_p \left(1 + \sum_{k=1}^{\infty} \varphi(p^k) p^{-sk}\right) \\ &= \prod_p \left(1 + \sum_{k=1}^{\infty} p^k (1 - p^{-1}) p^{-sk}\right) \\ &= \prod_p \left(1 + (1 - p^{-1}) \frac{p^{-s+1}}{1 - p^{-s+1}}\right) \\ &= \frac{\zeta(s-1)}{\zeta(s)}. \end{aligned}$$

□

**Lemma 2.4.** *For any arithmetic functions $f$ and $g$, let $F(s)$ and $G(s)$ be the Dirichlet series defined by*

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}, \quad G(s) = \sum_{n=1}^{\infty} g(n) n^{-s}.$$

*If $F(\sigma) = G(\sigma)$ for $\sigma > \sigma_0$, then $f(n) = g(n)$ for all $n$.*

*Proof.* Since both series converge uniformly for $\sigma > \sigma_0 + \epsilon$, if $\sigma \to \infty$, $f(1) = g(1)$ is obtained. Suppose that $f(n) = g(n)$ for $n = 1, 2, \cdots, N-1$. Then

$$f(N) = \lim_{\sigma\to\infty}\left(N^\sigma \sum_{n=N}^{\infty} f(n)n^{-\sigma}\right) = \lim_{\sigma\to\infty}\left(N^\sigma \sum_{n=N}^{\infty} g(n)n^{-\sigma}\right) = g(N).$$

Hence $f(n) = g(n)$ for all $n$. □

**Remark 2.5.** *Suppose that $F(s)$ and $G(s)$ are two Dirichlet series which converge absolutely for $\sigma > \sigma_1$. Then*

$$F(s)G(s) = \sum_{k=1}^{\infty} f(k)k^{-s} \sum_{l=1}^{\infty} g(l)l^{-s} = \sum_{n=1}^{\infty} h(n)n^{-s} = H(s),$$

*where $H(s)$ also converges absolutely for $\sigma > \sigma_1$. The arithmetical function $h(n)$ can be wrritten as*

$$h(n) = \sum_{kl=n} f(k)g(l) = \sum_{d|n} f(d)g(n/d) = \sum_{d|n} g(d)f(n/d).$$

**Theorem 2.6.** *Let $n$ be a positive integer. Then*

$$d(n) = \frac{1}{n}\sigma(n) * \varphi(n).$$

*Proof.* By Lemma 2.2 and Lemma 2.3,

$$\zeta(s)\zeta(s-1) \cdot \frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \sigma(n)n^{-s} \cdot \sum_{n=1}^{\infty} \varphi(n)n^{-s}.$$

Thus,

$$\zeta^2(s-1) = \sum_{n=1}^{\infty} (\sigma * \varphi)(n)n^{-s},$$

which means

$$\sum_{n=1}^{\infty} d(n)n^{-s+1} = \sum_{n=1}^{\infty}\left(\sum_{d|n} \sigma(d)\varphi(n/d)\right) n^{-s}.$$

By Lemma 2.4, we have

$$d(n) = \frac{1}{n}\sigma(n) * \varphi(n).$$

□

**Theorem 2.7.** *Let the notations be as above.*

$$\lambda(n) * 2^{\omega(n)} = 1.$$

*Proof.* Since

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \lambda(n) n^{-s} \quad \text{and} \quad \frac{\zeta(s)^2}{\zeta(2s)} = \sum_{n=1}^{\infty} 2^{\omega(n)} n^{-s},$$

it is easy to see

$$\zeta(s) = \sum_{n=1}^{\infty} \sum_{d|n} \lambda(d) 2^{\omega(n/d)} n^{-s}.$$

Thus it follows immediately that

$$\lambda(n) * 2^{\omega(n)} = 1.$$

□

**Theorem 2.8.** *(Möbius inversion formula) For any function $f : \mathbb{N} \to \mathbb{C}$, if the function $g : \mathbb{N} \to \mathbb{C}$ is defined by writing*

$$g(n) = \sum_{m|n} f(n)$$

*for every $n \in \mathbb{N}$, then for every $n \in \mathbb{N}$, we have*

$$f(n) = \sum_{m|n} \mu(m) g\left(\frac{n}{m}\right) = \sum_{m|n} \mu\left(\frac{n}{m}\right) g(m).$$

*Proof.* See [2]. □

Applying the Möbius inversion formula, we have

$$\varphi(n) = \sum_{m|n} \mu(m) \frac{n}{m}.$$

We are now in position to state the main results in this paper.

**Lemma 2.9.** *For every number $n \in \mathbb{N}$,*

$$\sum_{m=1,(m,n)=1}^{n} m = n \frac{\varphi(n)}{2}.$$

*Proof.* If $k < \frac{n}{2}$ is relatively prime to $n$, so is $n - k$. Thus there are $\frac{\varphi(n)}{2}$ $k'$s. That is, the sum of all numbers which are relative prime to $n$ is $n \cdot \frac{\varphi(n)}{2}$.

□

**Lemma 2.10.** *Let $n = p_1^{v_1} \cdots p_k^{v_k}$ for primes $p_i$, $i = 1, \cdots, k$,*

$$\sum_{m=1,(m,n)=1}^{n} m^2 = \frac{1}{3}\varphi(n)n^2 + \frac{1}{6}(-1)^k \varphi(n) p_1 \cdots p_k.$$

*Proof.* Let $\mathbb{Z}_m^\times$ be the multiplicative group of integers modulo $m$. Then it has $\varphi(m)$ elements. If we consider $C(m) = \{(n/m)^2 \cdot d^2 | d \in \mathbb{Z}_m^\times\}$, then claim that $\cup_{m|n} C(m) = \{1, 2^2, \cdots, n^2\}$. If $(n/m_1)^2 d_1 = (n/m_2)^2 d_2$ with $(m_1, d_1) = 1$ and $(m_2, d_2) = 1$, then $d_1 = m_2$ and $d_2 = m_1$, which is absurd because $m_i \geq d_i$ for $i = 1, 2$ and $m_1 \neq m_2$. Thus if $i \neq j$, $C(m_i) \cap C(m_j) = \emptyset$. Here, if we let $t(d) = \sum_{m=1,(m,n)=1}^{n} m^2$, then

$$\sum_{d|n} \left(\frac{n}{d}\right)^2 t(d) = \sum_{m=1}^{n} m^2 = \frac{n(n+1)(2n+1)}{6}.$$

Applying the Möbius inversion formula to $\sum_{d|n} \frac{t(d)}{d^2}$, we have

$$\begin{aligned} t(n) = \sum_{m=1,(m,n)=1}^{n} m^2 &= \sum_{m|n} \mu(m) \frac{2(n/m)^2 + 3(n/m) + 1}{6(n/m)} \\ &= \sum_{m|n} \mu(m) \frac{n}{3m} + \sum_{m|n} \frac{1}{2}\mu(m) + \sum_{m|n} \mu(m) \frac{m}{6n} \\ &= \frac{1}{3} \sum_{m|n} \mu(m) \frac{n}{m} + 0 + \frac{1}{6} \sum_{m|n} \mu(m) \frac{m}{n} \\ &= \frac{1}{3}\varphi(n)n^2 + \frac{1}{6}(-1)^k \varphi(n) p_1 \cdots p_k. \end{aligned}$$

□

Next, we consider the case $\sum_{m=1,(m,n)=1}^{n} m^3$.

**Theorem 2.11.** *For every number $n = p_1^{v_1} \cdots p_k^{v_k}$,*

$$\sum_{m=1,(m,n)=1}^{n} m^3 = \frac{1}{4}\varphi(n)n^3 + \frac{1}{4}(-1)^k \varphi(n) p_1 \cdots p_k.$$

*Proof.* Let $t(d) = \sum_{m=1,(m,n)=1}^{n} m^3$. Then as in the proof of Lemma 2.10,

$$\sum_{d|n} \left(\frac{n}{d}\right)^3 t(d) = \sum_{m=1}^{n} m^3 = \frac{n^2(n+1)^2}{4}.$$

Applying the Möbius inversion formula,

$$
\begin{aligned}
t(n) = \sum_{m=1,(m,n)=1}^{n} m^3 &= \sum_{m|n} \mu(m) \frac{n^2+2n+1}{4n} \\
&= \sum_{m|n} \mu(m) \frac{n}{4m} + \sum_{m|n} \frac{1}{2}\mu(m) + \sum_{m|n} \frac{m}{4n} \\
&= \frac{1}{4}\varphi(n) n^3 + \frac{1}{4}(-1)^k \varphi(n) p_1 \cdots p_k.
\end{aligned}
$$

□

In a similar way, if we evaluate the case $k = 4$ or over, we are able to find a formula for $\sum_{m=1,(m,n)=1}^{n} m^k$ for $k \geq 4$.

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Insuk Kim
Department of Mathematics Education ,
Wonkwang University
Iksan 570-749, Korea.
E-mail: iki@wonkwang.ac.kr