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ANALYTIC TRAVELLING WAVE SOLUTIONS OF NONLINEAR COUPLED EQUATIONS OF FRACTIONAL ORDER

Jeong Hyang An and Youho Lee*

Abstract. This paper investigates the issue of analytic travelling wave solutions for some important coupled models of fractional order. Analytic travelling wave solutions of the considered model are found by means of the Q-function method. The results give us that the Q-function method is very simple, reliable and effective for searching analytic exact solutions of complex nonlinear partial differential equations.

1. Introduction

The research of analytic exact solutions of nonlinear partial differential equations plays an important role in studying dynamical phenomena on physics, chemistry, biology and engineering. During the last few decades, directly finding for the analytic exact solutions of nonlinear partial differential equations has become more attractive in a variety of areas. Fractional calculus is a recent focus of interest to many researchers since a lot of systems in science and engineering can be modeled by using fractional derivatives. More precisely, finding explicit solutions to nonlinear partial differential equations of fractional order is of fundamental importance [1–3]. In particular, finding exact solutions of nonlinear partial differential equations of fractional order that are used in chemical models are more difficult than ordinary differential equations of fractional order.

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*Corresponding author.

Recently, it has shown that many phenomena in the diffusion of chemical and ecological models can be described by models using mathematical tools form fractional calculus. However, it is difficult to find exact analytic solutions for most fractional differential equations and hence an effective method for solving such equations is needed. The method of Q-functions is a very powerful approach for obtaining exact solutions of nonlinear ordinary and partial differential equations arising in mathematical physics [6] and also the advantage of this method is discussed in the papers [7, 8]. More recently, Kudryashov and Zakharchenko [9] obtained exact traveling wave solutions of a predator-prey system with diffusion by means of the Q-function method. In this paper, the Q-function method is employed to obtain some exact solutions of the nonlinear coupled equations of fractional order [4–7].

2. Method applied

In this section, we consider the main steps of the fractional subequation method based on the Kudrayshov method for searching exact solutions of nonlinear partial differential equations. In this work, we would used the modified Riemann-Liouville derivative of order α which is defined by Jumarie [10]:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, 0<\alpha<1, \quad (f^{(n)}(t))^{(\alpha-n)}, n \leqq \alpha < n+1, n \geqq 1. \tag{1}$$

Now, the main properties for the modified Riemann-Liouville derivative is shown in [10] and three important properties for the modified Riemann-Liouville derivative are given as follows:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{2}$$

$$D_t^\alpha(f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{3}$$

$$D_t^\alpha f[g(t)] = f_g'[g(t)]D_t^\alpha g(t) = D_t^\alpha f[g(t)](g'(t))^\alpha. \tag{4}$$

Consider the fractional Riccati equation in the following form

$$D_\eta^\alpha \phi(\eta) = \phi(\eta) - \phi^2(\eta), \tag{5}$$

where $D_\eta^\alpha \phi(\eta)$ is defined the modified Riemann-Liouville derivative of order α for $\phi(\eta)$ with respect to η. Eq.(5) is the fractional Riccati differential equation, where α is the order of the fractional derivative. In order to obtain the general solutions for Eq.(5), we transfer $\phi(\eta) = Q(\mathrm{z})$ and a nonlinear fractional complex transformation $\mathrm{z} = \frac{\eta^\alpha}{\Gamma(1+\alpha)}$. From Eq.(2), Eq(4) and Eq.(5), we can be transformed to the following second order ordinary differential equation

$$Q'(\mathsf{z}) = Q(\mathsf{z}) - Q^2(\mathsf{z}). \tag{6}$$

By solving Eq.(6), we get

$$Q(\mathsf{z}) = \frac{1}{1 + e^{-\mathsf{z}-\mathsf{z}_0}}, \tag{7}$$

where $Q(\mathsf{z})$ is the logistic function.

Also, we can be transform the following by the properties of the modified Riemann-Liouville derivative

$$\phi(\eta) = \frac{1}{1 + e^{-\frac{\eta^\alpha}{\Gamma(1+\alpha)}-\mathsf{z}_0}}, \tag{8}$$

Consider a nonlinear fractional partial differential equation(PDE), say in the independent variables $t, x_1, x_2, \cdots, x_n$;

$$\begin{aligned} &P(u_1, \cdots, u_k, D_t^\alpha u_1, \cdots, D_t^\alpha u_k, D_{x_1}^\alpha u_1, \cdots, D_{x_1}^\alpha u_k, \cdots, \\ &D_{x_k}^\alpha u_k, \cdots, D_{x_k}^\alpha u_k, D_t^{2\alpha} u_1, \cdots, D_t^{2\alpha} u_k, D_{x_1}^{2\alpha} u_1, \cdots) = 0, \end{aligned} \tag{9}$$

where P is a polynomial in u_i and their various partial derivatives including fractional derivatives;$u_i = U_i(t, x_1, \cdots, x_n), i = 1, \cdots, k$ are unknown functions. There are steps to find analytic travelling wave solutions of nonlinear partial differential equations [11].

Step 1. Suppose that

$$\begin{aligned} u_i(t, x_1, x_2, \cdots, x_n) &= U_i(\eta) \\ \eta = k_1x_1 + k_2x_2 + \cdot &+ k_nx_n + \eta_0 + \omega t. \end{aligned} \tag{10}$$

By using the second equality of Eqs.(4), (10) and (9) can be reduced into the following fractional ordinary differential equation with respect to the variable η:

$$\begin{aligned} &\widetilde{P}(U_1, \cdots, U_k, \omega D_\eta^\alpha U_1, \cdots, D_\eta^\alpha U_k, k_1 D_\eta^\alpha U_1, \cdots, k_1 D_\eta^\alpha U_k, \cdots, \\ &\qquad xk_n D_\eta^\alpha U_k, \cdots, k_n D_\eta^\alpha U_k, \omega^2 D_\eta^{2\alpha} U_1, \cdots, \\ &\qquad \omega^2 D_\eta^{2\alpha} U_k, k_1^2 D_\eta^{2\alpha} U_1, \cdots) = 0. \end{aligned} \tag{11}$$

Step 2. Assume that the solution of Eq.(11) can be expressed by a polynomial in $\phi(\eta)$ of the form

$$U_j(\eta) = \sum_{i=0}^{m_j} a_{ji}\phi^i(\eta), \tag{12}$$

where $\phi(\eta)$ satisfies Eq.(5), and $a_{ji}, i = 0, 1, \cdot, m_j, j = 1, 2, \cdots, k$ are unknown constants to be decided later, $a_{jm} \neq 0$. By considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq.(11), the positive integer m_j can be decided.

Step 3. Substituting Eq.(12) into Eq.(11) and using Eq.(5), collecting all terms with the same order of $\phi(\eta)$ together, the left-hand side of Eq.(11) is converted into another polynomial in $\phi(\eta)$. Equating each coefficient of this polynomial to zero, we can get a set of algebraic equations for $a_{ji}, i = 0, 1, \cdots, m_j, j = 1, 2, \cdots, k$.

Step 4. Solving the result system of algebraic equations in Step 3 and using solution (8), finally we can compose the family of exact travelling wave solutions for Eq.(9).

In this section, the fractional partial differential equation, which is the nonlinear coupled equations for the diffusion and reaction of the species in a one-dimensional system [4], can be converted into fractional ordinary differential equations. Then, by the Q-function method implementing the proposed modified fractional sub-equation method, we suggest the formulated fractional partial differential equations as follows:

$$\frac{\partial^\alpha U}{\partial t} = D_U \frac{\partial^2 U}{\partial X^2} - K'U(R - V) + KV,$$
$$\frac{\partial^\alpha V}{\partial t} = D_V \frac{\partial^2 V}{\partial X^2} + K'U(R - V) - KV, \tag{13}$$

where D_U and D_V are the diffusion coefficients of the species and K, K' and R are arbitrary constants.

The terms of Equation (13) substitute as follow:

$$u = K'U, v = \frac{K}{R}V, t = \frac{RT^\alpha}{\Gamma(1+\alpha)}, x = \sqrt{\frac{R}{D_V}}X, D = \frac{D_U}{D_V}. \tag{14}$$

We obtain the following system by Jummarie's modified Riemann-Liouville derivative:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - K'u(1 - \frac{v}{K}) + K'v,$$
$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{K}{R}u(1 - \frac{v}{K}) - \frac{K}{R}v, \tag{15}$$

Let us consider fractional diffusive nonlinear differential equation system (17), taking the traveling wave solutions. Using traveling wave variables,

$$u(x,t) = \mathsf{y}(\mathsf{z}), v(x,t) = \mathsf{w}(\mathsf{z}), \mathsf{z} = lx - \omega t (l \neq 0), \tag{16}$$

we obtain an ordinary differential equation) system

$$\begin{aligned} &\omega \mathsf{y}' + Dl^2 \mathsf{y}'' - K' \mathsf{y}(1 - \frac{1}{K}\mathsf{w}) + K'\mathsf{w} = 0, \\ &\omega \mathsf{w}' + l^2 \mathsf{w}'' + \frac{K}{R}\mathsf{y}(1 - \frac{1}{K}\mathsf{w}) - \frac{K}{R}\mathsf{w} = 0, \end{aligned} \tag{17}$$

where $\mathsf{y}' = \frac{d\mathsf{y}}{d\mathsf{z}}, \mathsf{y}'' = \frac{d^2\mathsf{y}}{d\mathsf{z}^2}, \mathsf{w}' = \frac{d\mathsf{w}}{d\mathsf{z}}$ and $\mathsf{w}'' = \frac{d^2\mathsf{w}}{d\mathsf{z}^2}$.

3. Analytic travelling wave solutions of Eq.(13)

Let us find the traveling wave exact solutions of partial differential equation system (13) using the values of parameters. At the present time, there are many methods for finding exact solutions of nonlinear differential equations. Here are some of them: the tanh-expansion method [12–14], the simplest equation method [15–17], and the G'/G-expansion method [18,19]. However, to obtain exact solutions of partial differential equation system (13),we use the method of Q-functions [6]. The advantage of this method was discussed in recent papers [7,20–22]. It was shown that partial differential equation system (13) has the second-order pole solution. Thus, we can look for solutions of system (17) in the form

$$\begin{aligned} \mathsf{y} &= A_0 + A_1 Q(\mathsf{z}) + A_2 Q^2(\mathsf{z}), \\ \mathsf{w} &= B_0 + B_1 Q(\mathsf{z}) + B_2 Q^2(\mathsf{z}), \end{aligned} \tag{18}$$

where $Q(\mathsf{z})$ is the logistic function in the form

$$Q(\mathsf{z}) = \frac{1}{1 + e^{-\mathsf{z} - \mathsf{z}_0}}, \tag{19}$$

and z_0 is an arbitrary constant.

One can see that $Q(\mathsf{z})$ satisfies the Riccati equation in the form;

$$Q_{\mathsf{z}} = Q - Q^2. \tag{20}$$

Taking into account expression (18) and Equation (20),we can obtain the derivatives $\mathsf{y}', \mathsf{w}', \mathsf{y}''$ and w'' expressed via the function $Q(\mathsf{z})$. Substituting $\mathsf{y}(\mathsf{z}), \mathsf{w}(\mathsf{z}), \mathsf{y}', \mathsf{w}', \mathsf{y}''$ and w'' expressed via $Q(\mathsf{z})$ into system (17) and equating to zero the expressions with the same degree of $Q(\mathsf{z})$, there is a system about coefficients in system (17) as follows:

$-A_2B_2 + 6Rl^2B_2 = 0,$
$6Dl^2KA_2 + K'A_2B_2 = 0,$

$$A_0K - A_0B_0 - KB_0 = 0,$$
$$-K'A_0K + K'A_0B_0 + K'KB_0 = 0,$$
$$-2\omega B_2 - A_1B_2 - A_2B_1 - 10Rl^2B_2 + 2Rl^2B_1 = 0,$$
$$2Dl^2KA_1 - 10Dl^2KA_2 - 2\omega KA_2 + K'A_1B_2 + K'A_2B_1 = 0,$$
$$A_1K - A_1B_0 - A_0B_1 - KB_1 + \omega B_1 + Rl^2B_1 = 0,$$
$$Dl^2KA_1 + \omega KA_1 + K'A_0B_1 - K'A_1K + K'A_1B_0 + K'KB_1 = 0,$$
$$2\omega B_2 - \omega B_1 - KB_2 - A_0B_2 - A_1B_1 + A_2K - A_2B_0 - 3Rl^2B_1 + 4Rl^2B_2 = 0,$$
$$-3Dl^2KA_1 + 4Dl^2KA_2 + 2\omega KA_2 - \omega KA_1 + K'A_0B_2 + K'A_1B_1 - K'A_2K + K'A_2B_0 + K'KB_2 = 0.$$

Solving the above system of algebraic equations with the aid of MAPLE, we obtain four possible coefficient sets of solution;

Case 1.

$$D = \frac{3}{2}, l = \pm\sqrt{\frac{2KK'R + 3(A_0+K)^2}{3R(A_0+K)}}, \omega = 0, A_0 = A_0,$$
$$A_1 = -\frac{2(2KK'R + 3(A_0+K)^2)}{A_0+K}, A_2 = \frac{2(2KK'R + 3(A_0+K)^2)}{A_0+K},$$
$$B_0 = \frac{A_0K}{A_0+K}, B_1 = \frac{3K(2KK'R + 3(A_0+K)^2)}{K'R(A_0+K)},$$
$$B_2 = -\frac{3K(2KK'R + 3(A_0+K)^2)}{K'R(A_0+K)}, \tag{21}$$

Case 2.

$$D = 1, l = \pm\sqrt{\frac{KK'R + (A_0+K)^2}{6R(A_0+K)}}\mathrm{i}, \omega = \frac{5(KK'R + (A_0+K)^2)}{6R(A_0+K)},$$
$$A_0 = A_0, A_1 = 0, A_2 = -\frac{KK'R + (A_0+K)^2}{A_0+K}, B_0 = \frac{A_0K}{A_0+K},$$
$$B_1 = 0, B_2 = \frac{K(KK'R + (A_0+K)^2)}{K'R(A_0+K)}, \tag{22}$$

Case 3.

$$D = 1, l = \pm\sqrt{\frac{KK'R + (A_0+K)^2}{R(A_0+K)}}, \omega = 0, A_0 = A_0,$$
$$A_1 = -\frac{6(KK'R + (A_0+K)^2)}{A_0+K}, A_2 = \frac{6(KK'R + (A_0+K)^2)}{A_0+K},$$
$$B_0 = \frac{A_0K}{A_0+K}, B_1 = \frac{6K(KK'R + (A_0+K)^2)}{K'R(A_0+K)},$$
$$B_2 = -\frac{6K(KK'R + (A_0+K)^2)}{K'R(A_0+K)}, \tag{23}$$

Case 4.

$$D = 1, l = \pm\sqrt{\frac{KK'R + (A_0 + K)^2}{R(A_0 + K)}}, \omega = \frac{5(KK'R + (A_0 + K)^2)}{6R(A_0 + K)},$$
$$A_0 = A_0, A_1 = -\frac{2(KK'R + (A_0 + K)^2)}{A_0 + K}, A_2 = \frac{KK'R + (A_0 + K)^2}{A_0 + K},$$
$$B_0 = \frac{A_0 K}{A_0 + K}, B_1 = \frac{2K(KK'R + (A_0 + K)^2)}{K'R(A_0 + K)},$$
$$B_2 = -\frac{K(KK'R + (A_0 + K)^2)}{K'R(A_0 + K)}, \tag{24}$$

Substituting Eq.(21) into (18), we can obtain the following travelling wave solution od Eq.(13) when $D_U = \frac{3}{2}D_V$:

$$U(X,T) = \frac{A_0}{K'} - \frac{2(2KK'R + 3(A_0 + K)^2)}{K'(A_0 + K)}Q(\mathsf{z}) + \frac{2(2KK'R + 3(A_0 + K)^2)}{K'(A_0 + K)}Q^2(\mathsf{z}),$$
$$V(X,T) = \frac{KA_0}{A_0 + K} + \frac{3K(2KK'R + 3(A_0 + K)^2)}{RK(A_0 + K)}Q(\mathsf{z}) - \frac{3K(2KK'R + 3(A_0 + K)^2)}{RK(A_0 + K)}Q^2(\mathsf{z}), \tag{25}$$

where $\mathsf{z} = \pm\sqrt{\frac{3A_0^2 + 3K^2 + 2KK'R + 6A_0K}{3R(A_0 + K)}}\sqrt{\frac{R}{D_V}}X$.

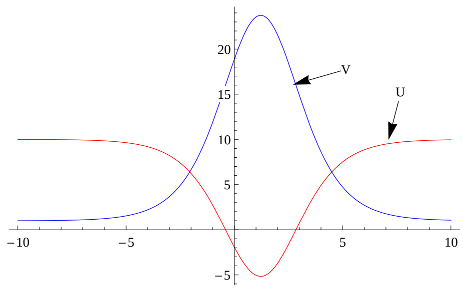

FIGURE 1. The figure of (25) represents the diffusion when $D_V = 1.5, R = 1, K = 0.01, K' = 0.1, \mathsf{z}_0 = 1, A_0 = 1$. This solution is independent on time process and no effect fractional order.

Substituting Eq.(22) into (18), we can obtain the following travelling wave solution od Eq.(13) when $D_U = D_V$:

$$U(X,T) = \frac{A_0}{K'} - \frac{(A_0+K)^2 + KK'R}{K'(A_0+K)}Q^2(\mathsf{z}),$$

$$V(X,T) = \frac{A_0R}{A_0+K} + \frac{R((A_0+K)^2 + KK'R)}{K'R(A_0+K)}Q^2(\mathsf{z}), \tag{26}$$

where $\mathsf{z} = \pm\sqrt{\frac{KK'R+(A_0+K)^2}{6R(A_0+K)}}\mathsf{i}\sqrt{\frac{R}{D_V}}X - \frac{5(KK'R+(A_0+K)^2)}{6R(A_0+K)}\frac{RT^\alpha}{\Gamma(1+\alpha)}$.

Substituting Eq.(23) into (18), we can obtain the following travelling wave solution od Eq.(13) when $D_U = D_V$:

$$U(X,T) = \frac{A_0}{K'} - \frac{6(KK'R + (A_0+K)^2)}{K'(A_0+K)}Q(\mathsf{z}) + \frac{6(KK'R + (A_0+K)^2)}{K'(A_0+K)}Q^2(\mathsf{z}),$$

$$V(X,T) = \frac{A_0R}{A_0+K} + \frac{6(KK'R + (A_0+K)^2)}{K'(A_0+K)}Q(\mathsf{z}) - \frac{6(KK'R + (A_0+K)^2)}{K'(A_0+K)}Q^2(\mathsf{z}), \tag{27}$$

where $\mathsf{z} = \pm\sqrt{\frac{KK'R+(A_0+K)^2}{R(A_0+K)}}\sqrt{\frac{R}{D_V}}X$.

Substituting Eq.(24) into (18), we can obtain the following travelling wave solution od Eq.(13) when $D_U = D_V$:

$$U(X,T) = \frac{A_0}{K'} - \frac{2(KK'R + (A_0+K)^2)}{K'(A_0+K)}Q(\mathsf{z}) + \frac{KK'R + (A_0+K)^2}{K'(A_0+K)}Q^2(\mathsf{z}),$$

$$V(X,T) = \frac{A_0R}{A_0+K} + \frac{2(KK'R + (A_0+K)^2)}{K'(A_0+K)}Q(\mathsf{z}) - \frac{KK'R + (A_0+K)^2}{K'(A_0+K)}Q^2(\mathsf{z}), \tag{28}$$

where $\mathsf{z} = \pm\sqrt{\frac{KK'R+(A_0+K)^2}{R(A_0+K)}}\sqrt{\frac{R}{D_V}}X - \frac{5(KK'R+(A_0+K)^2)}{6R(A_0+K)}\frac{RT^\alpha}{\Gamma(1+\alpha)}$.

4. Conclusion

In this work, we implement relatively new analytical technique, which is called the Q-function method, suggested by Kudrayshov [6] for finding the exact solutions of nonlinear fractional partial differential equation

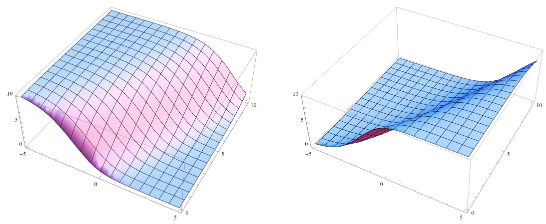

FIGURE 2. The figure of (28) represents a travelling wave solution that the population has different position as time varied under $D_V = 1, R = 0.01, K = 0.001, K' = 0.1, \alpha = 0.8, z_0 = 1; A_0 = 1$.

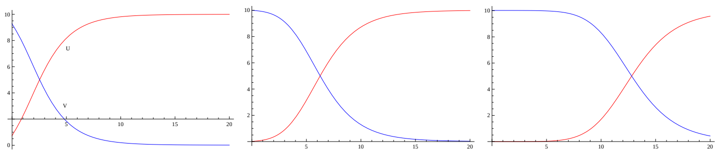

FIGURE 3. The figure of (28) represents the spatial behavior as time varied when $X = 0$, $X = 2$ and $X = 5$.

arising in a variety of areas, such as physics, chemistry, biology and engineering. Especially, in this paper, Eq.(13) is related to chemical dynamic models [4] and we have many kinds of exact solutions of Eq.(13). The graphical analysis of exact solutions of Eq.(13) is our work in the future.

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Youho Lee
Department of Internet Information, Daegu Haany University,
Kyeonsan 712-715, Korea
E-mail : youho@dhu.ac.kr

Jeong Hyang An
The Faculty of General Education, Daegu Haany University,
Kyeonsan 712-715, Korea
E-mail: jhan@dhu.ac.kr