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A NOTE ON A CLASS OF CONVOLUTION INTEGRAL EQUATIONS

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Abstract. This paper considers a class of new convolution integral equations whose kernels involve special functions such as the generalized Mittag-Leffler function and the extended Kummer hypergeometric function. Some basic properties of interconnection with the familiar Riemann-Liouville operators are obtained which are used in finding the solution of the main convolution integral equation. Several consequences are deduced from the main result by incorporating certain extended forms of hypergeometric functions in our present investigation.

1. Introduction and preliminaries

As early as in 1969, Prabhakar [11] discussed a convolution integral equation involving the generalized Mittag-Leffler function given by

$$(1)\quad \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left[\omega (x-t)^{\alpha}\right] f(t)\,\mathrm{d}t = g(x) \;\; (\Re(\beta)>0,\; a\geq 0),$$

where

$$(2)\qquad E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} \;\; (z,\; \alpha,\; \beta,\; \lambda\in\mathbb{C},\; \Re(\alpha)>0).$$

To solve the integral equation (1), a fractional integral operator was defined in the following form:

$$(3)\qquad \left(\mathbf{E}_{\alpha,\beta,\omega;a+}^{\gamma}\varphi\right)(x) = \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left[\omega (x-t)^{\alpha}\right] \varphi(t)\,\mathrm{d}t,$$

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which is bounded on $L\left(a,b\right)$ (see [11, p.9, Theorem 1.]) with

$$\left\|\left(\mathbf{E}^{\gamma}_{\alpha,\beta,\omega;a+}\varphi\right)(x)\right\|_{1}\leq M\left\|\varphi\right\|_{1}\ \ (M>0)\,. \tag{4}$$

The fractional integral operator (3) has a very important property given by

$$\mathbf{E}^{\gamma}_{\alpha,\beta,\omega;a+}\left(\widetilde{\mathbf{E}}^{-\gamma;-\lambda}_{\alpha,\lambda-\beta,a+;\omega}\varphi\right)(x)=\varphi\left(x\right),$$

where

$$\left(\widetilde{\mathbf{E}}^{-\gamma;-\lambda}_{\alpha,\lambda-\beta,a+;\omega}\varphi\right)(x)=\mathbf{E}^{-\gamma}_{\alpha,\lambda-\beta,a+;\omega}\left(I^{-\lambda}_{a+}\varphi\right)(x)\,.$$

Here, $I^{-\lambda}_{a+}$ $(\Re\left(\lambda\right)>0)$ is the inverse operator of the familiar Riemann-Liouville fractional integral operator of order λ given, for instance, by [7, p. 69, Eqn. (2.1.1)] (see also [9] and [14])

$$\left(I^{\lambda}_{a+}\varphi\right)(x)=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{x}\left(x-t\right)^{\lambda-1}\varphi\left(t\right)\mathrm{d}t\ \ (\Re\left(\lambda\right)>0)\,. \tag{5}$$

Over the last few decades, many researches on integral equations involving the operator (3) and its generalizations have been undertaken (amongst many others) in [4], [5], [6], [12] and [15]. The works in these references and similar other investigations are aptly mentioned and cited in the book by Srivastava and Buschman [16], which also describes in a comprehensive manner several other useful applications of the theory of convolution type integral equations.

The purpose of this paper is to consider and investigate a class of convolution integral equation given by

$$\int_{a}^{x}\left(x-t\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(x-t\right)^{\rho}\right]\varphi\left(t\right)\mathrm{d}t=g\left(x\right), \tag{6}$$

where the kernel function $\mathcal{F}^{\sigma}_{\rho,\lambda}\left(x\right)$ involved in (6) is explicitly defined by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}\left(x\right)=\mathcal{F}^{\{\sigma_0,\sigma_1,\cdots\}}_{\rho,\lambda}\left(x\right)=\sum_{k=0}^{\infty}\frac{\sigma\left(k\right)}{\Gamma\left(\rho k+\lambda\right)}x^{k} \tag{7}$$

$$\left(\rho,\lambda\in\mathbb{C}\ \ (\Re\left(\rho\right)>0)\,;\ |x|<\mathbb{R}\right),$$

where $\sigma\left(k\right)\left(k\in\mathbb{N}_0\right)$ is a suitably prescribed sequence of real (or complex) numbers and $\mathbb{R}$ is the set of real numbers. Many well-known special functions including the generalized Mittag-Leffler function (2) and the extended hypergeometric functions (given below) are the special cases of the function $\mathcal{F}^{\sigma}_{\rho,\lambda}$ with some suitably chosen coefficients $\sigma\left(k\right)$. The left-hand side of the above integral equation is actually the integral operator defined by (Raina [13, p. 194, Eqn. (2.2)]) as

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1}\,\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\,(x-t)^{\rho}\right]\varphi\,(t)\,\mathrm{d}t, \tag{8}$$

where $a \in \mathbb{R}_+\,(x > a)$; $\lambda,\ \rho,\ \omega \in \mathbb{C}$; $(\Re\,(\lambda) > 0,\ \Re\,(\rho) > 0)$, $\varphi\,(t)$ is such that the integral on the right side exists. The operator $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}$ and many other specific cases of the operator $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}$ can be found in [12], [13], [16] and [18].

For our investigation, we also need to mention the following specific cases of the function (7). The extended Gauss hypergeometric function is defined by (see [8])

$$ {}_2F_1\left[\begin{matrix} a,b \\ c\end{matrix};z;p,q\right] = \sum_{k=0}^{\infty}(a)_k\,\frac{B_{p,q}\,(b+k,c-b)}{B\,(b,c-b)}\,\frac{z^k}{k!}. \tag{9}$$

$$(|z|<1;\ \Re\,(c) > \Re\,(b) > 0;\ \Re\,(p)\geq 0,\ \Re\,(q)\geq 0)$$

This function can be easily obtained by setting

$$\sigma\,(k) = (a)_k\,\frac{\Gamma\,(\rho k+\lambda)}{k!}\,\frac{B_{p,q}\,(b+k,c-b)}{B\,(b,c-b)}$$

in (7). Similarly, we can define the extended Kummer hypergeometric function as

$$ {}_1F_1\left[\begin{matrix} b \\ c\end{matrix};z;p,q\right] = \sum_{k=0}^{\infty}\frac{B_{p,q}\,(b+k,c-b)}{B\,(b,c-b)}\,\frac{z^k}{k!}. \tag{10}$$

$$(\Re\,(c) > \Re\,(b) > 0;\ \Re\,(p)\geq 0,\ \Re\,(q)\geq 0)$$

In both (9) and (10), $B_{p,q}\,(x,y)$ is the extended beta function defined by (see [8, Eqn. (1.4)])

$$B_{p,q}\,(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\exp\left(-\frac{p}{t}-\frac{q}{1-t}\right)\mathrm{d}t, \tag{11}$$

$$(\Re\,(p)\geq 0,\ \Re\,(q)\geq 0)\,.$$

By setting $p = q = 0$ in (11), we get the usual beta function. It may also be noted that the function (11) is, in fact, a special case of the function defined by [17, p. 256, Eqn. (6.1)]. Functions such as (9), (10) and (11) with $p = q$ have been studied in [1, 2, 3]. Being generalizations of hypergeometric functions, these function classes have been widely studied, but their applications in fractional calculus are not much explored. In this paper, we will find that there is a very nice connection between the extended Kummer hypergeometric function (10) and our convolution integral equation (6).

2. Results involving class of operators $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}$

We begin this section by presenting the following theorem about the boundedness property for the class of fractional integral operators $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}$ on the space $L(a,b)$ of Lebesgue measurable functions:

$$L(a,b)=\left\{f:\|f\|_1:=\int_a^b |f(t)|\,\mathrm{d}t<\infty\right\}.$$

Theorem 2.1. *Let the function φ be in the space $L(a,b)$. Choose $\sigma(k)$ $(k\in\mathbb{N}_0)$ such that the series*

$$\mathfrak{M}:=\sum_{k=0}^{\infty}\frac{|\sigma(k)|\left|\omega(b-a)^{\Re(\rho)}\right|^k}{|\Gamma(\rho k+\lambda)|\left[\Re(\lambda)+\Re(\rho)k\right]} \tag{12}$$

is convergent. Then the fractional integral operator $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}$ is bounded on $L(a,b)$ and satisfies

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right\|_1\le\mathfrak{M}(b-a)^{\Re(\lambda)}\|\varphi\|_1.$$

Proof. It is sufficient to show that

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right\|_1=\int_a^b\left|\int_a^x(x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(x-t)^{\rho}\right]\varphi(t)\,\mathrm{d}t\right|\mathrm{d}x<\infty.$$

By using the Dirichlet formula ([14, p. 9, Eqn. (1.32)]), viz.

$$\int_a^b\mathrm{d}x\int_a^x f(x,y)\,\mathrm{d}y=\int_a^b\mathrm{d}y\int_y^b f(x,y)\,\mathrm{d}x, \tag{13}$$

we get

$$\begin{aligned}
\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right\|_1 &\le\int_a^b|\varphi(t)|\left(\int_t^b(x-t)^{\Re(\lambda)-1}\left|\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(x-t)^{\rho}\right]\right|\mathrm{d}x\right)\mathrm{d}t\\
&\le\int_a^b|\varphi(t)|\left(\int_0^{b-a}v^{\Re(\lambda)-1}\left|\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega v^{\rho}\right]\right|\mathrm{d}v\right)\mathrm{d}t\\
&\le\left(\int_0^{b-a}v^{\Re(\lambda)-1}\left|\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega v^{\rho}\right]\right|\mathrm{d}v\right)\|\varphi\|_1\\
&\le\left(\sum_{k=0}^{\infty}\frac{|\sigma(k)|\,|\omega|^k}{|\Gamma(\rho k+\lambda)|}\int_0^{b-a}v^{\Re(\lambda)+\Re(\rho)k-1}\mathrm{d}v\right)\|\varphi\|_1\\
&=\left(\sum_{k=0}^{\infty}\frac{|\sigma(k)|\,|\omega|^k(b-a)^{\Re(\lambda)+\Re(\rho)k}}{|\Gamma(\rho k+\lambda)|\left[\Re(\lambda)+\Re(\rho)k\right]}\right)\|\varphi\|_1
\end{aligned}$$

$$= \mathfrak{M}\,(b-a)^{\Re(\lambda)}\,\|\varphi\|_1\,. \tag{14}$$

This completes the proof. □

Remark 2.2. *By setting $\sigma(k) = (\gamma)_k/k!$, $\rho = \alpha$ and $\lambda = \beta$ in above theorem, we get* (4). *It also follows easily for suitably chosen parameters that the fractional integral operator $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}$ containing the function* (9) *or* (10) *as its kernel is also bounded on $L(a,b)$.*

In what follows, we assume throughout that the sequence $\sigma(k)$ satisfies the requirement (12) stated in Theorem 2.1 and that the function φ is in $L(a,b)$.

Proposition 2.3. *If $\Re(\alpha) > -\Re(\lambda)$, then*

$$\left(I^{\alpha}_{a+}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)\right)(x) = \left(\mathcal{J}^{\sigma}_{\rho,\lambda+\alpha,a+;\omega}\varphi\right)(x)\,. \tag{15}$$

Proof. The validity of (15) for $\Re(\alpha) > 0$ has been proved in [13, p. 196, Eqn. (2.8)]. So we only need to prove that (15) holds for $0 \geq \Re(\alpha) > -\Re(\lambda)$.

Suppose $0 > \Re(\alpha) > -\Re(\lambda)$. Since $\Re(\alpha+\lambda) > 0$, then for $\varphi \in L(a,b)$, we have

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda+\alpha,a+;\omega}\varphi\right)(x) \in L(a,b)\,. \tag{16}$$

Applying the operator $I^{-\alpha}_{a+}$ to both sides of (15), we have

$$I^{-\alpha}_{a+}\left(\mathcal{J}^{\sigma}_{\rho,\lambda+\alpha,a+;\omega}\varphi\right)(x) = \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(x)\,. \tag{17}$$

In view of the condition (16), we can write

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda+\alpha,a+;\omega}\varphi\right)(x) = I^{\alpha}_{a+}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(x)\,. \tag{18}$$

When $\Re(\alpha) = 0$, we have

$$\begin{aligned}\left(\mathcal{J}^{\sigma}_{\rho,\lambda+\alpha,a+;\omega}\varphi\right)(x) &= I^{\alpha-1}_{a+}\left(\mathcal{J}^{\sigma}_{\rho,\lambda+1,a+;\omega}\right)(x)\\ &= I^{\alpha}_{a+}\left(I^{-1}_{a+}\left(\mathcal{J}^{\sigma}_{\rho,\lambda+1,a+;\omega}\right)\right)(x)\\ &= I^{\alpha}_{a+}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(x)\,. \end{aligned} \tag{19}$$

This competes the proof. □

Proposition 2.4. *Let $\Re(\alpha) > 0$. Then we have*

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\left(I^{\alpha}_{a+}\varphi\right)\right)(x) = \left(I^{\alpha}_{a+}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)\right)(x)$$

Proof. Since $\varphi \in L(a,b)$ and $\left(I^{\alpha}_{a+}\varphi\right)(x) \in L(a,b)$, Therefore, on using (5) and (8), we have

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\left(I^{\alpha}_{a+}\varphi\right)\right)(x)$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega (x-t)^{\rho}\right] \left(\int_a^t (t-u)^{\alpha-1} \varphi(u)\, \mathrm{d}u \right) \mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)} \int_a^x \varphi(u)\, \mathrm{d}u \int_u^x (x-t)^{\lambda-1} (t-u)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega (x-t)^{\rho}\right] \mathrm{d}t \\
&\qquad \left(\begin{array}{l} \text{on evaluating the inner integral by} \\ \text{means of the substitution: } v = \frac{x-t}{x-u} \end{array} \right) \\
&= \int_a^x (x-u)^{\lambda+\alpha-1} \mathcal{F}^{\sigma}_{\rho,\lambda+\alpha,a+;\omega}\left[\omega (x-u)^{\rho}\right] \varphi(u)\, \mathrm{d}u \\
&= \left(\mathcal{J}^{\sigma}_{\rho,\lambda+\alpha,a+;\omega} \varphi \right)(x).
\end{aligned}$$

Now on using (15), we get

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \left(I^{\alpha}_{a+} \varphi \right) \right)(x) = \left(I^{\alpha}_{a+} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \varphi \right) \right)(x)$$

for $\Re(\alpha) > 0$. □

Theorem 2.5. *(commutativity) If* φ *and* $\left(I^{\alpha}_{a+} \varphi \right)(x)$ $(\alpha \in \mathbb{C})$ *exist in* $L(a,b)$, *then*

$$\left(I^{\alpha}_{a+} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \varphi \right) \right)(x) = \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \left(I^{\alpha}_{a+} \varphi \right) \right)(x).$$

Proof. The commutativity of I^{α}_{a+} and $\mathcal{J}^{\sigma}_{\rho,\lambda,a+,\omega}$ when $\Re(\alpha) > 0$ is considered in Proposition 2.4, so we just need to prove

$$\left(I^{\alpha}_{a+} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \varphi \right) \right)(x) = \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \left(I^{\alpha}_{a+} \varphi \right) \right)(x) \ \ (\Re(\alpha) \le 0).$$

Suppose $\Re(\alpha) < 0$, let $I^{\alpha}_{a+} \varphi(x) = f(x)$. By Proposition 2.4, we have

$$\left(I^{-\alpha}_{a+} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} f \right) \right)(x) = \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \left(I^{-\alpha}_{a+} f \right) \right)(x).$$

By applying I^{α}_{a+} on both the sides in this last equation above, we get

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} f \right)(x) = I^{\alpha}_{a+} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \left(I^{-\alpha}_{a+} f \right) \right)(x),$$

which implies that

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \left(I^{\alpha}_{a+} \varphi \right) \right)(x) = \left(I^{\alpha}_{a+} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \varphi \right) \right)(x) \ \ (\Re(\alpha) < 0).$$

When $\Re(\alpha) = 0$, we write

$$\left(I^{\alpha+1}_{a+} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \varphi \right) \right)(x) = \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \left(I^{\alpha+1}_{a+} \varphi \right) \right)(x),$$

that is,

$$\begin{aligned}
\left(I^{\alpha}_{a+} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \varphi \right) \right)(x) &= I^{-1}_{a+} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \left(I^{\alpha+1}_{a+} \varphi \right) \right)(x) \\
&= \mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \left(I^{-1}_{a+} \left(I^{\alpha+1}_{a+} \varphi \right) \right)(x) \\
&= \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} \left(I^{\alpha}_{a+} \varphi \right) \right)(x).
\end{aligned}$$

This completes the proof. □

Theorem 2.6. *If* $\Re(\lambda_1) > 0$, $\Re(\lambda_2) > 0$, *then*

$$\mathcal{J}^{\sigma_1}_{\rho,\lambda_1,a+;\omega}\left(\mathcal{J}^{\sigma_2}_{\rho,\lambda_2;a+;\omega}\varphi\right)(x) = \left(\mathcal{J}^{\Omega}_{\rho,\lambda_1+\lambda_2,a+;\omega}\varphi\right)(x), \tag{20}$$

where the sequences σ_1 *and* σ_2 *are suitably chosen such that*

$$\Omega := \Omega(m) = \sum_{n=0}^{m} \sigma_1(m-n)\,\sigma_2(n)$$

makes $\mathcal{J}^{\Omega}_{\rho,\lambda_1+\lambda_2,a+;\omega}$ *a bounded operator in* $L(a,b)$.

Proof. Since $\varphi \in L(a,b)$, therefore, $\left(\mathcal{J}^{\sigma_2}_{\rho,\lambda_2;a+;\omega}\varphi\right)(x)$ exists in $L(a,b)$, and upon using (8), we have

$$\begin{aligned}
&\mathcal{J}^{\sigma_1}_{\rho,\lambda_1,a+;\omega}\left(\mathcal{J}^{\sigma_2}_{\rho,\lambda_2;a+;\omega}\varphi\right)(x)\\
&= \int_a^x (x-u)^{\lambda_1-1}\,\mathcal{F}^{\sigma_1}_{\rho,\lambda_1}\left[\omega(x-u)^\rho\right]\\
&\qquad\left\{\int_a^u (u-t)^{\lambda_2-1}\,\mathcal{F}^{\sigma_2}_{\rho,\lambda_2}\left[\omega(u-t)^\rho\right]\varphi(t)\,\mathrm{d}t\right\}\mathrm{d}u\\
&= \int_a^x \varphi(t)\left\{\int_t^x (x-u)^{\lambda_1-1}(u-t)^{\lambda_2-1}\,\mathcal{F}^{\sigma_1}_{\rho,\lambda_1}\left[\omega(x-u)^\rho\right]\right.\\
&\qquad\left.\cdot\,\mathcal{F}^{\sigma_2}_{\rho,\lambda_2}\left[\omega(u-t)^\rho\right]\mathrm{d}u\right\}\mathrm{d}t.
\end{aligned}\tag{21}$$

Set $v = \frac{x-u}{x-t}$, the inner integral becomes

$$\begin{aligned}
&(x-t)^{\lambda_1+\lambda_2-1}\int_0^1 v^{\lambda_1-1}(1-v)^{\lambda_2-1}\,\mathcal{F}^{\sigma_1}_{\rho,\lambda_1}\left[\omega(x-t)^\rho v^\rho\right]\\
&\qquad\cdot\,\mathcal{F}^{\sigma_2}_{\rho,\lambda_2}\left[\omega(x-t)^\rho(1-v)^\rho\right]\mathrm{d}v\\
&= (x-t)^{\lambda_1+\lambda_2-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\sigma_1(m)\,\sigma_2(n)\,\omega^{m+n}(x-t)^{\rho(m+n)}}{\Gamma(\rho m+\lambda_1)\,\Gamma(\rho n+\lambda_2)}\\
&\qquad\cdot\int_0^1 v^{\lambda_1+\rho m-1}(1-v)^{\lambda_2 z+\rho n-1}\,\mathrm{d}v\\
&= (x-t)^{\lambda_1+\lambda_2-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\sigma_1(m)\,\sigma_2(n)\,\omega^{m+n}(x-t)^{\rho(m+n)}}{\Gamma(\rho(m+n)+\lambda_1+\lambda_2)}\\
&= (x-t)^{\lambda_1+\lambda_2-1}\sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}\sigma_1(m-n)\,\sigma_2(n)\right]\frac{\omega^m(x-t)^{\rho m}}{\Gamma(\rho m+\lambda_1+\lambda_2)}
\end{aligned}$$

$$= (x-t)^{\lambda_1+\lambda_2-1} \sum_{m=0}^{\infty} \frac{\Omega(m)}{\Gamma(\rho m+\lambda_1+\lambda_2)} [\omega(x-t)^{\rho}]^m$$

$$= (x-t)^{\lambda_1+\lambda_2-1} \mathcal{F}^{\Omega}_{\rho,\lambda_1+\lambda_2} [\omega(x-t)^{\rho}]. \tag{22}$$

The result (20) follows now on using (21) and (22). □

3. The convolution integral equation (6)

We apply the results of the previous sections to solve the integral equation (6).

Theorem 3.1. *Suppose sequences $\sigma_1(m), \sigma_2(n)$ $(m, n \in \mathbb{N})$ satisfies*

(i) $\sigma_1(0) = c_1, \sigma_2(0) = c_1$, *where c_1 and c_2 are nonzero constants;*

(ii) $\Omega := \Omega(m) = \sum_{n=0}^{m} \sigma_1(m-n)\sigma_2(n) = 0$ *for* $m \geq 1$ $(\Omega(0) = c_1c_2)$.

Then, if $\Re(\alpha) > \Re(\lambda) > 0$ and $I_{a+}^{-\alpha} g \in L(a,b)$, the integral equation

$$\frac{1}{c_1c_2} \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma_1}_{\rho,\lambda} [\omega(x-t)^{\rho}] \varphi(t)\,\mathrm{d}t = g(x) \quad (a < x \leq b) \tag{23}$$

possesses a solution φ in $L(a,b)$ given by

$$\varphi(x) = \int_a^x (x-t)^{\alpha-\lambda-1} \mathcal{F}^{\sigma_2}_{\rho,\alpha-\lambda} [\omega(x-t)^{\rho}] \left(I_{a+}^{-\alpha} g\right)(t)\,\mathrm{d}t. \tag{24}$$

Proof. The solution (24) of the integral equation (23) is already achieved on using the results of Theorems 2.5 and 2.6 in conjunction with the properties derived in the Propositions 2.3 and 2.4 subject to the boundedness condition (12) of Theorem 2.1. To complete the proof, it now only needs to verify the solution (24) of the integral equation (23). For this verification, we proceed as follows:

Following (8), equations (23) and (24) can be rewritten as

$$\frac{1}{c_1c_2} \left(\mathcal{J}^{\sigma_1}_{\rho,\lambda,a+,\omega} \varphi\right)(x) = g(x) \tag{25}$$

and

$$\varphi(x) = \left(\mathcal{J}^{\sigma_2}_{\rho,\alpha-\lambda,a+;\omega} \left(I_{a+}^{-\alpha} g\right)\right)(x). \tag{26}$$

Substituting (26) into (25) and applying (20), we have

$$\frac{1}{c_1c_2} \left(\mathcal{J}^{\sigma_1}_{\rho,\lambda,a+,\omega} \left(\mathcal{J}^{\sigma_2}_{\rho,\alpha-\lambda,a+;\omega} \left(I_{a+}^{-\alpha} g\right)\right)\right)(x) = \frac{1}{c_1c_2} \left(\mathcal{J}^{\Omega}_{\rho,\alpha,a+;\omega} \left(I_{a+}^{-\alpha} g\right)\right)(x),$$

where the operator $\mathcal{J}^{\Omega}_{\rho,\alpha,a+;\omega}$ is explicitly given by

$$\left(\mathcal{J}^{\Omega}_{\rho,\alpha,a+;\omega}f\right)(x)=\int_a^x (x-t)^{\alpha-1}\,\mathcal{F}^{\Omega}_{\rho,\alpha}\left[\omega\,(x-t)^{\rho}\right]f(t)\,\mathrm{d}t.$$

Under the conditions (i) and (ii), the kernel function $\mathcal{F}^{\Omega}_{\rho,\alpha}\left[\omega\,(x-t)^{\rho}\right]$ reduces to

$$\mathcal{F}^{\Omega}_{\rho,\alpha}\left[\omega\,(x-t)^{\rho}\right]=\sum_{m=0}^{\infty}\frac{\Omega\,(m)}{\Gamma\,(\rho m+\alpha)}\left[\omega\,(x-t)^{\rho}\right]^m=\frac{c_1c_2}{\Gamma\,(\alpha)}$$

and

$$\left(\mathcal{J}^{\Omega}_{\rho,\alpha,a+;\omega}f\right)(x)=\frac{c_1c_2}{\Gamma\,(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,\mathrm{d}t=c_1c_2\left(I^{\alpha}_{a+}f\right)(x).$$

Thus

$$\frac{1}{c_1c_2}\left(\mathcal{J}^{\sigma_1}_{\rho,\lambda,a+,\omega}\left(\mathcal{J}^{\sigma_2}_{\rho,\alpha-\lambda,a+;\omega}\left(I^{-\alpha}_{a+}g\right)\right)\right)(x)=I^{\alpha}_{a+}\left(I^{-\alpha}_{a+}g\right)(x)=g\,(x),$$

which completes the proof. □

Remark 3.2. *By using Theorem 2.5, it can be easily verified that* (23) *and* (24) *imply each other.*

We now consider some useful consequences of Theorem 3.1.

If we choose

$$\sigma_1\,(m)=\frac{(\gamma)_m}{m!}\text{ and }\sigma_2\,(n)=\frac{(-\gamma)_n}{n!}\quad(\sigma_1\,(0)=\sigma_2\,(0)=1),$$

in Theorem 3.1, then by applying the Chu-Vandermonde identity [10, p. 387, Eqn. (15.4.24)]:

$$\Omega\,(m)=\sum_{n=0}^{m}\frac{(\gamma)_{m-n}}{(m-n)!}\frac{(-\gamma)_n}{n!}=\frac{(\gamma-\gamma)_m}{m!}=\begin{cases}0, & m\geq 1;\\ 1, & m=0,\end{cases}$$

which obviously satisfies the conditions (i) and (ii) of Theorem 3.1, we get the result [11, p. 13, Theorem. 8], namely, the solution of (1) can be expressed by

$$f\,(t)=\mathbf{E}^{-\gamma}_{\alpha,\lambda-\beta,a+;\omega+}\left(I^{-\lambda}_{a+}g\right)(x),\tag{27}$$

where the operator $\mathbf{E}^{-\gamma}_{\alpha,\lambda-\beta,a+;\omega+}$ is given by (3).

In order to give the next result, we need the following summation formula concerning the extended Gauss hypergeometric function (9) obtained recently by Luo and Raina [8]

Theorem 3.3. *For $m \in \mathbb{N}$, $\Re(p) \geq 0$, $\Re(q) \geq 0$, the following summation formula holds:*

$$\sum_{n=0}^{m} \frac{(-m)_n (a)_n}{n! (1-c+a+b-m)_n} B_{p,q}(b+n, c-b) = \frac{e^{p-\frac{q}{2}} (c-a)_m}{(c-a-b)_m} B_{2q,\frac{1}{2}p}(c-b+m, b), \tag{28}$$

where the function $B_{p,q}(x,y)$ is defined by (11).

Employing elementary calculations, we rewrite (28) as

$$e^{-p+\frac{q}{2}} \sum_{n=0}^{m} \frac{(c-a-b)_{m-n}}{(m-n)!} \frac{(a)_n}{n!} B_{p,q}(b+n, c-b) = \frac{(c-a)_m}{m!} B_{2q,\frac{1}{2}p}(c-b+m, b). \tag{29}$$

Now, let

$$\sigma_1(m-n) = \frac{e^{-p+\frac{q}{2}}}{B_{2q,\frac{1}{2}p}(c-b,b)} \frac{(-b)_{m-n}}{(m-n)!}$$

and

$$\sigma_2(n) = \frac{(c)_n}{n!} B_{p,q}(b+n, c-b),$$

then it follows that

$$\sigma_1(0) = \frac{e^{-p+\frac{q}{2}}}{B_{2q,\frac{1}{2}p}(c-b,b)} \text{ and } \sigma_2(0) = B_{p,q}(b, c-b).$$

On using (29) (with $a = c$), we have

$$\Omega(m) = \frac{(c-c)_m}{m!} \frac{B_{2q,\frac{1}{2}p}(c-b+m,b)}{B_{2q,\frac{1}{2}p}(c-b,b)} = \begin{cases} 0, & m \geq 1; \\ 1, & m = 0 \end{cases} \tag{30}$$

which satisfies the condition (ii) of Theorem 3.1. Thus, we have the following result.

Corollary 3.4. *If $\Re(\alpha) > \Re(\lambda) > \Re(\beta) > 0$ and $I_{a+}^{-\alpha} g \in L(a,b)$, then the solution of the integral equation*

$$\frac{e^{p+\frac{q}{2}}}{B_{2q,\frac{1}{2}p}(\gamma-\beta,\beta)} \int_a^x (x-t)^{\lambda-1} E_{\rho,\lambda}^{-\beta}[\omega(x-t)^\rho] \varphi(t)\, \mathrm{d}t = g(x) \ (a < x \leq b)$$

is given by

$$\varphi(x) = \int_a^x (x-t)^{\alpha-\lambda-1} \mathcal{R}_{\rho,\alpha-\lambda}^{\beta,\gamma;p,q} [\omega(x-t)^\rho] \left(I_{a+}^{-\alpha} g\right)(t)\,\mathrm{d}t,$$

where $E_{\rho,\lambda}^{-\beta}[\omega(x-t)^\rho]$ *is defined by* (2) *and the function* $\mathcal{R}_{\rho,\alpha-\lambda}^{\beta,\gamma;p,q}[\omega(x-t)^\rho]$ *is given by*

$$\mathcal{R}_{\rho,\alpha-\lambda}^{\beta,\gamma;p,q}[\omega(x-t)^\rho] = \sum_{n=0}^{\infty} (\gamma)_n \frac{B_{p,q}(\beta+n,\gamma-\beta)}{\Gamma(\rho n+\alpha-\lambda)\, n!} [\omega(x-t)^\rho]^n. \tag{31}$$

We can further set $\gamma = \lambda$, $\alpha = 2\lambda$ and $\rho = 1$ in (31) to get the following:

$$\begin{aligned}
\mathcal{R}_{1,\lambda}^{\beta,\lambda;p,q}[\omega(x-t)] &= \sum_{n=0}^{\infty} (\lambda)_n \frac{B_{p,q}(\beta+n,\lambda-\beta)}{\Gamma(n+\lambda)\, n!} [\omega(x-t)]^n \\
&= \frac{1}{\Gamma(\lambda)} \sum_{n=0}^{\infty} B_{p,q}(\beta+n,\lambda-\beta) \frac{[\omega(x-t)]^n}{n!} \\
&= \frac{B(\beta,\lambda-\beta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{B_{p,q}(\beta+n,\lambda-\beta)}{B(\beta,\lambda-\beta)} \frac{[\omega(x-t)]^n}{n!} \\
&= \frac{B(\beta,\lambda-\beta)}{\Gamma(\lambda)} {}_1F_1\left[\begin{matrix}\beta \\ \lambda-\beta\end{matrix}; \omega(x-t); p, q\right], \qquad (32) \\
&\quad (\Re(\lambda-\beta) > 0)
\end{aligned}$$

where the extended Kummer hypergeometric function ${}_1F_1\left[\begin{matrix}\beta \\ \lambda-\beta\end{matrix}; z; p, q\right]$ is defined by (10). By using Corollary 3.3 and (32), we obtain the following corollary:

Corollary 3.5. *If* $\Re(\alpha) > \Re(\lambda) > \Re(\beta) > 0$ *and* $I_{a+}^{-\alpha} g \in L(a,b)$, *then the solution of the integral equation*

$$\frac{e^{p+\frac{q}{2}}}{B_{2q,\frac{1}{2}p}(\lambda-\beta,\beta)} \int_a^x (x-t)^{\lambda-1} E_{1,\lambda}^{-\beta}[\omega(x-t)]\,\varphi(t)\,\mathrm{d}t = g(x) \ \ (a < x \le b)$$

is given by

$$\varphi(x) = \frac{B(\beta,\lambda-\beta)}{\Gamma(\lambda)} \int_a^x (x-t)^{\alpha-\lambda-1} {}_1F_1\left[\begin{matrix}\beta \\ \lambda-\beta\end{matrix}; \omega(x-t); p, q\right] \left(I_{a+}^{-\alpha} g\right)(t)\,\mathrm{d}t.$$

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