Honam Mathematical J. **37** (2015), No. 4, pp. 387–395
http://dx.doi.org/10.5831/HMJ.2015.37.4.387

BLOCH-TYPE SPACES AND THEIR COMPOSITION OPERATORS

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Abstract. We investigate conditions under which a holomorphic self-map of the unit disk induces a bounded composition operator and study some properties of the composition operator from the Bloch-type spaces into a generalization of the Bloch-type spaces.

1. Introduction

For any real number α with $\alpha > -1$, we define $dA_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha dA$, where dA is the normalized area measure on the unit disk D. Since $\int_D (1-|z|^2)^\alpha dA = \frac{1}{1+\alpha}$, dA_α is a probability measure on D. For $p \geq 1$, the weighted Bergman space $L^p_a(dA_\alpha)$ consists of analytic functions on D which are also in $L^p(D, dA_\alpha)$. By the closedness of $L^2_a(dA_\alpha)$, for each $z \in D$, there is a function K^α_z in L^2_a such that $f(z) = < f, K^\alpha_z >$ for every f in $L^2_a(dA_\alpha)$. In fact, $K^\alpha_z(w) = \frac{1}{(1-\overline{z}w)^{2+\alpha}}$ is called the Bergman kernel, $||K^\alpha_z||_{2,\alpha} = \frac{1}{(1-|z|^2)^{1+\frac{\alpha}{2}}}$ and we define $k^\alpha_z(w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}$, where $||\cdot||_{p,\alpha}$ is the norm in the space $L^p(D, dA_\alpha)$ and $< \cdot, \cdot >$ is the inner product in the space $L^2(D, dA_\alpha)$. Moreover, we get the orthogonal projection P_α from $L^2(D, dA_\alpha)$ on $L^2_a(dA_\alpha)$, in fact, $P_\alpha(f)(z) = \int_D \frac{f(w)}{(1-z\overline{w})^{2+\alpha}} dA_\alpha(w) = < f, K^\alpha_z >$.

Received March 17, 2015. Accepted August 21, 2015.
2010 Mathematics Subject Classification. 47A38, 47B35.
Key words and phrases. weighted Bergman spaces, composition operators, Berezin transforms, integral operators.

Suppose $\beta > 0$. For any analytic function f on D, we define $||f||_\beta = \sup_{z\in D}(1-|z|^2)^\beta |f'(z)|$. Then $||\cdot||_\beta$ is a complete semi-norm on B_β, where B_β is the space of analytic function f on D such that $||f||_\beta$ is finite. For $\beta > 0$, the β-Bloch space B_β is a Banach space with norm of f equals to $||f|| = ||f||_\beta + |f(0)|$ and B_1 is the classical Bloch space.

For the space $H(D)$ of analytic functions on D, the composition operator $C_\phi : H(D) \to H(D)$ by $C_\phi(f) = f\circ\phi$, where ϕ is a holomorphic self-map of D.

For $z \in D$, let $\varphi_z(w) = \dfrac{z-w}{1-\overline{z}w}$. Then φ_z is an element of $\mathrm{Aut}(D)$ which is the set of all bianalytic maps of D onto D and $\mathrm{Aut}(D)$ is the Möbius group under composition because $\varphi_z \circ \varphi_z$ is the identity map on D.

For $\alpha > -1$ and $z \in D$, we define $U_z^\alpha : H(D) \to H(D)$ by $U_z^\alpha f(w) = f\circ\varphi_z(w)\dfrac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}$. We notice that $(U_z^\alpha)^{-1} = U_z^\alpha$ and for $f \in L_a^2$, $||U_z^\alpha f||_{2,\alpha} = ||f||_{2,\alpha}$ and hence U_z^α is a unitary operator on $L_a^2(dA_\alpha)$. For $f \in B_\beta$,

$$
\begin{aligned}
||f\circ\varphi_z||_\beta &= \sup_w (1-|w|^2)^\beta |(f\circ\varphi_z(w))'| \\
&= \sup_w (1-|\varphi_z(w)|^2)^\beta |f'(w)||\varphi'_z(\varphi_z(w))| \\
&= \sup_w \Big(\frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{z}w|^2}\Big)^\beta |f'(w)| \frac{|1-\overline{z}w|^2}{1-|z|^2} \\
&= \sup_w (1-|w|^2)^\beta |f'(w)|(1-|z|^2)^{\beta-1}|1-\overline{z}w|^{2-2\beta}.
\end{aligned}
$$

Thus $||\cdot||_1$ is Möbius invariant.

In this thesis, we prove that C_ϕ is a bounded linear operator on B_β under some condition and we study the bounded below property of C_ϕ. We introduce a generalization E_γ of the Bloch-type spaces and prove that the inclusion function and C_ϕ are bounded linear operators from B_β into E_γ where $\gamma = 2 + \frac{\alpha}{2} - \beta$.

Throughout the paper, we use the symbol $A \preceq B$ ($A \approx B$, respectively) for nonnegative constants A and B to indicate that A is dominated by B times some positive constant ($A \preceq B$ and $B \preceq A$, respectively).

2. Composition Operators

A nice survey of previously known results connecting composition operators and Bloch-type spaces can be found in [1],[2],[4].

We assume that ϕ is a holomorphic self-map of D and C_ϕ is the composition operator on $H(D)$.

Suppose $f \in B_\beta$ and $\beta > 1$. Then

$$|f(z) - f(0)| = \Big|z \int_0^1 f'(zt)dt\Big| \leq |z|||f||_\beta \Big|\int_0^1 \frac{1}{(1-|zt|)^\beta} dt\Big|$$
$$= |z|||f||_\beta \Big|\sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{n!\Gamma(\beta)} |z|^n \int_0^1 t^n dt\Big| \leq \frac{||f||_\beta}{(1-|z|)^{\beta-1}|1-\beta|}.$$

Thus

$$|f(z)| \preceq \frac{||f||_\beta}{(1-|z|)^{\beta-1}}.$$

In fact, if $\beta > 1$ then $(1-|z|^2)^{\beta-1} f(z)$ is bounded on D (see [3]).

Let $\tau_\phi^\beta(z) = \Big(\frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2}\Big)^\beta$. If ϕ is a Möbius transformation then Schwarz's lemma implies that $\phi(z) = \lambda \phi_a(z) = \lambda \frac{a-z}{1-\bar{a}z}$ for some unit modulus λ. Moreover, $\frac{|\phi'(z)|}{1-|\phi(z)|^2} \leq \frac{1}{1-|z|^2}$, that is,

$$\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| \leq 1.$$

Since $|\phi'_a(z)|^{1-\beta} \preceq \Big(\frac{1}{1-|a|}\Big)^{|1-\beta|}$,

$$(1-|z|^2)^\beta |(f\circ\phi_a)'(z)| = (1-|z|^2)^\beta |f'(\phi_a(z))||\phi'_a(z)|$$
$$= \tau_{\phi_a}^\beta(z)|\phi'_a(z)|^{1-\beta}(1-|\phi_a(z)|^2)^\beta |f'(\phi_a(z))|$$
$$\leq ||f||_\beta |\phi'_a(z)|^{1-\beta}, ||C_{\phi_a}|| \preceq \Big(\frac{1}{1-|a|}\Big)^{|1-\beta|}.$$

THEOREM 2.1. If $|\phi'(z)|^{1-\beta}$ is bounded and $\beta > 0$ then C_ϕ is a bounded linear operator on B_β.

Proof. Since ϕ is a holomorphic self-map of D, the Schwarz-Pick lemma implies that $\tau_\phi^1(z) \le 1$ and hence $\tau_\phi^\beta(z) \le 1$. If $f \in B_\beta$ then

$$\begin{aligned}(1-|z|^2)^\beta|(f\circ\phi)'(z)| &= (1-|z|^2)^\beta|f'(\phi(z))||\phi'(z)| \\ &= \tau_\phi^\beta(z)(1-|\phi(z)|^2)^\beta|f'(\phi(z))||\phi'(z)|^{1-\beta} \\ &\preceq ||f||_\beta.\end{aligned}$$

Thus $||C_\phi|| \preceq \sup_{z\in D}|\phi'(z)|^{1-\beta}$.

Corollary 2.2. Suppose ϕ is a holomorphic self-map of D. If $|\psi'(z)|^{1-\beta}$ is bounded and $\beta > 0$, where $\psi = \phi_a \circ \phi$ and $\phi(0) = a$ then C_ϕ is bounded on B_β.

Proof. Since $\phi(0) = a$, $\psi(0) = \phi_a(\phi(0)) = 0$ and hence C_ψ is bounded. If $f \in B_\beta$ then $C_\psi(C_{\phi_a}(f)) = f\circ\phi_a\circ\psi = f\circ\phi = C_\phi(f)$ and hence $||C_\phi|| \le ||C_\psi||||C_{\phi_a}||$. Since $|\psi'(z)|^{1-\beta}$ is bounded and $\beta > 0$, C_ψ is bounded and hence C_ϕ is bounded.

Notice that for each $f \in B_\beta$, $||f - f(0)|| = ||f - f(0)||_\beta = ||f||_\beta$ and hence let $B_a = \{f \in B_\beta : f(a) = 0\}$, where $a \in D$.

Theorem 2.3. Suppose $\phi(0) = 0$. Then $C_\phi : B_\beta \to B_\beta$ is bounded below if and only if $C_\phi : B_0 \to B_0$ is bounded below.

Proof. Take any f in B_0. Since $f\circ\phi(0) = f(0) = 0$, $f\circ\phi$ is in B_0 and hence $C_\phi : B_0 \to B_0$ is bounded below. Conversely, take any f in B_β. Since $f - f(0)$ is in B_0,there is $M > 0$ such that $||C_\phi(f - f(0))|| \ge M||f - f(0)|| = M||f||_\beta$. Since $||C_\phi(f - f(0))|| = \sup(1-|z|^2)^\beta|((f - f(0))\circ\phi)'(z)| = \sup(1-|z|^2)^\beta|(f\circ\phi)'(z)|$, $||C_\phi(f - f(0))|| + |f(0)| \ge M||f||_\beta + |f(0)|$ and hence $||C_\phi(f)|| \succeq ||f||$. Thus C_ϕ is bounded below.

Suppose $\phi(0) = a$ and let $\psi = \phi_a\circ\phi$. Then $\psi(0) = 0$. Notice that C_ψ is bounded below on B_β if and only if C_ϕ is bounded below on B_a if and only if C_ϕ is bounded below on B_β. Here the second equivalence comes from the fact that for each $f \in B_0$, $C_\psi(f)(0) = g(a)$ for some $g \in B_a$ and the third equivalence comes from $f - f(a) \in B_a$. So we assume that $\phi(0) = 0$ and C_ϕ is acting on B_0.

We say that a subset S of D is a sampling set for $\beta-$Bloch space B_β if there is $k > 0$ such that $\sup\{(1-|z|^2)^\beta |f'(z)| : z \in D\} \leq k$ $\sup\{(1-|z|^2)^\beta |f'(z)| : z \in S\}$ for all $f \in B_\beta$. For any $\varepsilon > 0$, let $\Omega_\varepsilon = \left\{z : \left(\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|\right)^\beta \geq \varepsilon\right\}$.

Theorem 2.4. Suppose $\beta \geq 0$ and $|\phi'(z)|^{1-\beta}$ is bounded by 1. Then the composition operator C_ϕ is bounded below on B_β if and only if there is $\varepsilon > 0$ and there is a subset Ω_ε of D such that $G_\varepsilon = \phi(\Omega_\varepsilon)$ is a sampling set for B_β, where $\Omega_\varepsilon = \{z \in D : \tau_\phi^\beta(z) \geq \varepsilon\}$. Moreover, for $0 < \varepsilon_1 < \varepsilon$, G_{ε_1} is also a sampling set for B_β.

Proof. Suppose that there is $\varepsilon > 0$ and there is a sampling set $G_\varepsilon = \phi(\Omega_\varepsilon)$ for B_β. Take any f in B_0. Then there is $k > 0$ such that

$$\begin{aligned} ||f|| = ||f||_\beta &\leq k \sup\{(1-|\phi(z)|^2)^\beta |f'(\phi(z))| : z \in \Omega_\varepsilon\} \\ &\leq k \sup\{(\tau_\phi^\beta(z))^{-1}(1-|z|^2)^\beta |f'(\phi(z))||\phi'(z)|^{1-\beta} : z \in \Omega_\varepsilon\} \\ &\leq \frac{k}{\varepsilon}\sup\{(1-|z|^2)^\beta |f'(\phi(z))| : z \in \Omega_\varepsilon\} \\ &= \frac{k}{\varepsilon}||f \circ \phi||_\beta = \frac{k}{\varepsilon}||C_\phi(f)||_\beta \preceq \frac{k}{\varepsilon}||C_\phi(f)||. \end{aligned}$$

Conversely, suppose that C_ϕ is bounded below on B_β, that is, C_ϕ is bounded below on B_0. Then there is $k > 0$ such that for any $f \in B_0$ with $||f||_\beta = 1$, $\sup\{(1-|z|^2)^\beta |(f \circ \phi)'(z)| : z \in D\} \geq k$. Then $(1-|z_f|^2)^\beta$ $|(f \circ \phi)'(z_f)| \geq \dfrac{k}{2}$ for some $z_f \in D$. Since $(1-|z_f|^2)^\beta |f'(\phi(z_f))| \geq \dfrac{k}{2}$ because $|\phi'(z_f)|^{1-\beta} \leq 1$. Then $G_\varepsilon = \phi(\{z_f : f \in B_0 \text{ with } ||f|| = 1\})$ is a sampling set for B_β.

Corollary 2.5. Suppose $\beta \geq 0$ and $|\phi'(z)|^{1-\beta}$ is bounded. Then C_ϕ is bounded below on B_β if and only if there is $\varepsilon > 0$ and there is Ω_ε such that $\phi(\Omega_\varepsilon)$ is a sampling set for B_β.

In Theorem 2.4, let $\Omega_\varepsilon = \{z \in D : |\tau_\phi^\beta(z)| \geq \frac{k}{2}\}$. Then $\phi(\Omega_\varepsilon)$ is a sampling set for B_β. We want to find some condition which makes the following statement true : G_ε is a sampling set for B_β.

A subset H of D is said to satisfy the reverse Carleson condition if there is $c > 0$ such that $\int_H |f(z)|^p dA(z) \geq c \int_D |f(z)|^p dA(z)$ for all $f \in L^p_a$.

Lemma 2.6. For $\beta \geq 0$ and $a \in D$, $|\phi'_a(z)|^\beta \leq \Big(\frac{2}{1-|a|}\Big)^\beta$.

Proof. Since $\phi'_a(z) = \frac{-1+|a|^2}{(1-\overline{a}z)^2}$, $1-|a|^2 \leq |\phi'_a(z)| \leq \frac{2}{1-|a|}$. This completes the proof.

Theorem 2.7. Suppose $H \subset D$ satisfies the reverse Carleson condition and $\beta \geq 1$. Then H is a sampling set for B_β.

Proof. Suppose $f \in B_0$ and $||f||_\beta = 1$. Take any a in D. Then

$$\begin{aligned}
&\int_D \big((1-|z|^2)^\beta |f'(z)|\big)^\beta (1-|a|^2)^{\beta^2} dA(z) \\
&\qquad \leq \int_D \big((1-|z|^2)^\beta |f'(z)|\big)^\beta |\phi_a'(z)|^{\beta^2} dA(z) \\
&\qquad \leq c \int_H \big((1-|z|^2)^\beta |f'(z)|\big)^\beta |\phi'_a(z)|^{\beta^2} dA(z) \\
&\qquad \leq c||f|_H||_\beta^\beta \int_D |\phi'_a(z)|^{\beta^2} dA(z) \\
&\qquad \preceq c||f|_H||_\beta^\beta \Big(\frac{1}{1-|a|}\Big)^{\beta^2}.
\end{aligned}$$

Thus $||f||_\beta \preceq \Big(\frac{1}{1-|a|}\Big)^{2\beta^2} ||f|_H||_\beta$, that is, $||f||_\beta$ is dominated by $||f|_H||_\beta$.

Corollary 2.8. Suppose $|\phi'(z)|^{1-\beta}$ is bounded and $1-\beta \geq 0$.

(1) C_ϕ is bounded below if and only if there is $\varepsilon > 0$ such that G_ε is a sampling set for B_β.

(2) If there is $\varepsilon > 0$ such that G_ε satisfies the reverse Carleson condition then C_ϕ is bounded below on B_β.

Let E_γ denote the space of analytic functions f on D sudh that $||f||_{E_\gamma} = \sup_{z\in D}(1-|z|^2)^\gamma ||U_z^\alpha f||_{2+\frac{\alpha}{2}-\gamma}$ is finite. We notice that for $f \in B_{2+\frac{\alpha}{2}-\gamma} = B_\beta$, $||f||_{E_\gamma} \preceq ||f||_\beta$ (see [3]).

Proposition 2.9. If $\dfrac{3+\alpha}{2} \geq \beta = 2 + \dfrac{\alpha}{2} - \gamma$ then the inclusion function is a bounded linear operator from B_β into E_γ.

Proof. Take any f in B_β. Then

$$\begin{aligned}
||f||_{E_\gamma} &= \sup_{z\in D}(1-|z|^2)^\gamma ||U_z^\alpha f||_\beta \\
&= \sup_{z\in D}(1-|z|^2)^\gamma \sup_{w\in D}(1-|w|^2)^\beta |(U_z^\alpha f)'(w)| \\
&= \sup_{z\in D}(1-|z|^2)^\gamma \sup_{w\in D}(1-|w|^2)^\beta \\
&\quad \times \left| f'(\varphi_z(w))\varphi'_z(w)\frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}} + f(\varphi_z(w))(2+\alpha)\overline{z}\frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{3+\alpha}}\right| \\
&= \sup_{z\in D}(1-|z|^2)^\gamma \sup_{w\in D}(1-|\varphi_z(w)|^2)^\beta \\
&\quad \times \left| f'(w)\varphi'_z(\varphi_z(w))\frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}\varphi_z(w))^{2+\alpha}} + f(w)(2+\alpha)\overline{z}\frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}\varphi_z(w))^{3+\alpha}}\right| \\
&\preceq \sup_{z\in D}(1-|z|^2)^\gamma \sup_{w\in D}\frac{(1-|z|^2)^\beta(1-|w|^2)^\beta}{|1-\overline{z}w|^{2\beta}} \\
&\quad \times \left| |f'(w)|\frac{|1-\overline{z}w|^2}{1-|z|^2}\frac{|1-\overline{z}w|^{2+\alpha}}{(1-|z|^2)^{1+\frac{\alpha}{2}}} + (2+\alpha)\frac{||f||_\beta}{(1-|w|^2)^{\beta-1}}\frac{|1-\overline{z}w|^{3+\alpha}}{(1-|z|^2)^{2+\frac{\alpha}{2}}}\right| \\
&\leq \sup_{z\in D}(1-|z|^2)^\gamma \sup_{w\in D}(1-|z|^2)^\beta ||f||_\beta |1-\overline{z}w|^{-2\beta} \\
&\quad \times \left\{ \frac{|1-\overline{z}w|^{4+\alpha}}{(1-|z|^2)^{2+\frac{\alpha}{2}}} + (2+\alpha)\frac{|1-\overline{z}w|^{3+\alpha}}{(1-|z|^2)^{2+\frac{\alpha}{2}}}\right\} \\
&= \sup_{z\in D}(1-|z|^2)^{\gamma+\beta-2-\frac{\alpha}{2}}||f||_\beta \sup_{w\in D}|1-\overline{z}w|^{-2\beta+3+\alpha}(|1-\overline{z}w|+2+\alpha) \\
&\leq 2^{-2\beta+3+\alpha}(4+\alpha)||f||_\beta.
\end{aligned}$$

Here the 5th inequality equivalence comes from $|f(w)| \preceq \dfrac{||f||_\beta}{(1-|w|^2)^{\beta-1}}$.

That is, $||f||_{E_\gamma} \preceq ||f||_\beta$. Thus the inclusion function is a bounded linear operator.

Theorem 2.10. If $\gamma = 2 + \frac{\alpha}{2} - \beta \geq \frac{1}{2}$ and $|\phi'(z)|^{1-\beta}$ is bounded on D then the composition operaotr $C_\phi : B_\beta \to E_\gamma$ is a bounded linear operator.

Proof. Notice that for any g in B_β, $||g||_{E_\gamma} \preceq ||g||_\beta$ (see [3]). Take any f in B_β. Then $||C_\phi(f)||_{E_\gamma}$

$$\begin{aligned}
&= ||f \circ \phi||_{E_\gamma} \\
&= \sup_{z \in D} (1 - |z|^2)^\gamma ||U_z^\alpha (f \circ \phi)||_\beta \\
&\preceq \sup_{z \in D} (1 - |z|^2)^\gamma ||f \circ \phi||_\beta \\
&= \sup_{z \in D} (1 - |z|^2)^\gamma \sup_{w \in D} (1 - |w|^2)^\beta |f'(\phi(w))||\phi'(w)| \\
&= \sup_{z \in D} (1 - |z|^2)^\gamma \sup_{w \in D} \tau_\phi^\beta(w)(1 - |\phi(w)|^2)^\beta |f'(\phi(w))||\phi'(z)|^{1-\beta} \\
&\preceq \sup_{z \in D} (1 - |z|^2)^\gamma ||f||_\beta \\
&\leq ||f||_\beta
\end{aligned}$$

Thus $C_\phi : B_\beta \to E_\gamma$ is a bounded linear operator.

Corollary 2.11. If $\gamma = 2 + \frac{\alpha}{2} - \beta \geq \frac{1}{2}$ and $|\phi'(z)|^{1-\beta}$ is bounded on D then the composition operator $C_\phi : B_\beta \to B_\beta$ is a bounded linear operator.

Proof. If follows from the fact that $||f \circ \phi||_\beta \preceq ||f||_\beta$ for all $f \in B_\beta$.

Theorem 2.12. Suppose ϕ is a holomorphic self-map of D, $\gamma = 2 + \frac{\alpha}{2} - \beta \geq \frac{1}{2}$ and $|\phi'(z)|^{1-\beta}$ is bounded on D. If for any $\varepsilon > 0$ there is r such that $0 < r < 1$ and $\tau_\phi^\beta(z) < \varepsilon$ whenever $|\phi(z)| > r$ then $C_\phi : B_\beta \to E_\gamma$ is a compact operator.

Proof. Suppose (f_n) is a bounded sequence in B_β and converges to 0 uniformly on compact subsets of D. Let $M = \sup_{n \in \mathbb{N}} ||f_n||$ and $K = \sup_{z \in D} |\phi'(z)|^{1-\beta}$, where $||f|| = ||f||_\beta + |f(0)|$. Take any $\varepsilon > 0$. By the assumption, there is r such that $0 < r < 1$ and for $|\phi(z)| > r$, $\tau_\phi^\beta(z) < \frac{\varepsilon}{2MK}$. Note that

$$\begin{aligned}(1-|w|^2)^\beta |f'(\phi(w))||\phi'(w)| &\leq \tau_\phi^\beta(w)(1-|\phi(w)|^2)^\beta |f'(\phi(w))||\phi'(w)|^{1-\beta} \\ &\preceq \tau_\phi^\beta(w)||f||_\beta,\end{aligned}$$

$f_n \circ \phi(0) \to 0$ and there is k such that for $n \geq k$, $(1-|w|^2)^\beta |(f_n \circ \phi)'(w)| < \frac{\varepsilon}{2}$ whenever $|\phi(w)| \leq r$. Hence $||f_n \circ \phi||_{E_\gamma} \to 0$ as $n \to \infty$. Thus C_ϕ is compact.

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