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SYMMETRY OVER CENTERS

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Abstract. The symmetric ring property was due to Lambek and provided many useful results in relation with noncommutative ring theory. In this note we consider this property over centers, introducing *symmetric-over-center*. It is shown that symmetric and symmetric-over-center are independent of each other. The structure of symmetric-over-center ring is studied in relation to various radicals of polynomial rings.

1. Introduction

Throughout this note every ring is associative with identity unless otherwise stated. Let R be a ring. We use $R[x]$ to denote the polynomial ring with an indeterminate x over R. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Use E_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). $C(R)$ denotes the center of R, i.e., $C(R) = \{a \in R \mid ar = ra \text{ for all } r \in R\}$.

A ring is usually called *reduced* if it has no nonzero nilpotent elements. Lambek introduced the concept of a symmetric right ideal, unifying the sheaf representation of commutative rings and reduced rings in [8]. Lambek called a right ideal I of a ring R *symmetric* if $rst \in I$ implies $rts \in I$ for all $r, s, t \in R$. If the zero ideal is symmetric then R is usually called *symmetric*; while Anderson-Camillo [3] used the term ZC_3 for this concept. It is proved by Lambek that a ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$ implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of

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the set $\{1, 2, \ldots, n\}$, where $n \geq 1$ and $r_i \in R$ for all i (see [8, Proposition 1]). Anderson-Camillo also obtained this result independently in [3, Theorem I.1]. Commutative rings clearly symmetric. Reduced rings are symmetric by [3, Theorem I.3]. There exist many non-reduced commutative rings (e.g., $\mathbb{Z}_{m^k}$ with $m, k \geq 2$), and many noncommutative reduced rings (e.g., direct products of noncommutative domains). A ring is usually called *Abelian* if every idempotent is central. It is simply checked that symmetric rings are Abelian.

Let R be a ring and $n \geq 2$. Following the literature, consider the extension rings

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in U_n(R) \mid a, a_{ij} \in R \right\},$$

$$N_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{ii} = 0 \text{ for all } i\}, \text{ and}$$

$$V_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{ij} = a_{(i+1)(j+1)} \text{ for } i = 1, \ldots, n-2 \text{ and } j = 2, \ldots, n-1\}.$$

It is easily checked that $V_n(R)$ is isomorphic to the factor ring $R[x]/x^n R[x]$. Note $D_2(R) = V_2(R)$.

A ring shall be called *symmetric-over-center* if $abc \in C(R)$ implies $acb \in C(R)$ for all $a, b, c \in R$. Symmetric-over-center rings are also Abelian by Lemma 2.2(1) to follow.

Lemma 1.1. (1) *Let A be a ring. The center of $D_3(A)$ is*

$$\left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b \in C(A) \right\}.$$

(2) *Let A be a ring. The center of $D_2(A)$ is*

$$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in C(A) \right\}.$$

(3) *A ring A is commutative if and only if so is $D_2(A)$.*

Proof. (1) Let $R = D_3(A)$ and $M = \begin{pmatrix} a & c & b \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \in C(R)$. Then $(rI_3)M = M(rI_3)$ implies $ra = ar$ for all $r \in A$, where I_3 is the identity matrix in R. Thus $a \in C(A)$.

First note that the subring $\left\{ \begin{pmatrix} s & 0 & t \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} \mid s, t \in C(A) \right\}$ of R is contained in $C(R)$. Then $N = \begin{pmatrix} 0 & c & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} = M - (aI_3)$ is contained in $C(R)$ since $aI_3 \in C(R)$. So $cE_{13} = NE_{23} = E_{23}N = 0$ and this yields $c = 0$. We also get $0 = NE_{12} = E_{12}N = dE_{13}$, entailing $d = 0$. These imply $M = \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$. Next we have

$$\begin{pmatrix} ra & 0 & rb \\ 0 & ra & 0 \\ 0 & 0 & ra \end{pmatrix} = (rI_3)M = M(rI_3) = \begin{pmatrix} ar & 0 & br \\ 0 & ar & 0 \\ 0 & 0 & ar \end{pmatrix},$$

where $r \in A$. This yields $br = rb$; hence $b \in C(A)$. This completes the proof.

(2) is shown by the method in the proof of (1), and (3) is an immediate consequence of (2). □

In fact the center of $D_3(A)$ is isomorphic to $D_2(C(A))$ in Lemma 1.1.

Proposition 1.2. *If A is a commutative ring then $D_3(A)$ is a (noncommutative) symmetric-over-center ring.*

Proof. Let A be a commutative ring and $R = D_3(A)$. Then the center of R is

$$\left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b \in A \right\}$$

by Lemma 1.1(1).

To show that R is symmetric-over-center, let $M_1M_2M_3 \in C(R)$ for $M_1 = (a_{ij}), M_2 = (b_{ij}), M_3 = (c_{ij}) \in R$. Then $M_1M_2M_3 = \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ for some $a, b \in A$. Here let

$$M_1' = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{11} \end{pmatrix}, M_2' = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{11} \end{pmatrix}, M_3' = \begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{11} \end{pmatrix} \in D_2(A).$$

But $D_2(A)$ is commutative by Lemma 1.1(3), and so $M_1'M_2'M_3' = M_1'M_3'M_2'$. This gives us that the $(1,2)$-entry of $M_1M_2M_3$ is equal to one of $M_1M_3M_2$; hence the $(1,2)$-entry of $M_1M_3M_2$ is zero. Similarly, the $(2,3)$-entry of

$M_1M_3M_2$ is also zero. Consequently $M_1M_3M_2$ is contained in the center of R, and thus R is symmetric-over-center. □

In the following we see that the concepts of symmetric and symmetric-over-center are independent of each other.

Example 1.3. (1) There exists a symmetric-over-center ring which is not symmetric. Let A be a commutative ring and $R = D_3(A)$. Then R is not symmetric by the computation that $E_{12}E_{23} = E_{13} \neq 0 = E_{23}E_{12}$. But R is symmetric-over-center by Proposition 1.2.

(2) There exists a symmetric ring which is not symmetric-over-center. Let K be a field and $A = K\langle a, b, c\rangle$ be the free algebra generated by the noncommuting indeterminates a, b, c over K. Let I be the ideal of A generated by

$$ab - c, ac - ca, bc - cb$$

and set $R = A/I$. We identify a, b, c with their images in R for simplicity. We first obtain $c \in C(R)$ from the fact that $ac = ca$ and $bc = cb$.

It is easily checked that R is a domain (hence symmetric). However R is not symmetric-over-center as can be seen by the computation that $1ab = ab = c \in C(R)$, but $aba \neq baa$ implies $1ba = ba \notin C(R)$.

Considering the domain in Example 1.3(2), one may hope division rings to be symmetric-over-center. However there exist division rings which are not symmetric-over-center as we see in the following.

Example 1.4. (1) Let R be the Hamilton quaternions over the real number field. Consider $a = i, b = i + j + k$ and $c = \frac{1}{3}(-1 + j - k)$ in R. Then $abc = 1 \in C(R)$. But $acb = \frac{1}{3}(-1 + 2j - 2k) \notin C(R)$. Thus R is not symmetric-over-center.

(2) Let K be a field of characteristic zero and $A = K\langle x, y\rangle$ be the free algebra generated by the noncommuting indeterminates x, y over K. Let I be the ideal of A generated by $yx - xy - 1$ and set $R = A/I$. We identify x, y with their images in R for simplicity. Recall that R is called the first Weyl algebra over K.

Note $C(R) = K$. Suppose that $abc \in K$ for $a, b, c \in R$. Then it is easily checked that $a, b, c \in K$. So $acb \in C(R)$. Thus R is symmetric-over-center.

But we claim that the quotient division ring of R is not symmetric-over-center, Q say. Note $C(Q) = K$. Consider $a = x, b = xy, c = (x^2y)^{-1}$ in Q. Then $abc = 1$, but

$$acb = x(x^2y)^{-1}xy = xy^{-1}x^{-2}xy = xy^{-1}x^{-1}y$$

is not contained in K. Thus Q is not symmetric-over-center.

2. Basic structure of symmetric-over-center rings

In this section we will study the basic structure of symmetric-over-center rings. Let R be a ring. $N_*(R)$, $N^*(R)$, $N_0(R)$, $N(R)$ and $J(R)$ denote the prime radical, the upper nilradical (i.e., sum of all nil ideals), the Wedderburn radical (i.e., the sum of all nilpotent ideals), the set of all nilpotent elements and the Jacobson radicalin R, respectively. Following [1, p.130], a subset of R is said to be *locally nilpotent* if its finitely generated subrings are nilpotent. Also due to [1, p.130], the Levitzki radical of R, written by $s\sigma(R)$, means the sum of all locally nilpotent ideals of R. It is well-known that $N^*(R) \subseteq J(R)$ and $N_0(R) \subseteq N_*(R) \subseteq s\sigma(R) \subseteq N^*(R) \subseteq N(R)$.

This symmetric-over-center property is left-right symmetric as follows.

Proposition 2.1. *Given a ring R, the following conditions are equivalent:*

(1) *R is symmetric-over-center;*

(2) *If $a_1a_2\cdots a_n \in C(R)$ for $a_1,\dots,a_n \in R$, then $a_{\theta(1)}a_{\theta(2)}\cdots a_{\theta(n)} \in C(R)$ for any permutation θ of the set $\{1,2,\dots,n\}$, where n is any positive integer;*

(3) *$abc \in C(R)$ implies $bac \in C(R)$ for $a,b,c \in R$.*

Proof. We apply the proof of [7, Theorem 1.2].

(1) $\Rightarrow$ (2): Let R be a symmetric-over-center ring and suppose that

$$a_1\cdots a_i\cdots a_j\cdots a_n \in C(R)$$

for $a_1,\dots,a_i,\cdots,a_j,\dots,a_n \in R$, where $i<j$. We will use freely R being symmetric-over-center. From

$$(a_1\cdots a_{i-1})(a_i\cdots a_{j-1})(a_j\dots a_n) \in C(R),$$

we get

$$(a_1\cdots a_{i-1})(a_j\dots a_n)(a_i\cdots a_{j-1}) \in C(R).$$

Next from

$$(a_1\cdots a_{i-1}a_j)(a_{j+1}\dots a_na_i)(a_{i+1}\cdots a_{j-1}) \in C(R),$$

we get

$$(a_1\cdots a_{i-1}a_j)(a_{i+1}\cdots a_{j-1})(a_{j+1}\dots a_na_i) \in C(R).$$

Similarly from

$$(a_1 \cdots a_{i-1} a_j a_{i+1} \cdots a_{j-1})(a_{j+1} \ldots a_n) a_i \in C(R),$$

we get

$$(a_1 \cdots a_{i-1}) a_j (a_{i+1} \cdots a_{j-1}) a_i (a_{j+1} \ldots a_n) \in C(R).$$

Note that any permutation is a product of finite number of transpositions, and so the preceding result implies that $a_{\theta(1)} a_{\theta(2)} \cdots a_{\theta(n)} \in C(R)$ for any permutation θ of the set $\{1, 2, \ldots, n\}$.

(2) $\Rightarrow$ (3) is clear, and (3) $\Rightarrow$ (2) is shown by a similar method to the proof of (1) $\Rightarrow$ (2).

□

Following the literature, the *index of nilpotency* of $a \in N(R)$ is the least positive integer n such that $a^n = 0$, write $i(a)$ for n. The *index of nilpotency* of a subset S of R is the supremum of the indices of nilpotency of all nilpotent elements in S, write $i(S)$; and if such a supremum is finite, then S is said to be *of bounded index of nilpotency.*

Lemma 2.2. *Let R be a symmetric-over-center ring. Then we have the following results.*

(1) *R is Abelian.*

(2) *If $a^2 = 0$ for $a \in R$ then $ara \in C(R)$ for all $r \in R$.*

(3) *If $a^2 = 0$ and $b^n = 0$ for $a, b \in R$ and $n \geq 1$, then $(ab)^{2n-1} = 0$ and $(ba)^{2n-1} = 0$.*

(4) *If $a^2 = 0$ for $a \in R$ then $(RaR)^3 = 0$.*

(5) *If R is of bounded index of nilpotency with $i(R) = 2$, then*

$$N_0(R) = N_*(R) = s\sigma(R) = N^*(R) = N(R).$$

Proof. (1) Let R be a symmetric-over-center ring, and $e^2 = e, r \in R$. Then $e(1-e)r = 0$. Since R is symmetric-over-center, $er(1-e) \in C(R)$. This yields $er(1-e) = e(er(1-e)) = (er(1-e))e = 0$, entailing $er = ere$. Similarly, we have $(1-e)re = 0$. So $re = ere$, and consequently $er = re$. Thus R is Abelian.

(2) Let $a^2 = 0$ for $a \in R$. Then $aar = 0 \in C(R)$. Since R is symmetric-over-center, $ara \in C(R)$.

(3) Suppose that $a^2 = 0$ and $b^n = 0$ for $a, b \in R$ and $n \geq 1$. Since R is symmetric-over-center, $aba \in C(R)$ by (2). This yields

$$0 = b^n (aba)^{n-1} = b(aba)b \cdots b(aba)b,$$

and this yields $(ab)^{2n-1} = 0$ and $(ba)^{2n-1} = 0$.

(4) Let $a^2 = 0$ for $a \in R$. Then $ara \in C(R)$ by (2), and so we have $(ara)sa = sa(ara) = 0$ for all $s \in R$. This yields $(RaR)^3 = 0$ since every element of $(RaR)^3$ is of the form $\sum_{\text{finite}} rasatau$ with $r, s, t, u \in R$.

(5) Let $a \in N(R)$. If $i(R) = 2$ then $a^2 = 0$. Since R is symmetric-over-center, $(RaR)^3 = 0$ by (4) and so $a \in N_0(R)$. This completes the proof.

□

In section 1, we see domains which are not symmetric-over-center, and so the converse of Lemma 2.2(1) need not hold since domains are clearly Abelian.

Lemma 2.3. (1) [2, Theorem 3] $N_*(R[x]) = N_*(R)[x]$ *for any ring* R.

(2) [2, Theorem 1] $J(R[x]) = N[x]$ *for any ring* R*, where* $N = J(R[x]) \cap R$ *is a nil ideal of* R *which contains* $s\sigma(R)$.

(3) [5, Corollary 4] $N_0(R[x]) = N_0(R)[x]$ *for any ring* R.

Proposition 2.4. (1) *Let* R *be a symmetric-over-center ring such that* R *is of bounded index of nilpotency with* $i(R) = 2$. *Then*

$$\begin{aligned} J(R[x]) &= N_0(R[x]) = N_*(R[x]) = s\sigma(R[x]) \\ &= N^*(R[x]) = N_0(R)[x] = N(R)[x] = N(R[x]). \end{aligned}$$

(2) *Let* R *be a symmetric-over-center ring such that* R *is of bounded index of nilpotency with* $i(R) = 2$. *Then* $R[x]/J(R[x])$ *is a reduced ring.*

Proof. (1) Let R be a symmetric-over-center ring such that R is of bounded index of nilpotency with $i(R) = 2$. Then we have

$$N_0(R) = N_*(R) = s\sigma(R) = N^*(R) = N(R) \quad (1)$$

by Lemma 2.2(5).

Next we get

$$J(R[x]) \subseteq N^*(R)[x]$$

by Lemma 2.3(2), entailing $J(R[x]) \subseteq N_*(R)[x]$ by the equality (1). But $N_*(R)[x] = N_*(R[x])$ by Lemma 2.3(1), and $N_*(R[x]) \subseteq J(R[x])$. Thus we have

$$\begin{aligned} J(R[x]) &= N^*(R)[x] = s\sigma(R)[x] = N_*(R[x]) \\ &= N_*(R)[x] = N(R)[x] = N_0(R)[x], \quad (2) \end{aligned}$$

combining the results above. Moreover $N_0(R)[x] = N_0(R[x])$ by Lemma 2.3(3), so we get the equality

$$\begin{aligned} J(R[x]) &= N^*(R)[x] = N_*(R[x]) = s\sigma(R)[x] \\ &= N_*(R)[x] = N(R)[x] = N_0(R)[x] = N_0(R[x]), \quad (3) \end{aligned}$$

from the equality (2).

Since $N_*(R) = N(R)$ by the equality (1), $N(R[x]) = N_*(R[x])$ by [4, Proposition 2.6]. So we finally obtain

$$\begin{aligned} J(R[x]) &= N_0(R[x]) = N_*(R[x]) = s\sigma(R[x]) \\ &= N^*(R[x]) = N_0(R)[x] = N(R)[x] = N(R[x]). \end{aligned}$$

(2) Let R be a symmetric-over-center ring such that R is of bounded index of nilpotency with $i(R) = 2$. Then $J(R[x]) = N(R[x])$ by (1), and so $R[x]/J(R[x])$ is a reduced ring. □

Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. The *left regular* can be defined similarly. An element is *regular* if it is both left and right regular (i.e., not a zero divisor).

Proposition 2.5. *Let R be a ring and M be a multiplicatively closed subset of R consisting of central regular elements. Then R is symmetric-over-center if and only if so is $M^{-1}R$.*

Proof. Set $E = M^{-1}R$. Then $C(E) = M^{-1}C(R)$ by the proof of [6, Proposition 2.2]. We use this fact freely.

Let $\alpha\beta\gamma \in C(E)$ for $\alpha = u^{-1}a, \beta = v^{-1}b, \gamma = w^{-1}c$ with $u, v, w \in M$ and $a, b, c \in R$. Then

$$\alpha\beta\gamma = z^{-1}(abc) \in C(E)$$

implies $abc \in C(R)$, where $z = uvw$.

If R is symmetric-over-center, then $abc \in C(R)$ implies $acb \in C(R)$. Thus we have

$$\alpha\gamma\beta = u^{-1}aw^{-1}cv^{-1}c = (uwv)^{-1}(acb) \in M^{-1}C(R) = C(E).$$

Therefore E is symmetric-over-center.

Conversely let $abc \in C(R)$ for $a, b, c \in R$. Since $C(R) \subset M^{-1}C(R) = C(E)$, we have $abc \in C(E)$. If E is symmetric-over-center, then $abc \in C(E)$ implies $acb \in C(E)$. But $acb \in R$, so $acb \in C(R) = R \cap C(E)$. Thus R is symmetric-over-center.

□

Let R be a ring. Recall that the ring of *Laurent polynomials*, in an indeterminate x over R, consists of all formal sums $\sum_{i=k}^{n} a_i x^i$ with obvious addition and multiplication, where $a_i \in R$ and k, n are (possibly negative) integers with $k \leq n$. We denote this ring by $R[x; x^{-1}]$.

Corollary 2.6. *Let R be a ring. Then $R[x]$ is symmetric-over-center if and only if $R[x; x^{-1}]$ is symmetric-over-center.*

Proof. The proof is an immediate consequence of Proposition 2.5, noting that $R[x; x^{-1}] = M^{-1}R[x]$ if $M = \{1, x, x^2, \ldots\}$. □

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