Communications for Statistical Applications and Methods
2015, Vol. 22, No. 6, 615–624
DOI: http://dx.doi.org/10.5351/CSAM.2015.22.6.615
Print ISSN 2287-7843 / Online ISSN 2383-4757

Estimation of Seasonal Cointegration under Conditional Heteroskedasticity

Byeongchan Seong[1,a]

[a]Department of Applied Statistics, Chung-Ang University, Korea

Abstract

We consider the estimation of seasonal cointegration in the presence of conditional heteroskedasticity (CH) using a feasible generalized least squares method. We capture cointegrating relationships and time-varying volatility for long-run and short-run dynamics in the same model. This procedure can be easily implemented using common methods such as ordinary least squares and generalized least squares. The maximum likelihood (ML) estimation method is computationally difficult and may not be feasible for larger models. The simulation results indicate that the proposed method is superior to the ML method when CH exists. In order to illustrate the proposed method, an empirical example is presented to model a seasonally cointegrated times series under CH.

Keywords: seasonal error correction model, seasonal unit root, reduced rank estimation, multivariate GARCH, feasible generalized least squares, maximum likelihood estimation, vector autoregressive model

1. Introduction

The seminal work of Engle and Granger (1987) allowed statistical methods to be applied to the analysis of cointegrated systems. Among others, the joint modeling of cointegration and generalized autoregressive conditional heteroskedasticity (GARCH), proposed by Bollerslev (1986) has attracted the attention of many researchers in finance and economics because the same model can explain two common economic variable characteristics of long-run equilibrium and time-varying volatility. For example, Seo (2007) developed the asymptotic distribution of the cointegrating vector estimator in the vector error correction model (VECM) with GARCH errors. Herwartz and Lütkepohl (2011) proposed a feasible generalized least squares (FGLS) estimator in the joint model and showed that it has superior small sample properties. However, all these studies are restricted to nonseasonal cointegration.

The VECM for seasonal cointegration is more complicated and has more parameters than nonseasonal cointegration. This is because the former has more than two error correction terms, composed of several seasonal filters including complex unit roots; see, for example, Seong (2013) and the references cited therein. Therefore, if we use a full maximum likelihood (ML) procedure that takes into account the GARCH errors, computation of the estimates will be quite demanding, and it may even be infeasible for larger models with a moderate number of variables and a realistic number of lags.

We explore the estimation of a seasonal VECM with conditional heteroskedasticity (CH) using a FGLS method that is easier to implement than the ML method. The proposed method relies on common linear regression methods, such as ordinary least squares (OLS) and generalized least squares

[1] Corresponding author: Department of Applied Statistics, Chung-Ang University, 221, Heukseok-dong, Dongjak-gu, Seoul 06974, Korea. E-mail: bcseong@cau.ac.kr

Published 30 November 2015 / journal homepage: http://csam.or.kr

(GLS), which do not require numerical optimization; subsequently, all the computations are based on closed form expressions. Monte Carlo experiments are conducted to evaluate the proposed procedure and for an empirical illustration, we analyze monthly U.S. Housing Starts and Sold using the procedure.

2. The Model

We consider a vector autoregressive (VAR) model for a K-dimensional process Y_t satisfying

$$\Pi(L)Y_t = \left(I_K - \sum_{j=1}^{p} \Pi_j L^j\right) Y_t = \varepsilon_t, \quad \text{for } t = 1, \ldots, T, \tag{2.1}$$

where I_K denotes a $K \times K$ identity matrix, $\Pi(L)$ is a polynomial matrix, and L is a lag operator such that $L^j Y_t = Y_{t-j}$. To allow for the possible presence of CH, the error process ε_t is assumed to be a vector martingale difference sequence with $E(\varepsilon_t|\mathcal{F}_{t-1}) = 0$ and $E(\varepsilon_t \varepsilon_t'|\mathcal{F}_{t-1}) = \Omega_t$, where $\mathcal{F}_t$ is the σ-field generated by $Y_t, Y_{t-1}, \ldots$. We assume that the roots of the determinant $|\Pi(z)| = 0$ are on or outside the unit circle and that Y_t is observed on a quarterly basis and has no deterministic terms. Models with other seasonal periods (e.g., monthly) and models with deterministic terms that may contain a constant, a linear trend, or seasonal dummies can easily be implemented as in Ahn *et al.* (2004).

Then, as in Ahn *et al.* (2004), if the series are cointegrated of order $(1, 1)$ at frequencies 0, π, $\pi/2$, and $3\pi/2$, model (2.1) can be rewritten as the following seasonal VECM:

$$\Psi(L)Z_t = A_1B_1U_{t-1} + A_2B_2V_{t-1} + (A_3B_4 + A_4B_3)W_{t-1} + (A_4B_4 - A_3B_3)W_{t-2} + \varepsilon_t, \tag{2.2}$$

where $Z_t = (1 - L^4)Y_t$, $U_t = (1 + L)(1 + L^2)Y_t$, $V_t = (1 - L)(1 + L^2)Y_t$, $W_t = (1 - L^2)Y_t$, $\Psi(L)$ is a matrix polynomial of order $p - 4$, A_j and B_j are $K \times r_j$ and $r_j \times K$ matrices, respectively, with rank equal to r_j for $j = 1, \ldots, 4$, and $r_3 = r_4$. For a unique parameterization, we need to normalize the B_js such that $B_1 = [I_{r_1}, B_{10}]$, $B_2 = [I_{r_2}, B_{20}]$, $B_3 = [I_{r_3}, B_{30}]$, and $B_4 = [O_{r_3}, B_{40}]$, where O_{r_j} is an $r_j \times r_j$ matrix of zeros, and the B_{j0}s are $r_j \times (K - r_j)$ matrices of unknown parameters. Lütkepohl (2005) explained that this normalization does not imply a loss of generality from a practical point of view. Note that r_1, r_2, and $r_3(r_4)$ denote the seasonal cointegrating ranks at frequencies 0, π, and $\pi/2$ $(3\pi/2)$, respectively, and that B_1U_t, B_2V_t, $(B_3 + B_4L)W_t$, and $(B_4 - B_3L)W_t$ are stationary processes, that is, they represent cointegrating relationships or long-run equilibrium relationships.

3. Estimation of a Seasonal VECM with CH

Before we present a FGLS method for estimating seasonal cointegration under CH, we consider a Gaussian ML estimation.

3.1. ML estimation

For a sample with T observations and p presample values, ML estimation of VECM (2.2) is theoretically straightforward if we assume $\varepsilon_t|\mathcal{F}_{t-1} \sim N(0, \Omega_t)$. Collecting all parameters in a vector η, the log-likelihood function is given by

$$l(\eta) = T^{-1} \sum_{t=1}^{T} l_t(\eta), \quad \text{where } l_t(\eta) = -\frac{1}{2} \log|\Omega_t(\eta)| - \frac{1}{2}\varepsilon_t(\eta)'\Omega_t(\eta)^{-1}\varepsilon_t(\eta). \tag{3.1}$$

Therefore, full ML estimator with allowance for CH (FML) can be obtained by maximizing the function. Ahn and Reinsel (1994) obtained the ML estimator ignoring CH, that is, assuming $\varepsilon_t \sim N(0, \Omega)$, by the Gaussian reduced rank ML estimation (RRML).

3.2. A FGLS estimation

Given the practical difficulties in computing the ML estimator in the presence of CH, we propose the following three-step procedure for a FGLS estimator.

Step 1. Estimate the parameters in the model

$$\Psi(L)Z_t = C_1 U_{t-1} + C_2 V_{t-1} + C_3 W_{t-1} + C_4 W_{t-2} + \varepsilon_t \tag{3.2}$$

by OLS and denote the residuals by $\hat{\varepsilon}_t$. Then, estimate the CH parameters, say θ, from a pseudo ML estimation based on maximizing $\hat{l}(\theta) = T^{-1} \sum_{t=1}^{T} \hat{l}_t(\theta)$, where

$$\hat{l}_t(\theta) = -\frac{1}{2} \log |\Omega_t(\theta)| - \frac{1}{2} \hat{\varepsilon}_t' \Omega_t(\theta)^{-1} \hat{\varepsilon}_t.$$

Denote the resulting estimate by $\hat{\theta}$ and define $\hat{\Omega}_t = \Omega_t(\hat{\theta})$.

Step 2. Obtain a feasible estimator $\hat{\alpha}$ for $\alpha = \text{vec}(C_1, C_2, C_3, C_4, \Psi_1, \ldots, \Psi_{p-4})$ from the transformed model, which is derived by premultiplying VECM (2.2) by the $\hat{\Omega}_t^{-1/2}$ from Step 1,

$$\hat{\Omega}_t^{-\frac{1}{2}} Z_t = \left[P_t \otimes \hat{\Omega}_t^{-\frac{1}{2}} \right] \alpha + \tilde{\varepsilon}_t, \tag{3.3}$$

where $P_t = [U_{t-1}', V_{t-1}', W_{t-1}', W_{t-2}', Z_{t-1}', \ldots, Z_{t-p+4}']$, $\tilde{\varepsilon}_t = \hat{\Omega}_t^{-1/2} \varepsilon_t$ and vec($\cdot$) vectorizes a matrix columnwise from left to right, and $\otimes$ denotes the Kronecker product. The FGLS estimator $\hat{\alpha}$ is then given by

$$\hat{\alpha} = \text{vec}\left(\hat{C}_1, \hat{C}_2, \hat{C}_3, \hat{C}_4, \hat{\Psi}_1, \ldots, \hat{\Psi}_{p-4}\right) = \left[\sum_{t=1}^{T} \left(P_t{}' P_t \otimes \hat{\Omega}_t^{-1} \right) \right]^{-1} \text{vec} \left\{ \sum_{t=1}^{T} \left(\hat{\Omega}_t^{-1} Z_t P_t \right) \right\}.$$

Then, we can obtain $\hat{A}_1 = \hat{C}_{11}$, $\hat{A}_2 = \hat{C}_{12}, \hat{A}_3 = -\hat{C}_{14}$, and $\hat{A}_4 = \hat{C}_{13}$ as the FGLS estimators for A_1, A_2, A_3, and A_4, respectively, where $\hat{C}_{1j}$ is the matrix with the first r_j columns of $\hat{C}_j$ for $j = 1, 2, 3, 4$, by using the following relationships between models (2.2) and (3.2).

$$C_j = \begin{cases} A_j B_j = [A_j, A_j B_{j0}], & \text{for } j = 1, 2, \\ A_3 B_4 + A_4 B_3 = [A_4, A_3 B_{40} + A_4 B_{30}], & \text{for } j = 3, \\ A_4 B_4 - A_3 B_3 = [-A_3, -A_3 B_{30} + A_4 B_{40}], & \text{for } j = 4. \end{cases}$$

If there is no CH so that $\Omega_t = \Omega$, the FGLS estimator reduces precisely to the RRML estimator considered by Ahn and Reinsel (1994). As usual, the covariance matrix of $\hat{\alpha}$ is calculated by var($\hat{\alpha}$) = $[\sum_{t=1}^{T} (P_t{}' P_t \otimes \hat{\Omega}_t^{-1})]^{-1}$.

Step 3. In order to compute the FGLS estimator $\hat{\beta}$ for $\beta = \text{vec}(B_{10}, B_{20}, B_{30}, B_{40})$, we take a multivariate linear regression model (obtained from the model in (3.3) by replacing the A_js and the Ψ_js by their respective FGLS estimators, $\hat{A}_j$s and $\hat{\Psi}_j$s). Thus,

$$\tilde{Z}_t = \hat{\Omega}_t^{-\frac{1}{2}} Q_t \beta + \tilde{\varepsilon}_t,$$

where

$$\tilde{Z}_t = \hat{\Omega}_t^{-\frac{1}{2}} \left(Z_t - \hat{A}_1 U_{1t-1} - \hat{A}_2 V_{1t-1} - \hat{A}_4 W_{1t-1} + \hat{A}_3 W_{1t-2} - \sum_{j=1}^{p-4} \hat{\Psi}_j Z_{t-j} \right),$$

$$Q_t = \left[U'_{2t-1} \otimes \hat{A}_1,\ V'_{2t-1} \otimes \hat{A}_2,\ W'_{2t-1} \otimes \hat{A}_4 - W'_{2t-2} \otimes \hat{A}_3,\ W'_{2t-1} \otimes \hat{A}_3 + W'_{2t-2} \otimes \hat{A}_4 \right],$$

U_{1t}, V_{1t}, and W_{1t} are the first r_1, r_2, and r_3 components of U_t, V_t, and W_t, respectively, and U_{2t}, V_{2t}, and W_{2t} are the last $K - r_1$, $K - r_2$, and $K - r_3$ components of U_t, V_t, and W_t, respectively. Then, the FGLS estimator for β can be expressed by

$$\hat{\beta} = \left(\sum_{t=1}^{T} Q'_t \hat{\Omega}_t^{-1} Q_t \right)^{-1} \left(\sum_{t=1}^{T} Q'_t \hat{\Omega}_t^{-1} \dot{Z}_t \right),$$

where $\dot{Z}_t = Z_t - \hat{A}_1 U_{1t-1} - \hat{A}_2 V_{1t-1} - \hat{A}_4 W_{1t-1} + \hat{A}_3 W_{1t-2} - \sum_{j=1}^{p-4} \hat{\Psi}_j Z_{t-j}$. Similarly in Step 2, if there is no CH, the FGLS estimator reduces to the estimator in Ahn and Reinsel (1994). The covariance matrix of $\hat{\beta}$ is calculated as $\mathrm{var}(\hat{\beta}) = (\sum_{t=1}^{T} Q'_t \hat{\Omega}_t^{-1} Q_t)^{-1}$.

4. Monte Carlo Simulations

Monte Carlo simulations are conducted to evaluate the finite sample properties of the FGLS estimators suggested in this article. We compare these properties with those of the two ML-based estimators, FML and RRML, which are explained in Section 3.1.

We consider a bivariate data-generating process (DGP) as follows:

$$Z_t = -A_3 B_3 W_{t-2} + \varepsilon_t, \tag{4.1}$$

where $Z_t = (1 - L^4)Y_t$, $A_3 = (0, a_3)' = (0, 0.5)'$ and $B_3 = (1, b_3) = (1, -1)$. We also assume $\varepsilon_t = Le_t$ and $E(e_t e_t' | \mathcal{F}_{t-1}) = \Sigma_t$, where

$$L = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \qquad \Sigma_t = \begin{pmatrix} \sigma_{1t}^2 & 0 \\ 0 & \sigma_{2t}^2 \end{pmatrix}, \tag{4.2}$$

$$\sigma_{jt}^2 = \omega_j + \psi_j e_{j,t-1}^2 + \phi_j \sigma_{j,t-1}^2, \quad e_{jt} = \sigma_{jt} \xi_{jt}, \quad \text{and} \quad \xi_{jt} \sim N(0,1) \quad \text{for } j = 1, 2. \tag{4.3}$$

The Y_ts in equation (4.1) are seasonally cointegrated with a cointegrating rank of one only at frequency $\pi/2$, ε_t is a generalized orthogonal (GO-) GARCH process (van der Weide, 2002), and λ determines the correlation between the two components of ε_t. Note that Y_t is cointegrated at a single seasonal frequency but if one performs a usual cointegration analysis by Johansen (1988), we cannot obtain the correct cointegrating relationship. It is related to a different form of VECM for seasonal cointegration.

The cointegrating parameter $b_3 = -1$ is in line with typical parameter in economic models. Table 1 summarizes the of values of the parameters λ, ψ_j and ϕ_j used in the simulations. As in most of the DGPs in Seo (2007) and Herwartz and Lütkepohl (2011), we choose identical values for the GARCH parameters of the two error components, and we set the value of $\psi_j + \phi_j$ to be close to unity. It is suitable for investigating possible estimation efficiency losses due to model misspecification since the case of $(\psi_j, \phi_j) = (0, 0)$ has no GARCH and assumes GARCH when there is none. Note that for higher frequency data, the GARCH parameter ψ_j tends to be closer to zero while ϕ_j is often close to unity.

Table 1: Parameter values used in Monte Carlo simulations

DGPs	λ	ψ_j	ϕ_j
1	0	0	0
2	−0.5	0.10	0.85
3	0.5	0.10	0.85
4	−0.5	0.25	0.70
5	0.5	0.25	0.70
6	−0.5	0.40	0.55
7	0.5	0.40	0.55
8		BEKK	
9		DCC	

DGP = data-generating process.

For GARCH processes which do not have a GO-GARCH structure, we consider two more multivariate GARCH processes. DGP 8 is a BEKK model (Engle and Kroner, 1995), i.e.,

$$\Omega_t = DD' + F\varepsilon_t\varepsilon_t'F' + H\Omega_{t-1}H',$$

with

$$D = \begin{pmatrix} 2.5 \times 10^{-3} & 0 \\ -8.4 \times 10^{-4} & 8.3 \times 10^{-5} \end{pmatrix}, \quad F = \begin{pmatrix} 0.229 & -0.173 \\ 0.005 & 0.174 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 0.954 & 0.033 \\ 0.008 & 0.981 \end{pmatrix}.$$

The parameter values are identical to those used in Herwartz and Lütkepohl (2011). Note that this BEKK process has high persistence in second moments but remains covariance stationary.

As another GARCH process, DGP 9 is a dynamic conditional correlation (DCC) model where the volatility matrix of ε_t, Ω_t, is defined by

$$\Omega_t = D_t\rho_t D_t,$$

where $D_t = \text{diag}(\sigma_{1t}, \sigma_{2t})$ and $\sigma_{jt}(j = 1, 2)$ follows GARCH(1,1) with $(\psi_1, \phi_1) = (\psi_2, \phi_2) = (0.1,\ 0.85)$ as in equation (4.3). The conditional correlation matrix ρ_t is defined as

$$\rho_t = \text{diag}(J_t)^{-\frac{1}{2}} J_t\, \text{diag}(J_t)^{-\frac{1}{2}},$$
$$J_t = (1 - \theta_1 - \theta_2)\bar{J} + \theta_1 s_{t-1} s_{t-1}^T + \theta_2 J_{t-1}, \quad \theta_1 = 0.05, \quad \theta_2 = 0.93,$$

where $\bar{J}$ is the unconditional correlation matrix of $s_t = D_t^{-1}\varepsilon_t$ with $\text{corr}(s_{1t}, s_{2t}) = 0.5$. Engle (2002) proposed a DCC model that would divide multivariate volatility processes into volatility series σ_{jt}s and the correlation matrix ρ_t.

The series lengths considered are $T = 100$ and 200, and the replications are set at 1,000 each. For each DGP, the adjustment parameter a_3 and the cointegration parameter b_3 are estimated using the three methods–FGLS, FML, and RRML. To obtain FML, the log-likelihood in (3.1) is maximized using the Berndt-Hall-Hall-Hausman (BHHH) algorithm, which is justified under the assumption of conditional normality (Berndt *et al.*, 1974). Table 2 summarizes the estimators that are computed by the mean squared errors (MSEs) and the mean absolute errors (MAEs).

The MSEs and MAEs in Table 2 show that estimation precision increases with the sample size, reflecting consistency of the estimators. It does not make much difference for the ranking of the estimators whether MSEs or MAEs are considered. A comparison of FGLS (and FML) with RRML shows that it pays to account for GARCH errors. In almost all DGPs where CH is allowed for, FGLS

Table 2: Simulation results for estimators of a_3 and b_3 based on DGPs 1–9: MSE ($\times 10^{-3}$) and MAE ($\times 10^{-2}$) for three different estimation methods, FGLS, FML and RRML, based on 1,000 replications for each sample size T

DGP	Estimators	a_3				b_3			
		MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE
		T = 100		T = 200		T = 100		T = 200	
	FGLS	4.538	5.313	2.161	3.668	2.454	3.489	0.621	1.784
1	FML	4.334	5.243	**2.098**	**3.615**	2.059	3.252	**0.552**	**1.723**
	RRML	**4.323**	**5.235**	2.110	3.633	**1.972**	**3.197**	0.556	1.726
	FGLS	3.947	4.919	**1.852**	**3.403**	2.563	**3.507**	**0.643**	**1.844**
2	FML	3.971	4.915	1.987	3.503	2.686	3.577	0.679	1.863
	RRML	**3.895**	**4.912**	1.995	3.514	**2.539**	3.523	0.703	1.887
	FGLS	**8.899**	7.564	**4.829**	**5.467**	2.886	**3.589**	**0.529**	**1.734**
3	FML	9.042	**7.506**	5.071	5.580	2.832	3.590	0.564	1.784
	RRML	9.076	7.535	5.066	5.575	**2.703**	3.611	0.573	1.793
	FGLS	**4.124**	**4.923**	**1.472**	**3.058**	**3.166**	**3.407**	**0.420**	**1.509**
4	FML	4.924	5.375	2.137	3.683	3.345	3.787	0.697	1.907
	RRML	5.115	5.468	2.350	3.828	3.173	3.704	0.692	1.869
	FGLS	**9.724**	**7.692**	**4.327**	**5.174**	**2.447**	**3.348**	**0.595**	**1.584**
5	FML	11.945	8.437	6.654	6.337	3.032	3.735	0.963	1.973
	RRML	12.146	8.443	6.915	6.409	3.533	3.781	0.801	1.919
	FGLS	**3.522**	**4.622**	**1.610**	**3.157**	**2.857**	**3.283**	**0.612**	**1.526**
6	FML	5.853	5.734	2.788	3.974	4.420	4.205	1.819	2.671
	RRML	6.511	5.871	3.507	4.359	4.045	3.905	0.872	1.966
	FGLS	**10.159**	**7.917**	**4.745**	**5.396**	**2.045**	**2.940**	**0.528**	**1.541**
7	FML	13.561	9.146	7.669	6.777	3.035	3.658	1.812	2.801
	RRML	13.961	9.253	8.400	7.095	2.901	3.394	2.516	2.153
	FGLS	9.061	7.438	1.901	**3.006**	**3.676**	**4.824**	**0.527**	**1.681**
8	FML	8.845	7.405	**1.881**	3.021	3.730	4.836	0.548	1.711
	RRML	**8.709**	**7.360**	1.903	3.041	3.733	4.832	0.571	1.724
	FGLS	9.004	7.380	1.423	2.696	3.885	4.883	**0.365**	**1.395**
9	FML	**8.818**	**7.339**	**1.397**	**2.671**	3.843	4.817	0.372	1.421
	RRML	8.959	7.360	1.411	2.685	**3.823**	**4.786**	0.369	1.409

Note: The underlined bold numbers denote the minimum MSEs or MAEs in estimating a_3 and b_3.
DGP = data-generating process; MSE = mean squared error; MAE = mean absolute error; FGLS = feasible generalized least squares; FML = full maximum likelihood estimator; RRML = reduced rank maximum likelihood estimation.

is substantially better than both FML and RRML in terms of MSE and MAE. However, FML seems to perform at the same level as RRML, even though the former is theoretically better than the latter.

In DGP 1 where there is no CH, FGLS and FML are expected to perform worse than RRML because of the loss in estimation efficiency associated with these methods. However, poorer performance is observed only for the case with $T = 100$. It is remarkable that FML tends to be superior than RRML to estimate the parameters, even when there is no CH. It may imply that the estimation of the VECM parameters are not sensitive to the misspecification of GARCH errors.

Focusing on DGPs 4–7 with GO-GARCH processes, FGLS produces the best performance to estimate the parameters a_3 and b_3, irrespective of the sample size. In some cases of DGPs 2 and 3, FGLS performs worse than FML or RRML when $T = 100$. We conjecture that, in smaller sample size, the closer the parameter ϕ_j is to zero, the higher the efficiency of FGLS relative to FML and RRML. This phenomenon may occur because the DGPs have a low (quarterly) frequency. When $T = 200$, FGLS still shows the best performance. In all cases with GO-GARCH, the positive correlation between the components ($\lambda > 0$) deteriorates the performance of the estimation methods but the sign of λ has no effect on the ranking of the estimators.

FGLS performs better than ML in estimating the nonstationary parameter DGPs 8 and 9 when the true GARCH process is not a GO-GARCH type, especially when $T = 200$. However, FGLS ap-

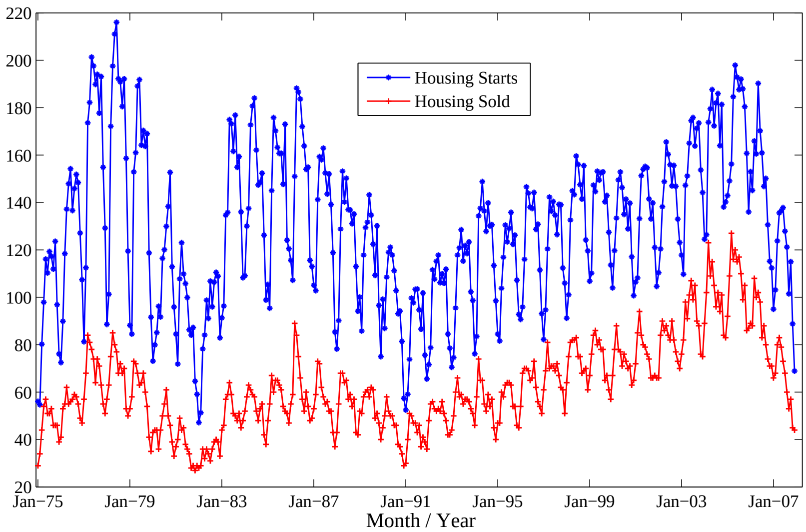

Figure 1: *Monthly U.S. Housing Starts and Housing Sold (in thousands) for the period January 1975 to December 2007.*

pears more sensitive to misspecification of the GARCH structure than ML in estimating the stationary parameter a_3 with smaller sample sizes.

In summary, these observations underpin that the FGLS accounting for CH is overall superior to ML in terms of MSE and MAE. ML with allowance for CH may produce very unreliable estimates especially if the GARCH assumption is more complex than the true dynamics of the DGP.

5. An Empirical Example

In this section, an example is provided to model a seasonally cointegrated time series under CH. We analyze the monthly U.S. Housing Starts y_{1t} and Housing Sold y_{2t} (in thousands) for the period January 1975 to December 2007 with data obtained from the U.S. Census Bureau. Figure 1 displays the time series plot of the raw data. We take the natural logarithm of the raw data and identify a model as a VAR(14) with a constant term as:

$$\begin{aligned}\left(1-L^{12}\right)Y_t = \sum_{j=1}^{7}\left\{\left(A_{jR}B_{jI}+A_{jI}B_{jR}\right)W_{t-1}^{(j)}+\left(A_{jI}B_{jI}-A_{jR}B_{jR}\right)V_{t-1}^{(j)}\right\}\\ +\Psi_1\left(1-L^{12}\right)Y_{t-1}+\Psi_2\left(1-L^{12}\right)Y_{t-2}+\delta+\varepsilon_t,\end{aligned} \tag{5.1}$$

where $Y_t = (\log y_{1t}, \log y_{2t})'$; the subscript j for $j = 1, \ldots, 7$ corresponds to seven monthly frequencies $\theta_j = (j-1)\pi/6$; A_{jR}, A_{jI}, B'_{jR}, and B'_{jI} are $2\times r_j$ matrices; δ is a 2×1 vector; and $W_t^{(j)}$ and $V_t^{(j)}$ are the monthly filtered series of Y_t with all the seasonal unit roots except $z_j = \exp(i\theta_j)$ and $\bar{z}_j = \exp(-i\theta_j)$ for $i = \sqrt{-1}$. Note that the r_js denote seasonal cointegrating ranks at frequencies $\theta_j = (j-1)\pi/6$ for $j = 1, \ldots, 7$. Ahn *et al.* (2004) also analyzed similar time series for illustrating the RRML under the homoskedastic Gaussian assumption of ε_t.

As an initial step, in order to examine the possible presence of CH, we conduct Engle's ARCH test using the residuals $\hat{\varepsilon}_t = (\hat{\varepsilon}_{1t}, \hat{\varepsilon}_{2t})'$ from the estimated model in (5.1) with $r_j = 2$ for $j = 1, \ldots, 7$. The

Table 3: LR statistics and critical values for identifying seasonal cointegrating ranks in (5.1)

j	LR statistics		Critical values at the 5% level		Seasonal cointegrating ranks identified
	$r_j = 0$	$r_j \leq 1$	$r_j = 0$	$r_j \leq 1$	
1	14.2	0.6	12.3	4.1	1
2	53.1	3.7	20.5	6.2	1
3	51.0	3.8	20.5	6.2	1
4	63.6	24.2	20.5	6.2	2
5	73.3	49.8	20.5	6.2	2
6	71.0	16.8	20.5	6.2	2
7	54.7	22.6	12.3	4.1	2

LR = likelihood ratio.

two obtained p-values, 5.4×10^{-6} and 7.8×10^{-4}, imply that there exists a statistically significant CH in the error process. Seasonal cointegrating ranks are identified by computing the likelihood ratio (LR) statistics (Table 3); see Seong *et al.* (2006) details. Accordingly, under the 5% level, cointegrating ranks ($r_j = 1$) are found only at frequencies 0, $\pi/6$, and $\pi/3$ while full ranks ($r_j = 2$) are found in other frequencies.

Assuming that ε_t follows a GO-GARCH in (4.2) and (4.3), we estimate the CH parameters through the first step explained in Section 3.2 and obtain

$$\sigma^2_{1t} = \underset{(<0.0001)}{0.0001} + \underset{(0.2121)}{0.0457}\, e^2_{1,t-1} + \underset{(<0.0001)}{0.9393}\, \sigma^2_{1,t-1},$$
$$\sigma^2_{2t} = \underset{(<0.0001)}{2\times10^{-5}} + \underset{(0.0492)}{0.0123}\, e^2_{2,t-1} + \underset{(0.0191)}{0.9823}\, \sigma^2_{2,t-1},$$

where the numbers in parentheses denote the standard errors of the corresponding estimators. The correlation between the error components is estimated as $\hat{\lambda} = \underset{(0.8933)}{0.49}$. It indicates that shocks to volatility have a persistent effect σ^2_{1t} and σ^2_{2t} since the sums of the ARCH and GARCH coefficients are close to unity.

Through the second and third steps of Section 3.2, we obtain the FGLS estimators for α and β, which are based on the ranks $(r_1, \ldots, r_7) = (1, 1, 1, 2, 2, 2, 2)$. Table 4 shows them with their standard errors in parentheses. We see that a single cointegration vector at the zero frequency exists as $(1, -0.2781)'$. At frequency $\theta_j = \pi/6$, the two long-run equilibrium relationships estimated are $(1, -1.0058 - 1.0669L)'$ and $(-L, -1.0669 + 1.0058L)'$, which imply polynomial cointegrations at the annual frequency; in addition, the equilibrium relationships at frequency $\theta_j = \pi/3$ are estimated as $(1, -0.7305 - 1.3233L)'$ and $(-L, -1.3233 + 0.7305L)'$. Regarding adjustment parameters A_{jR} and A_{jI}, the off-diagonal coefficients at frequencies $\theta_j = (j-1)\pi/6$ for $j = 4, 5, 6, 7$ are not significant under the 5% level, with the exception of one coefficient in A_{5I}.

This brief example shows that the proposed FGLS procedure enables us to simultaneously model long-run equilibrium and time-varying volatility in the seasonal VECM. It is also expected that the FGLS will provide more exact forecasts than ML, because the former produces more reliable estimates than the latter, especially when the series has a time-varying volatility.

6. Conclusion

In contrast to the case of nonseasonal cointegration, it is not simple to apply the ML method to seasonal cointegration, especially under CH. In this article, we considered a FGLS procedure for the estimation of a seasonal VECM with CH. This procedure can be easily implemented using common methods such as OLS and GLS. We showed that the proposed procedure performs well compared to

Table 4: FGLS estimators in (5.1) with $(r_1, \ldots, r_7) = (1, 1, 1, 2, 2, 2, 2)$

j	$\hat{A}_{jR}$		$\hat{A}_{jI}$		$\hat{B}'_{jR}$	$\hat{B}'_{jI}$
1	−0.0083*				1	
	(0.0032)					
	−0.0064				−0.2781*	
	(0.0033)				(0.0089)	
2	−0.0490*		0.0165		1	0
	(0.0149)		(0.0137)			
	−0.0046		−0.0583*		−1.0058*	−1.0669*
	(0.0150)		(0.0137)		(0.0589)	(0.0616)
3	−0.0652*		−0.0077		1	0
	(0.0183)		(0.0187)			
	−0.0432*		−0.0836*		−0.7305*	−1.3233*
	(0.0184)		(0.0188)		(0.0968)	(0.0959)
4	−0.1046*	−0.0117	0.0145	0.0427		
	(0.0267)	(0.0292)	(0.0267)	(0.0290)		
	−0.0233	−0.1357*	−0.0430	0.1282*		
	(0.0266)	(0.0300)	(0.0266)	(0.0297)		
5	−0.0987*	−0.0265	0.0334	0.0843*		
	(0.0231)	(0.0335)	(0.0239)	(0.0354)		
	−0.0162	−0.2169*	0.0317	0.0984*		
	(0.0236)	(0.0349)	(0.0244)	(0.0371)		
6	−0.1695*	0.0486	0.0303	−0.0368		
	(0.0307)	(0.0309)	(0.0306)	(0.0311)		
	0.0565	−0.1357*	−0.0196	0.0562		
	(0.0313)	(0.0320)	(0.0314)	(0.0324)		
7	0.1153*	0.0594				
	(0.0223)	(0.0315)				
	0.0045	0.1654*				
	(0.0227)	(0.0332)				

Note: 1. $\hat{\Psi}_1 = \begin{bmatrix} 0.0045 & 0.1573* \\ 0.0963* & 0.1714* \end{bmatrix}$, $\mathrm{se}\left(\hat{\Psi}_1\right) = \begin{bmatrix} 0.0267 & 0.0292 \\ 0.0266 & 0.0300 \end{bmatrix}$

2. $\hat{\Psi}_2 = \begin{bmatrix} 0.2594* & 0.0448 \\ 0.0366 & 0.0324 \end{bmatrix}$, $\mathrm{se}\left(\hat{\Psi}_2\right) = \begin{bmatrix} 0.0239 & 0.0354 \\ 0.0244 & 0.0371 \end{bmatrix}$

3. $\hat{\delta} = \begin{pmatrix} 0.3436* \\ 0.3316* \end{pmatrix}$, $\mathrm{se}\left(\hat{\delta}\right) = \begin{pmatrix} 0.1200 \\ 0.1186 \end{pmatrix}$, $\hat{\Omega} = \begin{bmatrix} 67.92 & 22.03 \\ 22.03 & 89.09 \end{bmatrix} \times 10^{-4}$

4. * denotes significance at the 5% level, and the numbers in parentheses denote standard errors of the upper estimators.

FGLS = feasible generalized least squares.

the ML method and we illustrated the procedure with monthly time series data that uses a small-scale Monte Carlo simulation.

References

Ahn, S. K. and Reinsel, G. C. (1994). Estimation of partially nonstationary vector autoregressive models with seasonal behavior, *Journal of Econometrics*, **62**, 317–350.

Ahn, S. K., Cho, S. and Seong, B. C. (2004). Inference of seasonal cointegration: Gaussian reduced rank estimation and tests for various types of cointegration, *Oxford Bulletin of Economics and Statistics*, **66**, 261–284.

Berndt, E. K., Hall, B. H., Hall, R. E. and Hausman, J. A. (1974). Estimation and inference in nonlinear structural models, *Annals of Economic and Social Measurement*, **3/4**, 653–665.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics*, **31**, 307–327.

Engle, R. F. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models, *Journal of Business & Economic Statistics*, **20**, 339–350.

Engle, R. F. and Granger, C. W. (1987). Co-integration and error correction: Representation, estimation, and testing, *Econometrica*, **55**, 251–276.

Engle, R. F. and Kroner, K. F. (1995). Multivariate simultaneous generalized ARCH, *Econometric Theory*, **11**, 122–150.

Herwartz, H. and Lütkepohl, H. (2011). Generalized least squares estimation for cointegration parameters under conditional heteroskedasticity, *Journal of Time Series Analysis*, **32**, 281–291.

Johansen, S. (1988). Statistical analysis of cointegration vectors, *Journal of Economic Dynamics and Control*, **12**, 231–254.

Lütkepohl, H. (2005). *New Introduction to Multiple Time Series Analysis*, Springer-Verlag, Berlin.

Seo, B. (2007). Asymptotic distribution of the cointegrating vector estimator in error correction models with conditional heteroskedasticity, *Journal of Econometrics*, **137**, 68–111.

Seong, B. (2013). Semiparametric selection of seasonal cointegrating ranks using information criteria, *Economics Letters*, **120**, 592–595.

Seong, B., Cho, S. and Ahn, S.K. (2006). Maximum eigenvalue test for seasonal cointegrating ranks, *Oxford Bulletin of Economics and Statistics*, **68**, 497–514.

van der Weide, R. (2002). GO-GARCH: A multivariate generalized orthogonal GARCH model, *Journal of Applied Econometrics*, **17**, 549–564.

Received July 26, 2015; Revised October 11, 2015; Accepted October 12, 2015