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On testing NBUL aging class of life distribution

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Abstract. Let X and X_t denote the lifetime and the residual life at age t , respectively. X is said to be a NBUL (new better than used in Laplace transform order) random variable if X_t is smaller than X in Laplace order, i.e., $X_t \leq_{LT} X$. We propose a new test statistics for testing exponentiality versus NBUL class of life distribution. The tests by Hollender and Proschan (1975) and the generalized Hollender and Proschan test by Ains and Mitra (2011) are considered as special cases of the our of test statistics. Our proposed test statistics is simple, consistent and asymptotically normal. Efficiency and powers of the test statistics for some commonly used distributions in reliability are discussed. Finally, real examples are presented to illustrate the theoretical results.

Key Words: *Asymptotic normal, consistent, efficiency, Hollender and Proschan test, NBUL classes of life distributions*

1. INTRODUCTION

In the last decade, various classes of life distributions have been proposed in order to model different aspects of aging. We get the well-known classes of increasing failure rate (IFR), increasing failure rate on average (IFRA), and new better than used (NBU), new better than used in expectation (NBUE) and harmonic new better than used in expectation (HNBUE). The following are the relation between these classes: IFR $\Rightarrow$ IFRA $\Rightarrow$ NBU $\Rightarrow$ NBUE $\Rightarrow$ HNBUE. Properties and applications of these aging notions can be found, for instance, in Bryson and Siddiqui (1969), Barlow and Proschan (1981), Rolski (1975), Klefsjö (1982), and Stoyan (1983). In this paper, we interest in the new better than use in the Laplace transform order aging class NBUL which introduced by Wang (1996), where

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the preservation properties for this new class under reliability operations and Poisson shock models are discussed.
We review some common notions of stochastic orderings and aging notions are considered in this paper (see Shaked and Shanthikumar (1994), Barlow and Proschan (1981).

Definition 1.1. Let X and Y be two random variables with distributions F and G (survival $\overline{F}; \overline{G}$), respectively, then X is smaller than Y in

- $Stochastic\ order\ (\ denoted\ by\ X \leq_{st} Y\ )\ if,\ and\ only,\ if\ \overline{F}(x) \leq \overline{G}(x)\ \forall x$
- $Increasing\ concave\ order\ (\ denoted\ by\ X \leq_{icv} Y\ )\ if,\ and\ only,\ if$ $\int_0^x \overline{F}\,(u)du \leq \int_0^x \overline{G}\,(u)du\ \forall x.$
- $Laplace\ transform\ order\ (\ denoted\ by\ X \leq_{LT} Y\ )\ if,\ and\ only,\ if$ $\int_0^\infty e^{-sx}\,\overline{F}(x)\,dx \leq \int_0^\infty e^{-sx}\,\overline{G}(x)\,dx\ \ \forall x.$

Applications, properties and interpretations of the Laplace transform order in the statistical theory of reliability, and in economics can be found in Denuit (2001), Muller and Stoyan (2002), Klefsjö (1982), Shaked and Shanthikumar (1994) and Ahmed and Kayid (2004). Some of the above orderings have been used to give new definitions of aging classes.

Definition 1.2. A non-negative random variable X is said to be the new better than used $(NBU)\ if,\ \ \overline{F}(x+t) \leq \overline{F}(x)\overline{F}(t)\ \forall x, t > 0$.

Definition 1.3. A non-negative random variable X is said to be the new better than used $\big(NBU(2)\big)\ if, \int_0^x \overline{F}\,(u+t)du \leq \overline{F}(t) \int_0^x \overline{F}\,(u)du\ \forall x, t > 0$.

Based on the Laplace transform order, Wang (1996) introduced a new aging class of life distributions. Its definition is

Definition 1.4. A non-negative random variable X is said to be the new better than used in Laplace order (NBUL) if,

$$\int_0^\infty e^{-sx}\,\overline{F}(x+t)\,dx \leq \overline{F}(t) \int_0^\infty e^{-sx}\,\overline{F}(u)du \forall x, t > 0. \tag{1.1}$$

It is obvious that equation (1.1) is equivalent to $X_t \leq_{LT} X\ \forall t \geq 0$ where $X_t = [X - t | X > t]$ and $t \in x : F(x) \leq 1$. The relations between the above stochastic orderings and aging classes are as follows: $X \leq_{st} Y \Rightarrow X \leq_{icv} Y \Rightarrow X \leq_{LT} Y$ and $NBU \subset NBU(2) \subset NBUL.$ Yue and Cao (2001) studied some interpretations for the NBUL. Some properties of the NBUL class including some characterizations and closure properties have been discussed by Yue and Cao (2001) and Gao et al. (2002), Belzunce et al. (1999) showed that the NBUL class is preserved under both the pure birth shock model and Poisson shock model. Finally, Al-Wasel et al. (2007) investigated the test statistic based on U-statistic by using the expected value method. Our goal in this paper is find a new test statistics for testing exponentiality versus NBUL. It is interesting to note that our proposed test statistics includes the test statistics proposed by Hollander and Proschan (1975) and generalized

Hollander and Proschan by Ains and Mitra (2011) as special cases. This is introduced in section 2. In section3, we find the asymptotic distribution of the proposed test statistics. The test is shown to be consistent in section 4. Pitman efficiency is discussed in section 5. The power estimates of the test is given for some well known alternatives in section 6. Finally, examples in medical sciences are presented in section 7.

2. TESTING EXPONENTIALITY VERSUS NBUL

The test depends on random sample X_1,...,$X_n$from a population with distribution function F. We test H_0: F is exponential (μ) against H_1: NBUL and not exponential. In order to test H_0against H_1, we may use the following measure of departure from

$$\gamma_j(s)=\int_0^\infty e^{-sjt}\,\overline{F}^j(t)[e_s(0)-e_s(t)]d(e^{-st}F(t)) \quad (2.1)$$

where $e_s(t)=\frac{1}{\overline{F}(t)}\int_t^\infty e^{-s(x-t)}\,\overline{F}(x)dx$. Observe that $\gamma_j(s)=0$ if and only if H_0 is true. Otherwise H_1 is true. Now,

$$\gamma_j(s)=I_1-I_2 \quad (2.2)$$

where

$$I_1=\frac{1}{(j+1)}\int_0^\infty e^{-sx}\,\overline{F}(x)dx$$

and

$$I_2=\frac{1}{j}[\int_0^\infty e^{-sx}\,\overline{F}(x)dx-\int_0^\infty e^{-s(j+1)x}\,\overline{F}^{j+1}(x)dx]$$

We, therefore, re−write (2.2) as:

$$\gamma_j(s)=\int_0^\infty[\frac{1}{j(j+1)}e^{-sx}\overline{F}(x)+\frac{1}{j}e^{-s(j+1)x}\overline{F}^{j+1}(x)]dx. \quad (2.3)$$

We now replace the unknown life distribution function F by empirical distribution function F_n to obtain

$$\hat{\gamma}_j(s)=\int_0^\infty[\frac{-1}{j(j+1)}e^{-sx}\overline{F_n}(x)+\frac{1}{j}e^{-s(j+1)x}\,\overline{F_n}^{j+1}(x)]dx. \quad (2.4)$$

which yields

$$\hat{\gamma}_j(s)=\sum_{k=1}^n X_{(k)}[\frac{-1}{j(j+1)n}e^{-sX_{(k)}}+\frac{1}{j}((\frac{n-k+1}{n})^{j+1}-(\frac{n-k}{n})^{j+1})e^{-s(j+1)X_{(k)}}] \quad (2.5)$$

where $X_{(1)}\leq ...\leq X_{(n)}$ denoted the order statistics based on the random sample X_1,...,X_n and $\overline{F_n}=\widehat{\overline{F}}=\sum_{k=1}^n\frac{n-k+1}{n}$. Hence, the empirical form of $\widehat{\gamma_j}(s)$ is given by (2.5).

Remark 2.1. It is important to note that, the statistic is given by (2.5) reduces to the test statistic is given by Ains and Mitra (2011) when s = 0, and reduces to the test statistic is given by Hollander and Proschan (1975) when s = 0 and j = 1.

3. THE ASYMPTOTIC DISTRIBUTION OF TEST STATISTICS

Now, we will derive the asymptotic distribution of our test statistics in (2.4). First, we state the following theorem.

Theorem 3.1. Let X be a continuous non-negative random variable, such that $E[X^2] < \infty$ and let $\mu(J_j, e^{-sx}F(x)) = \int_0^\infty xJ_j\, e^{-sx}F(x))d(e^{-sx}F(x))$ and

$$\sigma^2\,(J_j, e^{-sx}F(x), e^{-sy}F(y)) = \int_0^\infty \int_0^\infty J_j(e^{-sx}F(x))J_j(e^{-sy}F(y))$$
$$[min(J_j(e^{-sx}F(x)), J_j(e^{-sy}F(y))) - (e^{-sx}F(x))\,(\,e^{-sy}F(y))]dxdy$$

where, $J_j(u) = \frac{-1}{j(j+1)} + \frac{1}{j}(1\, -\, u)^j$. Then, $\frac{\sqrt{n}(\gamma_j(s) - \mu(J_j, e^{-sx}F(x))}{\sigma^2(J_j, e^{-sx}F(x), e^{-xy}F(y)))} \rightarrow N(0,1)$.

The above theorem is consequence of theorems 2 and 3 of Stigler (1974).
Now, we obtain the limiting distribution of our test statistics in (2.4) under the null hypothesis of exponentiality. Let $H_0 : F(x) = 1 - e^x,\ x \geq 0$ and H_1: $F(x) \in NBUL$. Thus, the the limiting distribution of the test statistic in (2.4) $(1 + s)(j\ + 1)\,\sqrt{n(2j + 1)}\hat{\gamma}_j(s)$ is $N(0,1)$. Hence, for large values of n, we reject the null hypothesis of exponentiality if ,

$$(1 + s)(\,j + 1)\sqrt{n(2j + 1)}\hat{\gamma}_j(s) > z\alpha \text{ or } (1 + s)(j + 1)\,\sqrt{n(2j + 1)}\hat{\gamma}_j(s) < -\,z\alpha,$$

where $z\alpha$ is the upper $\alpha - Percentile$ of $N(0, 1)$.

Remark 3.1. Not that, for $s = 0$ and $j = 1$ in (2.4), we obtained the same result in Hollander and Proschan (1975) and for $s = 0$ in (2.4) , we obtained the same result in Ains and Mitra (2011).

4. CONSISTENCY

Now, we prove that proposed test in (2.1) is consistent.

Theorem 4.1. Let X be a continuous non-negative random variable and let $lim_{x\rightarrow\infty}\, x\overline{F}(x)e^{-sx} = 0$. Then $\gamma_j(s) = \mu(J_j, e^{-sx}F(x))$ and $\gamma(s) = \mu(J_1, e^{-sx}F(x))$ when j = 1.

Proof 4.1. $\mu(J_j, e^{-sx}F(x)) = \int_0^\infty xJ_j(e^{-sx}F(x))d(e^{-sx}F(x)) = \gamma_j(s)$.

5. PITMAN EFFICIENCY

To compare our consistent tests on the basis of Pitman asymptotic relative efficiency. Consider a sequence of alternative distribution function indexed by parameters θ_n, where $\theta_n = \theta_0 + cn^{\frac{-1}{2}}$, c is an arbitrary positive constant, and θ_0 corresponds to the exponential distribution.

Let T_{n1} and T_{n2} be two test statistics for testing $H_0 : F_\theta \in F_{\theta_n}$; $\theta_n = \theta + \frac{c}{\sqrt{n}}$ then the asymptotic relative efficiency of T_{n1} relative to T_{n2} can be defined as

$$e(T_{n1}\ , T_{n2}) = \frac{\mu'_1(\theta_{0)/\sigma_1}(\theta_{0)}}{\mu'_2(\theta_{0)/\sigma_2}(\theta_{0)}}$$

where $\mu'_i(\theta_0) = [lim_{n\to\infty} \frac{\partial}{\partial\theta} E(T_{n_i})]_{\theta\to\theta_0}$ and $\sigma_i{}^2(\theta_0) = lim_{n\to\infty} Var(T_{ni}),\ i = 1, 2.$
In our case the asymptotic relative efficiency is defined as

$$e(\gamma_j(s)) = (\frac{\mu'(J_j, e^{-sx}F(x))}{\sigma(J_j, e^{-sx}F(x), e^{-sy}F(y))})_{\theta=\theta_0}$$

where θ_0 corresponds to the exponential distribution.

Three of the most commonly used alternatives (see Hollander and Proschan (1975)) are
(i) Linear failure rate family: $\overline{F}_1(x) = e^{-x-\frac{\theta x^2}{2}}, x \geq 0, \theta \geq 0.$
(ii) Makeham family: $\overline{F}_2(x) = e^{-x-\theta(x-1+e^{-x})}, x \geq 0, \theta \geq 0$
(iii) Weibull family: $\overline{F}_3(x) = e^{-x^\theta}, x \geq 0, \theta \geq 0$

The null hypothesis is at $\theta = 0$ for linear failure rate and Makeham families and $\theta = 1$ for Weibull family, respectively. Direct calculations of the efficiencies of these families give the following results :
(i) Linear failure rate family: $(\gamma_j(s)) = \frac{\sqrt{(1+2s)(1+j+(2+j)s)(1+2s+2j(1+s))}}{(1+j)(1+s)^2\sqrt{1+2s(1+s)+j(1+s)(1+2s)}}.$
(ii) Makeham family: $(\gamma_j(s)) = \frac{\sqrt{(1+2s)(1+j+2s+js)(1+2s+2j(1+s))}}{(2+s)(2+j+s+js)\sqrt{1+j+2s+3js+2s^2+2js^2}}.$
(iii) Weibull family: $e(\gamma_j(s)) = \frac{Log[(1+j)\sqrt{(1+2s)(1+j+2s+js)(1+2s+2j(1+s))}}{j\sqrt{1+j+2s+3js+2s^2+2js^2}}$

The maximum values are located at approximately at $(j, s) = (1, 1), (0.1, 0.1)$ and (0.63, 0.1) for Weibull, LFR and Makeham families respectively; and they are 2.14; 0.97 and 0.289. The bench mark test for NBUE due to Hollander and Proschan (1975) has PAE values equal to 1.228; 0.866 and 0.289 and generalized Hollander and Proschan by Ains and Mitra (2011) test has PAE values equal to 1.288; 0.98 and 0.289. Hence our test is more efficient than test NBUE due to Hollander and Proschan (1975) and generalized Hollander and Proschan by Ains and Mitra (2011) test.

6. THE POWER ESTIMATES OF THE TEST STATISTICS

The power estimate of the test statistic in (2.3) are considered for the significant level at 95 th upper percentile in Table (1) in Appendix, for two of the most commonly used alternatives (see Hollander and Proschan (1975), which are

- (i) Linear failure rate family: $\overline{F}_1(x) = e^{-x-\frac{\theta x^2}{2}}, x \geq 0, \theta \geq 0.$
- (ii) Makeham family: $\overline{F}_2(x) = e^{-x-\theta(x-1+e^{-x})}, x \geq 0, \theta \geq 0$

It is clear from Table (1) the power of the proposed test increases when θ increases.

7. EXAMPLES

Example 1: Bryson and Siddiqui (1969) have analyzed data which are survival times, in days from diagnosis, of patients suffering from chronic granulocytic leukemia. Here $n =$ 43 and the order statistic $X_1 < X_2 < \cdots < X_{43}$ are: 7, 47, 58, 74, 177, 232, 273, 285, 317, 429, 440, 445, 455 , 468, 495, 497, 532, 571, 579, 581, 650, 702, 715, 779, 881 , 900, 930, 968, 1077, 1109, 1314, 1334, 1367, 1534, 1712, 1784 , 1877, 1886, 2045, 2056, 2260, 2429, 2509 . We compute blew the value of test statistic and the associated p-value for different values of j and s in Table 2 in Appendix. We observe that the null hypothesis of exponentiality is reject for all values of j and s. This agrees with the conclusion of Hollander and Proschan (1975) and Ains and Mitra (2011).

Example 2: In an experiment at the Florida State University to study the effect of methyl mercury poisoning on the life lengths of goldfish, goldfish were subjected to various dosages of methyl mercury Kochar (1985). At one dosage level, the ordered times in days to death are: 0.86, 0.88, 1.04, 1.24, 1.35, 1.41, 1.45, 1.65, 1.67, 1.67. We compute below the value of test statistic in (2.5) and the associated p-value for different values of j and s in Table 3 in Appendix. We observe that the null hypothesis of exponentiality is accepted for all values of j and s.

8. CONCLUSIONS

In this paper we have studied a family of test statistics, intended for testing exponen-tiality against NBUL, which includes the test statistic proposed by Hollander and Proschan (1975) and the generalized Hollender and Proschan test by Ains and Mitra (2011) as special cases. We have found the asymptotic distribution of the test statistic. We have shown that our test is consistent and we have also studied the effciency of our proposed test. we get it is more effcient than test NBUE due to Hollander and Proschan (1975) and generalized Hollander and Proschan by Ains and Mitra (2011) test.

APPENDIX

Table 1: Empirical Power Estimates

Distribution	j	θ	$s=1$	$s=1.5$	$s=2$
Linear failure rate family $\overline{F}_1$ (L.F.R)	1	0.5	0.71412	0.713876	0.714344
		0.75	0.71754	0.716596	0.716435
		1	0.719798	0.718549	0.718036
	1.5	0.5	0.708565	0.0.708495	0.709405
		0.75	0.713158	0.712084	0.71204
		1	0.716236	0.714694	0.714149
	2	0.5	0.702806	0.702968	0.745966
		0.75	0.708569	0.707419	0.707534
		1	0.712469	0.71068	0.710141
Makeham family $\overline{F}_2$	1	0.5	0.717256	0.717272	0.717609
		0.75	0.719085	0.718696	0.718712
		1	0.720492	0.719852	0.719644
	1.5	0.5	0.705657	0.705693	0.706407
		0.75	0.709536	0.708711	0.708746
		1	0.71251	0.711157	0.710718
	2	0.5	0.687398	0.687463	0.688783
		0.75	0.694556	0.693036	0.693101
		1	0.700024	0.697539	0.696732

Table 2: Summary statistics for example (1)

j	s	$(1+s)(1+j)\sqrt{n(2j+1)}\hat{\gamma}_j(s)$	$P-Value$
1	0.25	-0.393424	0.652997
	0.5	-0.0837409	0.533369
	.75	-0.0169792	0.506773
	1	-0.00337204	0.501345
2	0.25	-0.258971	0.602171
	0.5	-0.0540604	0.521556
	.75	-0.01096	0.504372
	1	-0.00217664	0.500868
3	0.25	-0.204549	0.581038
	0.5	-0.0426434	0.517007
	.75	-0.00864537	0.503449
	1	-0.00171696	0.500685

Table 3: Summary statistics for example (2)

j	s	$(1+s)(1+j)\sqrt{n(2j+1)}\hat{\gamma}_j(s)$	$P-Value$
1	0.25	11.27	0
	0.5	7.39474	0
	.75	5.35493	0
	1	4.08873	0.0000216871
2	0.25	6.96412	0
	0.5	4.28926	0
	.75	3.16296	0.000780876
	1	2.4888	0.00640867
3	0.25	5.14586	0
	0.5	-3.13007	0.00087383
	.75	2.39389	0.00833545
	1	1.92833	0.0269071

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