http://dx.doi.org/10.7468/jksmed.2014.18.3.171
J. Korean Soc. Math. Educ., Ser. D, Res. Math. Educ.
Vol. 18, No. 3, September 2014, 171–185 ⓒ 2014 Korean Society of Mathematical Education

Prospective Teachers' Competency in Teaching how to Compare Geometric Figures: The Concept of Congruent Triangles as an Example[1,2]

LEUNG, K. C. Issic*
Department of Mathematics and Information Technology, Hong Kong Institute of Education, Hong Kong SAR, China; Email: ikcleung@ied.edu.hk

DING, Lin
Department of Mathematics and Information Technology, Hong Kong Institute of Education, Hong Kong SAR, China; Email: lding@ied.edu.hk

LEUNG, Allen Yuk Lun
Department of Education Study, Hong Kong Baptist University, Kowloon Tong, Hong Kong SAR, China; Email: aylleung@hkbu.edu.hk

WONG, Ngai Ying
Department of Curriculum and Instruction, Faculty of Education, Chinese University of Hong Kong, Shatin, Hong Kong SAR, China; nywong@cuhk.edu.hk

(Received February 11, 2014; Revised March 31, 2014; Accepted September 25, 2014)

Mathematically deductive reasoning skill is one of the major learning objectives stated in senior secondary curriculum (CDC & HKEAA, 2007, page 15). Ironically, student performance during routine assessments on geometric reasoning, such as proving geometric propositions and justifying geometric properties, is far below teacher expectations. One might argue that this is caused by teachers' lack of relevant subject content knowledge. However, recent research findings have revealed that teachers' knowledge of teaching (e.g., Ball et al., 2009) and their deductive reasoning skills also play a crucial role in student learning. Prior to a comprehensive investigation on teacher competency, we use a case study to investigate teachers' knowledge competency on how to teach their students to mathematically argue that, for example, two triangles are congruent. Deductive reasoning skill is essential to geometry. The initial findings indicate that both subject and

[1] This project is supported by the GRF grant 2013–2014 (Hong Kong UGC).
[2] A draft version of the article was presented at the KSME 2014 Spring Conference on Mathematics Education held at Hankuk Univ. of Foreign Studies, Seoul 130-791, Korea; April 1–2, 2014 (*cf.* Leung, Ding, Leung & Wong, 2014).
* Corresponding author

pedagogical content knowledge are essential for effectively teaching this challenging topic. We conclude our study by suggesting a method that teachers can use to further improve their teaching effectiveness.

Keywords: mathematics teachers' competency, pedagogical content knowledge, congruent triangles
MESC Classification: G19
MSC2010 Classification: 97C70

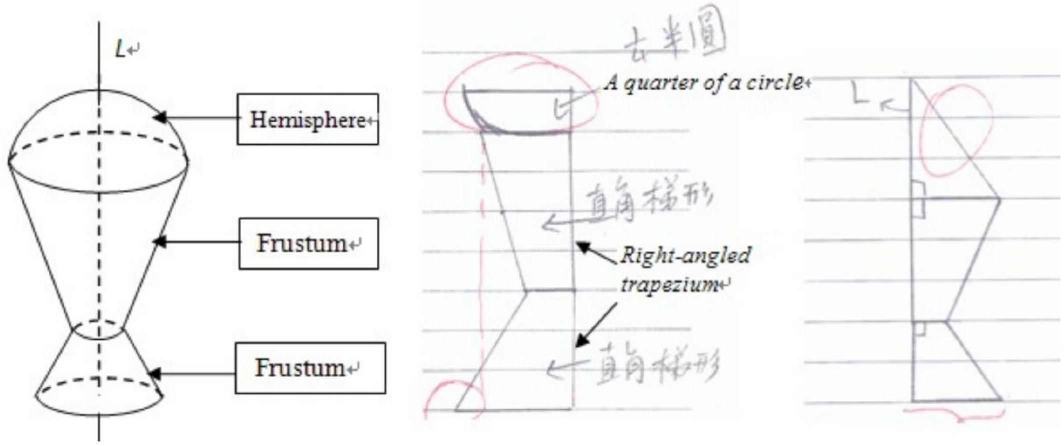
INTRODUCTION

It is often challenging for teachers to teach students to use logical deduction when proving geometric propositions. The difficulty of formally proving geometric propositions frustrated students; they often find it difficult to write coherent and logical arguments, even though they have an idea of what the final result should be. This occurs even during a simple proof of the verification of two similar or congruent triangles in an irregularly-oriented geometric figure. Writing down a systematic sequence of arguments in proper mathematical language requires a skilful mastery of mathematical symbols and logic. Sharpen students' deductive reasoning skills is challenging.

When we mention a mathematics learning disability, we intuitively refer to a student's difficulty in learning abstract mathematical concepts. By contrast, in his frequently cited paper "*Why Jonny Can't Prove*", Dreyfus (1999) explained why it is difficult to train students to present a logical proof. He stated that students cannot correctly and logically write down a proof because they lack training on how to reach, from an explanation through an argument to justify the proof of a theorem or proposition.

Deductive argument and reasoning are necessary to prove geometric propositions. Proving the congruence of two triangles is a topic that both teachers and students commonly find difficult, to teach and learn respectively.

From the time students learn to compare two mathematical quantities, they are asked to determine the values of these two quantities, and identify which quantity is greater or less than the other. When asked to compare two figures, such as, two triangles, we inevitably measure their sizes (areas). However, other than the measurement and the abstract aspects of topological equivalence, another method exists for comparing the *equality* of two triangles — whether they are similar or congruent. When justifying similarity and congruence, one must identify the condition (same shape, for similarity) to prove similarity and its size to prove congruency (congruent shapes are of the same shape and size). Geometry learners and teachers must be able to initially visualize a geometric image to justify common features during comparison. This ability involves the capability to identify figures with the same shape even if they are different sizes, to deduce geometric properties from the figures that students perceive. Unfortunately they do not necessarily possess the ability to see through the geometric imagery to obtain the elements of comparison. To recreate a typical example, we tested a small group of prospective teachers and asked them to compare the shape of a solid figure projected onto a plane, and vice versa. The test problem required the prospective teachers to construct a plane figure that would generate the illustrated solid. To our surprise, some of the teachers could not answer the problem completely correctly. Two of the teachers could not even visualize the images of revolution (See Figure 1(b), (c)). For some information on visualization and imagery, refer to studies by Jones & Bills (1998).



(a) (b) (c)

Figure 1. Deficiencies in interpreting a three dimensional solid figure and its two dimensional representation, which reflects difficulties depicting geometric features by comparing of shapes under dimensional exchange of geometric figures.

The congruency and similarity of triangles are two major topics included in the local secondary mathematics curriculum. Deductive reasoning skills are essential for justifying the congruence of two triangles. Two triangles are congruent if they are of the same size and shape. The sizes of the interior angles (A), the lengths of the corresponding sides (S), and the special case: (a right angle (R) and the length of hypotenuse (H) and another side (S) contribute to the prerequisite conditions for congruence. It is always challenging to teach this topic. The underlying reason why a minimal number of conditions (e.g. A.S.A.) sufficient for congruence is a key concept to understand and many students finds it difficult in justify the reason. A competent teacher is supposed to clearly explain the following conditions to justify why a pair of triangles is congruent:

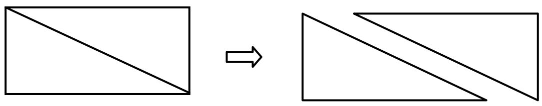
(1) Any one of four conditions: S.S.S., A.S.A., S.A.S., and R.H.S. is sufficient to show that two triangles are congruent;
(2) Comparing three features (S and/or A) is sufficient to show that two triangles are congruent;
(3) Neither condition A.S.S. nor A.A.A. are sufficient to show that two triangles are congruent; and
(4) R.H.S. is a special sufficient condition.

These four propositions included all aspects of what students should learn regarding congruent triangles in the secondary school curriculum. Teachers usually find it difficult to teach condition (2) to (4), and they tend to teach students to check the measures (angles and sides) of two given triangles before concluding that they are congruent according to one of the four conditions stated in (1). Our teaching experience tells us that few students can properly explain why condition (4) is a special case of (3). That is, in a special case, insufficient condition A.S.S. becomes sufficient condition R.H.S.

THEORETICAL BACKGROUND

Mathematics learners may easily acquire knowledge from multiple representations of a described concept through the inferential connections between corresponding elements and structures in those representations (Seufert 2003; Mayer 2003). When learning plane geometry, these representations are mainly plane figures. Research findings have indicated that students may not be able to perceive a geometric figure and comprehend its properties without having the real object physically placed in front of them (e.g., Piaget & Inhelder, 1967); Jones, 1998); Fujita, Jones & Yamamoto, 2004a; Fujita, Jones & Yamamoto, 2004b; Jones & Bills, 1998). When justifying that two triangles are congruent, teachers may show students two identical paper-cut triangle, placing them on top of each other to demonstrate that they are the same. However, this is not a rigorous proof justified through geometric reasoning. Teaching geometry requires teachers to present dual picture demonstrations and narrative explanations of their abstract images of real objects to students. The bridge between a real object and a student's mental image is a relevantly constructed geometric figure that is well perceived by the student. A crucial step required for geometry teachers to deliver effective instruction is to unpack the mathematical knowledge and ideas and scaffold them for students (Ball et al., 2009). However, our experience shows that misconceptions of geometric properties sometimes occur because students are not able to transform the image of the observed structures into a meaningful geometric interpretation. Decreasing the number of opportunities for believing misconceptions is challenging to teachers when teaching abstract geometric concepts. In particular, when teaching the concept of congruent of triangles, students must learn to justify the pre-conditions, and deduce the conclusion by checking the sufficiency of these conditions. Real object presentation is hardly a helpful metaphor for learning to use logical arguments to make this justification. For example, placing two right-angled triangles together as a pair of half rectangles (formed by cutting a rectangle along a diagonal; see the figure 2(a)) would be straight forward for students to see that the two triangles are identical. This is a procedure of "seeing first, justifying later" is easy. However, justifying that triangles that appear in a complex diagram with different orientations or backgrounds are congruent (Figure 2(b), two triangles are circumscribed by a semi-circle) takes some deductive reasoning to reduce uncertainty and to lead the argument logically toward the conclusion.

(a)



(b)

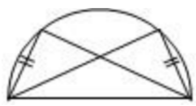


Figure 2. Any of the four conditions: S.S.S., A.S.A., S.A.S., and R.H.S. can justify the congruence of a pair of half rectangles in (a), but justifying the congruence of the two circumscribed triangles in (b) requires rigorous deductions.

Apart from student learning disabilities, teaching disabilities may also exist, in the sense that teachers themselves may not possess the relevant competency required to teach. This competency consists of a certain standard of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) (Ball et al., 2009; Hill, Ball & Schilling, 2008; Shulman, 1986). Ball (2005) also included common content knowledge (CCK) and specialized content knowledge (SCK) in their construct of SMK. In particular, they defined SCK, which is different from CCK, as something "that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide explanations of common rules and procedures, and examine and understand unusual solution methods to problem" (Ball, 2005). By contrast, Buchholtz et al. (2013) emphasized Klein's concept of elementary mathematics from an advanced standpoint (EMFAS) as another essential component of teachers' professional knowledge. The results of their international comparative study (Buchholtz et al., 2013) indicated that prospective teachers from countries where students achieve top mathematics performance in international assessments, still find it difficult to systematically connect school mathematics and university knowledge. The EMFAS construct is different to the concept of SCK. EMFAS is more related to mathematical concept, and SCK refers to mathematics knowledge applied to teaching. However, we hypothesize that EMFAS and SCK share common content, both of them lead to conceptual-understanding oriented mathematics teaching.

This paper presents the result of a portion of a larger study. We investigated how capable the Hong Kong (HK) prospective mathematics teachers were at delivering the concepts related to the sufficiency of congruent triangles. In particular we aimed to answer the followings questions:

- What SMK and PCK do HK prospective teachers possess for teaching the topic of congruent triangles?
- How does the geometric thinking of the HK prospective teachers relate to the richness of PCK when teaching the concept of congruent triangles?

Student achievement is closely connected to teachers' knowledge and skill when teaching mathematics. (Ma, 1999; Tchoshanov, 2011). Barlow & Reddish (2006) asserted that mathematical ideas are initially based on intuitive notions and that deduction is an essential skill required for mathematical arguments and proofs. We also believe that it is essential for geometry learners to undergo a rigorous process to learn deductive reasoning skills before they can master geometric arguments (Jones & Mooney, 2003; Brown, Jones & Taylor, 2003). Using the example of identifying a pair of congruent triangles, the capability of mathematics teachers relevant to effectively delivering the concepts of congruence and justification through deductive reasoning of the conditions of congruency reflects the richness of their SMK and PCK.

METHOD

This paper describes a portion of a larger project in which two groups of prospective secondary mathematics teachers from HK participated. The prospective teacher were in either their third or fourth year of study toward a Bachelor of Education majoring in mathematics, or were full-time or part-time students enrolled in the Postgraduate Diploma in Education in mathematics program. The project comprised both quantitative and qualitative data collection. Quantitative data were collected using a questionnaire on prospective teachers' beliefs on the nature of mathematics, mathematics teaching and mathematics knowledge. Based on the results of this phase, five participants were selected to take part in the second phase, which constituted an interview aimed at capturing the prospective teachers' PCK and SMK relevant to teaching three topics. They were assigned the pseudonyms: Shing, Fai, Alan, Henry and Patrick. Video-based interviews were employed in second phase. The video was a TIMSS 1999 HK video and the video was of a 40-minute Grade 8 lesson. During the interview, both the researcher and interviewee sat next to each other. The researcher controlled the video play and asked questions (Figure 3).

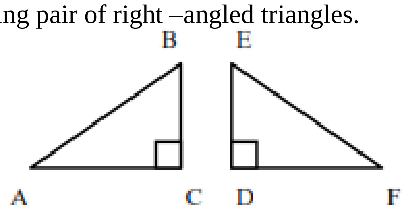
Figure 3. The context of video-based interview

The candidates were asked hypothetical questions based on the teaching in the video teaching, where illustrative examples and diagrams were used in the classroom. They were required to respond to the teaching approaches used by the demo teacher in the video clip of the lesson. The hypothetical classroom environment enables prospective teachers to experience an intermediate stage of professionalism because professional teachers are pragmatic, application directed, and survival oriented (Vermunt, 2007). We modified and adopted questions in the interview that were developed by Wong & Su (1995). The questions, were hypothetical in nature. The questions were:

- *"As a start, how would you introduce this topic?"*
- *"What do you think the expected teaching and learning goal of that particular sub-routine was?"*
- *"What would you do if a student indicated that she or he didn't understand this?"*
- *"If you must proceed in a way that is different from that shown in the video, what other teaching strategies can you think of?"*
- *"How would you proceed at this stage of the lesson?"*

The interviewees watched the video and sometimes wrote answers on the whiteboard. The whole process was recorded. We adopted the interview questions from Wong & Su (1995), referenced Hill, Ball & Schilling (2008) and Ball et al. (2009), and used the MKT framework (and incorporating EMFAS). The elements of knowledge are depicted in Table 1. According to the framework and knowledge of content and teaching (KCT), knowledge of content and students (KCS) constitute most PCK, while CCK and SCK constituent most of SMK.

Table 1. Interview questions and how they correspond to PCK categories

		Interview questions	Video context
PCK	KCT	How do you view the teacher's 10 warm-up exercises that are used as the introduction in the video clip? If you were teaching this topic, how would you introduce this topic on congruent triangles?	The teacher in the TIMSS video assigned the students 10 warm-up exercises as part of the introduction to this lesson based on the following pair of right –angled triangles.  *Figure* 4. A pair of right-angled triangles used by the teacher in the video
	KCT	Why did the teacher ask the students to write down the steps that they had completed and asked the other students in their group follow these steps? If you were the teacher, would you use similar methods? If not, what alternative methods would you use to achieve your goals?	To understand the different requirements to prove that two triangles are congruent. Teacher in the TIMSS video organized group work. Each person in the group constructed a triangle and wrote down the steps that were followed to construct it. The group members followed the instructions to construct a new triangle to check if the two triangles were congruent. They determined the minimal numbers of sides and angles required to construct the congruent triangles.
	KCS KCT	How did you interpret the student's thinking when she said "I could write all my angles and all my sides"? If you were the teacher, how would you respond to your student?	During the group work, one student suggested writing down all of the sides (lengths) and angles of the triangle she had constructed for her group members to reconstruct.
SMK	CCK SCK	Why is R.H.S. a condition and a special case (special case) of A.S.S.?	After the group activity, the teacher explained the R.H.S. method. Some students were confused about the difference between R.H.S. and A.S.S

RESULTS AND ANALYSIS

KCT- How would you introduce the topic regarding congruent triangles?

All five HK prospective teachers had a negative opinion of the teacher's approach (in the TIMSS 1999 video) to introducing the topic by assigning 10 warm-up problems because they were concerned with the lesson time constraint. Instead, four prospective teachers, excluding Alan, suggested that the relevant information should be introduced in a more straightforward manner. All four of teacher would have prepared pairs of triangles to allow the students to identify the properties of congruent triangles.

The types of triangle pairs that they preferred to draw on the chalkboard also reflected their various understandings of what should be viewed as prerequisite knowledge for the taught topic. For example, Patrick indicated that the "same angles and same sides" could apply to all polygons. He suggested drawing a pair of pentagons and a pair of hexagons with same corresponding angles and sides for students to observe and explore.

In addition, he also stated that the orientation of the pair of right-angled triangles (in Figure 4) might confuse students. He suggested drawing the triangles with the same orientation. Henry suggested preparing the four pairs of triangles to introduce the different conditional requirements for congruence.

> "There are four types [of requirement], R.H.S., S.S.S., S.A.S., and A.A.S., and she [the teacher in the TIMSS 1999 video] didn't point out which type it was. If I had taught the class, I would have given some sets of triangles to my students so that they would be able to remember which type they were. Is that S.S.S.? Is that R.H.S.? Is that ….? Something like that. I think that this would help the students remember which type is which. Also, one type is S.S.A. I would also give them two types of S.S.A. to see whether they are congruent because this is a common mistake." (Henry)

Only Alan agreed with the teacher's approach, but he suggested that extra questions should be provided for different needs. As he mentioned,

> "Maybe some students can finish 10 questions in 1 minute. Maybe some students need 10 minutes. So… I would provide extra questions for the high-ability students." (Alan)

KCT: What methods would you use to help students learn the different conditions to justify that the given triangles are congruent?

Before asking the prospective teachers' about their methods of teaching the concept of congruent triangles, the five prospective teachers were invited to comment on the function of group work adopted by the teacher in the video clip. Similar to their attitudes toward the warm-up exercises, all five prospective teachers tended to reject this type of group-activity. In addition to their time limitation concern, they claimed that the activity was to complex. They were probably influenced by the capability of the students they were currently teaching. Two prospective teachers Patrick and Shing, suggested providing more hints and guidelines to facilitate the students' own discovery.

> "First, I let the student use *your* method… their method to design one… a pair of congruent triangles. If they cannot draw them, then I guide them to focus on the angle relationship, the side relationships … This can help them to discover it." (Shing)

Two prospective teachers, Fai and Henry, proposed other manipulating activities. Henry suggested cutting different triangles out of paper. Some of their sides or angles should be fixed, and students could be asked to rotate the triangles to see that some of them overlapped. Fai suggested using toothpicks to form different pairs of triangles with some of fixed sides or angles.

> "We can have a toothpick that is 10 cm long, and use it as one side to form two triangles…. It is easier to change the other sides or angles of the two triangles and see if they are congruent. I think it will be easier to understand by manipulating it with real objects." (Fai)

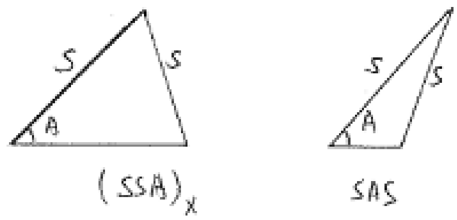


Figure 5. Fai's toothpick-forming examples

KCS and KCT: How did you interpret the student's thinking when she (the student in the TIMSS video) suggested writing down all the angles and all the sides? How would you respond to this student?

The KCS of the five prospective teachers when responding to the student's (in the TIMSS video) suggestion to "…write down all sides and angles for the other group members to follow" was demonstrated to be insufficient. All the prospective teachers misinterpreted her statement as a reflection of her poor capabilities (e.g., Shing) or they felt that the student did not know what to do during the group activity. Correspondingly, the methods that they selected to respond to the student tended to be practical and straightforward. Here is one typical response:

> "Of course, three corresponding angles and sides that are equal can prove that the triangles are congruent. I would talk about some special requirements, for example, if two corresponding sides are equal, and the corresponding angles between the two sides are equal, even though we do not know whether the third sides of two triangles are equal, we can still justify that the triangles are congruent." (Fai)

CCK and SCK: Why RHS is a condition and a special case (special case) of A.S.S.?

No interviewees identified that R.H.S. is actually a condition and a special case of A.S.S. Most of them only claimed that R.H.S. is a rule specifically for the case of right angled triangles or they explained the R.H.S. rule. Alan explicitly admitted that his knowledge on teaching the topic was weak and he did not know why. Only Shing attempted to compare the A.S.S. and R.H.S. rule, however, his explanation reflected that his knowledge was merely procedural.

> "In comparing A.S.S. and R.H.S., we find that for R.H.S., we know one right- angle one side and the hypotenuse; for A.S.S., the length of the last side cannot be fixed." (Shing)

What Shing intended to say was that the last condition (the length of side) for A.S.S. could not be determined. He might have known that the corresponding sides of the two triangles may exhibit different orientations, hence they could have different shapes even though they are the same length. However, Shing still could not properly explain the sufficiency of R.H.S. This knowledge is typically a type of SCK.

CONCLUSION

The teacher on the TIMSS 1999 video intended to teach the concept of sufficiency through an activity that allowed students to explore the minimal conditions required to justify the congruence of triangles. The first objective appeared to have been met: identifying the three sufficient features (a combination of sides and angles). The second objective was to identify the four conditions: S.S.S., A.S.A., S.A.S., and R.H.S. The TIMSS teacher did not clearly explain why R.H.S. is special (Item (4) in the Introduction), nor did the prospective teachers. A competent teacher should understand why R.H.S. is special and they should be able to explain why A.S.S. (Item (3) in the Introduction) is an insufficient condition. In other words, a competent teacher should be able to disprove the proposition "A.S.S. is a condition for congruence" by using an illustrative counter example, and simultaneously demonstrating that R.H.S. is a special case of A.S.S.

Examples and counter examples are common teaching approaches used to illustrate an abstract concept when teaching mathematics. In an experiment, Tapan (2009) classified various types of example according to the level of rigor:

- Naïve empiricism — conclusion based on a small number of cases
- Crucial example — a consideration based on the examination of a number of cases
- Generic example — a thought experiment, such as intellectual proofs and arguments based on concepts and language
- Counter example — to disprove a false statement or an incorrect proposition

A competent teacher should be able to achieve at least the last two levels of rigor when illustrating an example while teaching.

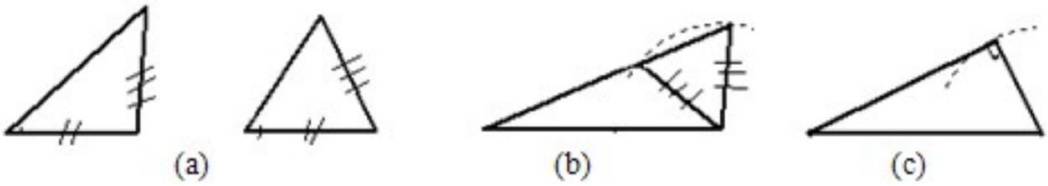


Figure 6. Comparing two triangles in an illustrative example (a), a counter example (b), and the special A.S.S case (c)

During the interview, four of the prospective teachers (except Alan) demonstrated that two triangles of different shapes that are next to each other satisfy the A.S.S. condition (Figure 6(a)), but they did not show that the two triangles could be put together to demonstrate the different shapes in one diagram. They only reach the second level (using crucial examples) defined by Tapan (2009).

Figure 6(b) is a more useful illustration because students can easily grasp the idea that the two triangles come from the same family, because of the construction of an arc intersecting the third side, instead of two isolated and disconnected triangles. Students can see that the two intersecting points on the arc connected to the bottom right vertex are the radii of the same circle. (see Figure 6(b), and the experiment on a geometric counter example conducted by Potarri, Zachariades & Zaslavaky (2009).

If the two intersecting points move closer to each other, they meet at one point. That is, the third side becomes a tangent to the constructed arc. The two radii overlap and becomes a single side (H) that is perpendicular to the third side, producing the R.H.S. condition (See Figure 6(c)). This is the optimal method of unpacking (Hill, Ball & Schilling, 2008) such a critical geometric idea personally experiencing construction. Using a counter example to illustrate insufficiency by showing a pair of triangles with different shapes and applying deductive argument to demonstrate the special case if R.H.S. from the same pair of triangles requires teachers to exhibit high level of KCS, KCT and SCK. Unfortunately, none of the five prospective teachers can point out that R.H.S. was a special case.

It is easy to identify that the diagrams of (in Figure 6) the arc intersecting the third side of the triangle to its tangent reflects the knowledge of geometric construction. During this construction process, students go through the rigorous procedure of constructing a specific figure with accurate dimensional measures, such as the equal length of some lines and edges, parallel and perpendicular lines, and equally sized interior angles. Geometric construction was also the intention of the teacher in the TIMSS 1999 video. Weak knowledge of geometric construction implies that prospective teachers may be restricted when transmitting their geometrical knowledge during activity of a construction activity or justification task (Tapan, 2009). Through construction, even a rough sketch, students must cognitively perceive many relevant ideas. A geometry teacher should be able to unpack and transmit geometric knowledge to students through classroom activities that involve geometric construction. The question is, how well equipped are our teachers to teach geometry through geometric construction. Although current information technology is so sophisticated that we seldom use traditional tools, such as compasses and straight ruler, construction skills can be used to unpack the mathematical knowledge and ideas that scaffold student learning. Using knowledge of construction and its geometric implications as a tool to justify congruence is also a type of EMFAS.

It is difficult to teach the concept of congruent triangles. Diagrams (Figure 6) also demonstrate how to move from insufficient (A.S.S.) to sufficient (R.H.S.) congruency conditions. When the height of a triangle is fixed (two sides overlap), all of its angles and sides are fixed. This tells the learners that three identical properties are sufficient for congruence. This type of logical reasoning is vital for deducing geometric propositions. It is more useful to justify congruence by performing cross-checking to compare the angles and sides of two triangles by placing them side by side (Figure 5). Although we only used a small sample of interviewees, we believe that the SMK and PCK of our HK prospective teachers are relatively weak. Their geometric thinking related to PCK can be substantially improved. Unfortunately, geometric construction was removed from the school mathematics curriculum at the turn of the millennium.

REFERENCES

Ball, D. (2005). Effects of teachers mathematical knowledge for teaching on student achievement. *American Educational Research Journal* **42(2)**, 371–406.

Ball, D. L.; Sleep, L.; Boerst, T. A. & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *Elementary School Journal* **109(5)**, 458–474.

Barlow, A. T. & Reddish, J. M. (2006). Mathematical myths: Teacher candidates' beliefs and the implication for teacher educators. *Teacher Educator* **41(3)**, 145–157.

Brown, M., Jones, K., & Taylor, R. (2003). *Developing geometrical reasoning in the secondary school: Outcomes of trialling teaching activities in classrooms (a report to the QCA).* London, U. K.: Qualifications and Curriculum Authority (QCA).

Buchholtz, N.; Leung, F.; Ding, L.; Kaiser, G.; Park, K. & Schwarz, B. (2013). Future mathematics teachers' professional knowledge of elementary mathematics from an advanced standpoint. *ZDM, Int. J. Math. Educ.* **45(1)**, 107–120. ME **2013c.**00123

Curriculum Development Council, Hong Kong & Hong Kong Examinations and Assessment Authority (CDC & HKEAA) (2007). Mathematics education key learning area: mathematics curriculum and assessment guide (Secondary 4–6), Hong Kong: Government Printer. Retrieved from:

http://334.edb.hkedcity.net/doc/eng/curriculum/Math%20C&A%20Guide_updated_e.pdf

Dreyfus, T. (1999). Why Johnny can't prove. *Educational studies in mathematics* **38(1–3)**, 85–109. ME **2000d.**02330

Fujita, T., Jones, K., & Yamamoto, S. (2004a). *Geometrical intuition and the learning and teaching of geometry.* Paper presented at the Topic Group on Research and Development in the Teaching and Learning of Geometry, 10th International Congress on Mathematical Education (ICME-10), Copenhagen, Denmark.

_____ (2004b). *The role of intuition in geometry education: Learning from the teaching practice in the early 20th Century*. Paper presented at the Topic Group on the History of the Teaching and the Learning of Mathematics, 10th International Congress on Mathematical Education (ICME-10), Copenhagen, Denmark.

Hill, H. C.; Ball, D. L. & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *J. Res. Math. Educ.* **39(4)**, 372–400. ME **2009d.**00095

Jones, K. (1998). Theoretical frameworks for the learning of geometrical reasoning. In: *Proceedings of the British Society for Research into Learning Mathematics*, 18(1&2) (pp. 29–34). London, U.K.: British Society for Research into Learning Mathematics.
http://eprints.soton.ac.uk/41308/1/Jones_BSRLM_18_1998.pdf

Jones, K. & Bills, C. (1998). Visualisation, imagery and the development of geometrical rasoning. In: *Proceedings of the British Society for Research into Learning Mathematics*, 18(1&2) (pp. 123–128). Birmingham, U.K.: British Society for Research into Learning Mathematics.
http://eprints.soton.ac.uk/41306/1/Jones_Bills_BSRLM_18_1998.pdf

Jones, K. & Mooney, C. (2003). Making space for geometry in primary mathematics. In: I. Thompson (Ed.), *Enhancing primary mathematics teaching and learning* (pp. 3–15). London, U.K.: Open University Press. ME **2003f.**04977

Leung, K. C. I.; Ding, L.; Leung, A. Y. L. & Wong, N. Y. (2014). Prospective Teachers' Competency in Teaching how to Compare Geometric Figures: The Concept of Congruent Triangles as an Example. In: H. J. Hwang, S.-G. Lee, Y. H. Choe, (Eds.), *Proceedings of the KSME 2014 Spring Conference on Mathematics Education held at Hankuk Univ. of Foreign Studies, Seoul 130-791, Korea; April 4–5, 2014* (pp. 545–558). Seoul, Korea: Korean Society of Mathematical Education.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ, USA: Lawrence Erlbaum Associates. ME **1999d.**02835

Mayer, R. E. (2003), The promise of multimedia learning: Using the same instructional design methods across different media. *Learning and Instruction* **13,** 125–139.

Piaget, J. & Inhelder, B (1967). *The child's conception of geometry*. New York. W.W. Norton & Co.

Potarri, D.; Zachariades, T. & Zaslavaky, O. (2009). Mathematics teachers’ reasoning for refuting students’ invalid claims. In: *Proceedings of CERME* (pp. 281–290). Lyon, France:

Seufert, T. (2003). Supporting coherence formation in learning from multiple representations. *Learning and Instruction* **13(2)**, 227–237.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher* **57(1)**, 1–22.

Tapan, M. S. (2009). Preserivce teachers’ use of spatio-visual elements and their level of justification dealing with a geometrical construction problem. *US-China Education Review* **6(3)**, 54–60.

Tchoshanov, M. A. (2011). Relationship between teacher knowledge of concepts and connections, teaching practice, and student achievement to middle grades mathematics. *Educ. Stud. Math.* 76(2), 141–164. ME **2011b.**00269

Vermunt, J. D. (2007). The Power of teaching-learning environments to influent student learning. In: Entwistle, N. and Tomlinson, P. (eds.), *Student Learning and University Teaching*. *British Journal of Educational Psychological Society Monograph Series II*, *Number* 4 (pp.73–90). Leicester, UK: British Psychological Society.

Wong. N. Y. & Su, S. D. (1995). Universal education and teacher preparation: The new challenge of mathematics teachers in the changing times. In: G. Bell (Ed.), *Review of mathematics education in Asia and the Pacific* (pp. 137–142). Lismore, Australia: Southern Cross Mathematical Association. ME **1997b.**00803