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A Coupled Fixed Point Theorem for Mixed Monotone Mappings on Partial Ordered G-Metric Spaces

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Abstract. In this paper, we establish coupled fixed point theorems for mixed monotone mappings satisfying nonlinear contraction involving a pair of altering distance functions in ordered G-metric spaces. Via presented theorems we extend and generalize the results of Harjani et al. [J. Harjani, B. López and K. Sadarangani, Fixed point theorems for mixed monotone operators and applications to integral equations, Nonlinear Anal. 74 (2011) 1749-1760] and Choudhury and Maity [B.S. Choudhury and P. Maity, Coupled fixed point results in generalized metric spaces. Math. Comput. Model. 54 (2011), 73-79].

1. Introduction and Preliminaries

Mustafa and Sims [21] introduced the notion of G-metric spaces. The structure of G-metric spaces is a generalization of metric spaces. Mustafa and Sims [21] initiated the theory of fixed points in G-metric spaces and established the Banach contraction principle in this generalized structure. Afterwards, different authors proved several fixed point results in this space (see, e.g., [2, 3, 6, 7, 10, 11, 18, 19, 20, 22, 27, 28]).

Definition 1.1.([21]) Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}_+ = [0, \infty)$ satisfies:
(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(G4) (symmetry in all three variables)
$$G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots, ;$$
(G5) (rectangle inequality)
$$G(x, y, z) \leq G(x, a, a) + G(a, y, z) \text{ for all } x, y, z, a \in X.$$

Then G is called a *G-metric* on X and (X, G) is called a *G-metric space* or a *generalized metric space by G*.

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The following are examples of G-metric spaces.

Example 1.2. Let $(\mathbb{R}, d)$ be the usual metric space. Define G_1 and G_2 by

$$G_1(x,y,z) = d(x,y) + d(y,z) + d(x,z),$$
$$G_2(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$$

for all $x, y, z \in \mathbb{R}$. Then it is clear that $(\mathbb{R}, G_1)$ and $(\mathbb{R}, G_2)$ are G-metric spaces.

Example 1.3. Let $X = \{a, b\}$ and $G : X \times X \times X \to [0, \infty)$ be defined by

$$G(a,a,a) = G(b,b,b) = 0,$$
$$G(a,a,b) = G(a,b,a) = G(b,a,a) = 1,$$
$$G(a,b,b) = G(b,a,b) = G(b,b,a) = 2.$$

It is easy to show that the function G satisfies all properties of Definition .

Definition 1.4.([21]) Let X be a G-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the *limit of a sequence* $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$ and we say in this case that the sequence $\{x_n\}$ is said to be G-convergent to x.

Thus, $x_n \to x$ in a G-metric space X if for any $\epsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$. It has been shown in [21] that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology.

Lemma 1.5.([21]) If X is a G-metric space, then the following are equivalent:
(i) $\{x_n\}$ is G-convergent to x,
(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$.

Definition 1.6.([21]) Let X be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$ there is a positive integer N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

Lemma 1.7.[[21]] If X is a G-metric space, then the following are equivalent:
(i) The sequence $\{x_n\}$ is G-Cauchy.
(ii) For every $\epsilon > 0$, there exists a positive integer N such that
$G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 1.8.([21]) A G-metric space X is said to be G-*complete* (or a *complete* G-*metric space*) if every G-Cauchy sequence is G-convergent in (X, G).

Definition 1.9.([21]) Let (X, G) and (X', G') be two generalized metric spaces. A mapping $f : X \to X'$ is G-*continuous at a point* $x \in X$ if and only if it is G sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G'-convergent to $f(x)$.

Definition 1.10.([21]) Let X be a G-metric space. A mapping $F : X \times X \to X$ is said to be *continuous* if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y, respectively, $\{F(x_n, y_n)\}$ is G-convergent to $F(x, y)$.

In recent years, there has been a lot of interest in establishing fixed point theorems on ordered metric spaces with a contractive condition which holds for all points that are related by partial ordering. This trend was initiated by Ran and Reurings in [26] where they extended the Banach contraction principle in partially ordered sets with some applications to matrix equations. Subsequently, Nieto and Rodríguez-López [24] extended the results in [26] for non-decreasing mappings and applied them to obtain a unique solution for a first-order ordinary differential equation with periodic boundary conditions. Recently, many researchers have obtained common fixed point results on partially ordered metric spaces (see, e.g., [4, 5, 8, 9, 14, 23, 24, 25]).

Bhaskar and Lakshmikantham [12] introduced the notions of a mixed monotone mapping and a coupled fixed point, and proved some coupled fixed point theorems for mixed mappings in ordered metric spaces. Afterwards, Lakshmikantham and Ćirić [16] have established coupled coincidence and coupled fixed point theorems for two mappings F and g, where F has the mixed g-monotone property. Many other results on coupled fixed point theory exist in the literatures [1, 13, 16, 17, 29, 30].

Definition 1.11.([12]) Let $(X, \leq)$ be a partial ordered set. A mapping $F : X\times X \to X$ is said to have the a *mixed monotone property* if F is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X,\ x_1 \leq x_2\ \Rightarrow F(x_1, y) \leq F(x_2, y) \tag{1.1}$$

and

$$y_1, y_2 \in X,\ y_1 \leq y_2\ \Rightarrow F(x, y_1) \geq F(x, y_2). \tag{1.2}$$

Definition 1.12.([12]) An element $(x, y) \in X \times X$ is called a *coupled fixed point* of mapping $F : X \times X \to X$ if

$$x = F(x, y) \ \text{ and } \ y = F(y, x).$$

An altering distance function was introduced by Khan et al. in [15] where they present some fixed point theorems.

Definition 1.13. An *altering distance function* is a map $\Psi : [0,\infty) \to [0,\infty)$ satisfying:

(i) Ψ is continuous and nondecreasing;

(ii) $\Psi(t) = 0$ if and only if $t = 0$.

Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Harjani, López and Sadarangani [13] established some coupled fixed point theorems for the mixed monotone mapping $F : X \times X \to X$ involving a pair of altering distance functions under a contractive condition of the form

$$\phi(d(F(x,y), F(u,v))) \leq \phi(\max\{d(x,u), d(y,v)\}) - \psi(\max\{d(x,u), d(y,v)\})$$

for $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$, where ϕ and ψ are altering distance functions. The purpose of this work is to extend this theorem to the set of G-metric spaces.

2. Coupled Fixed Point in G-Metric Spaces

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete G-metric on X and $F : X \times X \to X$ be a continuous mapping having the mixed monotone property. Suppose that there exist altering distance functions φ and ψ such that*

$$\begin{aligned} &\varphi(G(F(x,y), F(u,v), F(w,z))) \\ &\leq \varphi(\max\{G(x,u,w), G(y,v,z)\}) - \psi(\max\{G(x,u,w), G(y,v,z)\}) \end{aligned} \tag{2.1}$$

for all $x, y, z, u, v, w \in X$ for which $x \geq u \geq w$ and $y \leq v \leq z$ where either $x \neq u$ or $y \neq v$. If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0),$$

then F has a coupled fixed point.

Proof. We construct sequences (x_n) and (y_n) putting

$$x_{n+1} = F(x_n, y_n) \ \text{ and } \ y_{n+1} = F(y_n, x_n) \ \text{ for } \ n \geq 0.$$

In order that the proof is more comprehensive, we will divide it in several steps.

Step 1. $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$, for $n \geq 0$.
In fact, we use mathematical induction.
As $x_0 \leq F(x_0, y_0) = x_1$ and $y_0 \geq F(y_0, x_0) = y_1$ our claim is satisfied for $n = 0$.
Again by the induction hypothesis and the mixed monotone property of F, we have

$$x_{n+1} = F(x_n, y_n) \geq F(x_{n-1}, y_{n-1}) = x_n$$

and

$$y_{n+1} = F(y_n, x_n) \leq F(y_{n-1}, x_{n-1}) = y_n.$$

This proves our claim.

Step 2. $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = 0$.

From the contractive condition (2.1) and Step 1, we obtain

$$\begin{aligned}
&\varphi\left(G\left(x_n,x_{n+1},x_{n+1}\right)\right)\\
&=\varphi\left(G\left(F\left(x_{n-1},y_{n-1}\right),F\left(x_n,y_n\right),F\left(x_n,y_n\right)\right)\right)\\
&\le\varphi\left(\max\{G(x_{n-1},x_n,x_n),G(y_{n-1},y_n,y_n)\}\right)\\
&\qquad-\psi\left(\max\{G(x_{n-1},x_n,x_n),G(y_{n-1},y_n,y_n)\}\right)\\
&\le\varphi\left(\max\{G(x_{n-1},x_n,x_n),G(y_{n-1},y_n,y_n)\}\right).
\end{aligned}\tag{2.2}$$

Using the fact that φ is nondecreasing, we have

$$G\left(x_n,x_{n+1},x_{n+1}\right)\le\max\{G(x_{n-1},x_n,x_n),G(y_{n-1},y_n,y_n)\}.$$

Similarly, we get

$$G\left(y_n,y_{n+1},y_{n+1}\right)\le\max\{G(x_{n-1},x_n,x_n),G(y_{n-1},y_n,y_n)\}.$$

Hence, the sequence $\{\max\{G\left(x_n,x_{n+1},x_{n+1}\right),G\left(y_n,y_{n+1},y_{n+1}\right)\}\}_{n=0}^{\infty}$ is nonnegative and decreasing. This implies that there exists $\alpha\ge0$ such that

$$\lim_{n\to\infty}\max\{G\left(x_n,x_{n+1},x_{n+1}\right),G\left(y_n,y_{n+1},y_{n+1}\right)\}=\alpha.$$

It is easily seen that if $\varphi:[0,\infty)\to[0,\infty)$ is nondecreasing, $\varphi(\max\{a_1,a_2\})=\max\{\varphi(a_1),\varphi(a_2)\}$ for $a_1,a_2\in[0,\infty)$. Taking into account this and (2.2) we get

$$\begin{aligned}
&\varphi\left(\max\{G\left(x_n,x_{n+1},x_{n+1}\right),G\left(y_n,y_{n+1},y_{n+1}\right)\}\right)\\
&=\max\{\varphi\left(G\left(x_n,x_{n+1},x_{n+1}\right)\right),\varphi\left(G\left(y_n,y_{n+1},y_{n+1}\right)\right)\}\\
&\le\varphi\left(\max\{G(x_{n-1},x_n,x_n),G(y_{n-1},y_n,y_n)\}\right)\\
&\quad-\psi\left(\max\{G(x_{n-1},x_n,x_n),G(y_{n-1},y_n,y_n)\}\right)\\
&\le\varphi\left(\max\{G(x_{n-1},x_n,x_n),G(y_{n-1},y_n,y_n)\}\right).
\end{aligned}$$

Since φ is a continuous function, letting $n\to\infty$ in the above inequalities yields

$$\varphi\left(\alpha\right)\le\varphi\left(\alpha\right)-\psi\left(\alpha\right)\le\varphi\left(\alpha\right).$$

and this implies $\psi(\alpha)=0$. Since ψ is an altering distance function, $\alpha=0$ and

$$\lim_{n\to\infty}\max\{G\left(x_n,x_{n+1},x_{n+1}\right),G\left(y_n,y_{n+1},y_{n+1}\right)\}=0,$$

and this proves our claim.

Step 3. $\{x_n\}$ and $\{y_n\}$ are G-Cauchy sequences.
Assume that at least one of the sequences $\{x_n\}$ and $\{y_n\}$ is not a G-Cauchy sequence. By Lemma , this implies that

$$\lim_{n,m\to\infty}G\left(x_n,x_m,x_m\right)\ne0\quad\text{or}\quad\lim_{n,m\to\infty}G\left(y_n,y_m,y_m\right)\ne0$$

and, consequently,

$$\lim_{n,m\to\infty}\max\left\{G(x_n,x_m,x_m),G(y_n,y_m,y_m)\right\}\ne0.$$

This means that there exists an $\varepsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_k\}$ such that $n(k)$ is the smallest index for which

$$\max\left\{G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right), G\left(y_{n(k)}, y_{m(k)}, y_{m(k)}\right)\right\} \geq \varepsilon \tag{2.3}$$

for $n(k) > m(k) > k$.
This means that

$$\max\left\{G\left(x_{n(k)-1}, x_{m(k)}, x_{m(k)}\right), G\left(y_{n(k)-1}, y_{m(k)}, y_{m(k)}\right)\right\} < \varepsilon. \tag{2.4}$$

The rectangle inequality and (2.4) give us, for each k,

$$\begin{aligned} &G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right) \\ &\leq G\left(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}\right) + G\left(x_{n(k)-1}, x_{m(k)}, x_{m(k)}\right) \\ &< G\left(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}\right) + \varepsilon \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &G\left(y_{n(k)}, y_{m(k)}, y_{m(k)}\right) \\ &\leq G\left(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}\right) + G\left(y_{n(k)-1}, y_{m(k)}, y_{m(k)}\right) \\ &< G\left(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}\right) + \varepsilon \end{aligned} \tag{2.6}$$

Using (2.3), (2.5) and (2.6), we get

$$\begin{aligned} \varepsilon &\leq \max\left\{G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right), G\left(y_{n(k)}, y_{m(k)}, y_{m(k)}\right)\right\} \\ &< \max\left\{G\left(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}\right), G\left(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}\right)\right\} + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the last inequality and using Step 2, we obtain that

$$\lim_{k\to\infty} \max\left\{G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right), G\left(y_{n(k)}, y_{m(k)}, y_{m(k)}\right)\right\} = \varepsilon. \tag{2.7}$$

Again, the rectangle inequality and (2.4) give us, for each k,

$$\begin{aligned} &G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right) \\ &\leq G\left(x_{n(k)-1}, x_{m(k)}, x_{m(k)}\right) + G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \\ &< \varepsilon + G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &G\left(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}\right) \\ &\leq G\left(y_{n(k)-1}, y_{m(k)}, y_{m(k)}\right) + G\left(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}\right) \\ &< \varepsilon + G\left(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}\right). \end{aligned} \tag{2.9}$$

By (2.8) and (2.9) we get

$$\begin{aligned} &\max\left\{G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right), G\left(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}\right)\right\} \\ &< \max\left\{G\left(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}\right), G\left(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}\right)\right\} + \varepsilon. \end{aligned} \tag{2.10}$$

Using the rectangle inequality we have

$$\begin{aligned}&G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right)\\&\leq G\left(x_{n(k)},x_{n(k)-1},x_{n(k)-1}\right)+G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right)\\&\quad+G\left(x_{m(k)-1},x_{m(k)},x_{m(k)}\right)\end{aligned}$$

and

$$\begin{aligned}&G\left(y_{n(k)},y_{m(k)},y_{m(k)}\right)\\\leq\ &G\left(y_{n(k)},y_{n(k)-1},y_{n(k)-1}\right)+G\left(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1}\right)\\&\quad+G\left(y_{m(k)-1},y_{m(k)},y_{m(k)}\right).\end{aligned}$$

By the two last inequalities and (2.3) we get

(2.11)
$$\begin{aligned}\varepsilon&\leq\max\left\{G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right),G\left(y_{n(k)},y_{m(k)},y_{m(k)}\right)\right\}\\&\leq\max\left\{G\left(x_{n(k)},x_{n(k)-1},x_{n(k)-1}\right),G\left(y_{n(k)},y_{n(k)-1},y_{n(k)-1}\right)\right\}\\&\quad+\max\left\{G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right),G\left(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1}\right)\right\}\\&\quad+\max\left\{G\left(x_{m(k)-1},x_{m(k)},x_{m(k)}\right),G\left(y_{m(k)-1},y_{m(k)},y_{m(k)}\right)\right\}.\end{aligned}$$

By (2.10) and (2.11) we have

$$\begin{aligned}&\varepsilon+\max\left\{G\left(x_{m(k)},x_{m(k)-1},x_{m(k)-1}\right),G\left(y_{m(k)},y_{m(k)-1},y_{m(k)-1}\right)\right\}\\&>\max\left\{G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right),G\left(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1}\right)\right\}\\&\geq\varepsilon-\max\left\{G\left(x_{n(k)},x_{n(k)-1},x_{n(k)-1}\right),G\left(y_{n(k)},y_{n(k)-1},y_{n(k)-1}\right)\right\}\\&\quad-\max\left\{G\left(x_{m(k)-1},x_{m(k)},x_{m(k)}\right),G\left(y_{m(k)-1},y_{m(k)},y_{m(k)}\right)\right\}.\end{aligned}$$

Letting $k\to\infty$ in the last inequality, and by Step 2, we obtain that

$$\max\left\{G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right),G\left(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1}\right)\right\}=\varepsilon. \tag{2.12}$$

Since $n(k)>m(k)>k$, by Step 1

$$x_{n(k)-1}\leq x_{n(k)}\quad\text{and}\quad y_{n(k)-1}\geq y_{n(k)}$$

and using the contractive condition we can obtain

$$\begin{aligned}&\varphi\left(G\left(x_{n(k)},x_{m(k)},x_{m(k)}\right)\right)\\&=\varphi\left(G\left(F\left(x_{n(k)-1},y_{n(k)-1}\right),F\left(x_{m(k)-1},y_{m(k)-1}\right),F\left(x_{m(k)-1},y_{m(k)-1}\right)\right)\right)\\&\leq\varphi\left(\max\left\{G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right),G\left(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1}\right)\right\}\right)\\&\quad-\psi\left(\max\left\{G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right),G\left(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1}\right)\right\}\right)\end{aligned}$$

and

$$\begin{aligned}&\varphi\left(G\left(y_{n(k)},y_{m(k)},y_{m(k)}\right)\right)\\&=\varphi\left(G\left(F\left(y_{n(k)-1},x_{n(k)-1}\right),F\left(y_{m(k)-1},x_{m(k)-1}\right),F\left(y_{m(k)-1},x_{m(k)-1}\right)\right)\right)\\&\leq\varphi\left(\max\left\{G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right),G\left(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1}\right)\right\}\right)\\&\quad-\psi\left(\max\left\{G\left(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}\right),G\left(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1}\right)\right\}\right).\end{aligned}$$

Thus,
(2.13)

$$\begin{aligned}
&\varphi\left(\max\left\{G\left(x_{n(k)}, x_{m(k)}, x_{m(k)}\right), G\left(y_{n(k)}, y_{m(k)}, y_{m(k)}\right)\right\}\right) \\
&\leq \varphi\left(\max\left\{G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right), G\left(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}\right)\right\}\right) \\
&\quad - \psi\left(\max\left\{G\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right), G\left(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}\right)\right\}\right).
\end{aligned}$$

Finally, letting $k \to \infty$ in (2.13) and using (2.7), (2.12), and the continuity of φ and ψ, we get

$$\varphi(\varepsilon) \leq \varphi(\varepsilon) - \psi(\varepsilon)$$

and, consequently, $\psi(\varepsilon) = 0$. Since ψ is an altering distance function, $\varepsilon = 0$, and this is a contradiction. This proves our claim.

Since (X, G) is a complete G-metric space there exist $x, y \in X$ such that the sequences $\{x_k\}$ and $\{y_k\}$ are G-convergent to x and y, respectively.

In fact, using the continuity of F we have

$$\begin{aligned}
x &= \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} F\left(x_k, y_k\right) = F\left(\lim_{k\to\infty} x_k, \lim_{k\to\infty} y_k\right) = F\left(x, y\right) \\
y &= \lim_{k\to\infty} y_{k+1} = \lim_{k\to\infty} F\left(y_k, x_k\right) = F\left(\lim_{k\to\infty} y_k, \lim_{k\to\infty} x_k\right) = F\left(y, x\right).
\end{aligned}$$

This proves that (x, y) is a coupled fixed point F. □

In the following result, the continuity of F is not required.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete G-metric on X and $F : X \times X \to X$ be a mapping having the mixed monotone property. Suppose that there exist altering distance functions φ and ψ such that*

$$\begin{aligned}
&G(F(x, y), F(u, v), F(w, z)) \\
&\leq \varphi(\max\{G(x, u, w), G(y, v, z)\}) - \psi(\max\{G(x, u, w), G(y, v, z)\})
\end{aligned} \tag{2.14}$$

for all $x, y, z, u, v, w \in X$ for which $x \geq u \geq w$ and $y \leq v \leq z$ where either $x \neq u$ or $y \neq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0)$$

and X has the following property:

(i) *if a nondecreasing sequence $\{x_n\}$ is G-convergent to x, then $x_n \leq x$ for all $n \in N$,*

(ii) *if a nonincreasing sequence $\{y_n\}$ is G-convergent to y, then $y_n \geq y$ for all $n \in N$,*

then F has a coupled fixed point.

Proof. Following the proof of Theorem we only have to check that (x, y) is a coupled fixed point of F.

In fact, since $\{x_n\}$ is nondecreasing and $x_n \to x$, and $\{y_n\}$ is nonincreasing and $y_n \to y$, by our assumption, $x_n \leq x$ and $y_n \geq y$ for every $n \in N$.

Applying the contractive condition of altering distance functions φ and ψ we have

$$\begin{aligned} & \varphi\left(G\left(F\left(x,y\right),F\left(x_n,y_n\right),F\left(x_n,y_n\right)\right)\right) \\ \leq\ & \varphi\left(\max\left\{G\left(x,x_n,x_n\right)G\left(y,y_n,y_n\right)\right\}\right) - \psi\left(\max\left\{G\left(x,x_n,x_n\right)G\left(y,y_n,y_n\right)\right\}\right) \\ \leq\ & \varphi\left(\max\left\{G\left(x,x_n,x_n\right)G\left(y,y_n,y_n\right)\right\}\right). \end{aligned}$$

and, since φ is nondecreasing, we obtain

$$G\left(F\left(x,y\right),F\left(x_n,y_n\right),F\left(x_n,y_n\right)\right) \leq \max\left\{G\left(x,x_n,x_n\right)G\left(y,y_n,y_n\right)\right\}. \tag{2.15}$$

On the other hand, by the rectangle inequality and (2.15) we get

$$\begin{aligned} & G\left(x,F\left(x,y\right),F\left(x,y\right)\right) \\ \leq\ & G\left(x,x_{n+1},x_{n+1}\right) + G\left(x_{n+1},F\left(x,y\right),F\left(x,y\right)\right) \\ =\ & G\left(x,x_{n+1},x_{n+1}\right) + G\left(F\left(x_n,y_n\right),F\left(x,y\right),F\left(x,y\right)\right) \\ \leq\ & G\left(x,x_{n+1},x_{n+1}\right) + \max\left\{G\left(x_n,x,x\right),G\left(y_n,y,y\right)\right\}. \end{aligned}$$

Taking $n \to \infty$ in the last inequality, Lemma 1.3 yields

$$G\left(x,F\left(x,y\right),F\left(x,y\right)\right) = 0$$

and hence, $x = F\left(x,y\right)$.

Using a similar argument it can be proved that $y = F(y,x)$ and this finished the proof. □

Now, we will show that many results can be deduced from our previously obtained results.

Corollary 2.3. If in Theorem 2.1 (resp. Theorem 2.3) we replace the contractive condition by

there exists $\alpha \in [0,1)$ such that

$$G(F(x,y),F(u,v),F(w,z)) \leq \alpha \ \cdot \max\{G(x,u,w),G(y,v,z)\}$$

for all $x \geq u \geq w$and $y \leq v \leq z$ where either $x \neq u$ or $y \neq v$,

then F has a coupled fixed point of F.

Proof. Taking as $\varphi =$ identity and $\psi = (1-\alpha)\varphi$, we obtain the corollary. □

Corollary 2.4. If in Theorem 2.1 (resp. Theorem 2.2) we replace the contractive condition by there exist $\delta_1, \delta_2 \in [0,1)$ and $\delta_1 + \delta_2 < 1$ such that

$$G(F(x,y), F(u,v), F(w,z)) \le \delta_1\ G(x,u,w) + \delta_2\ G(y,v,z)$$

for all $x \ge u \ge w$and $y \le v \le z$ where either $x \ne u$ or $y \ne v$,

then F has a coupled fixed point of F.

Proof. We have

$$\begin{aligned} G(F(x,y), F(u,v), F(w,z)) &\le \delta_1\ G(x,u,w) + \delta_2\ G(y,v,z) \\ &\le (\delta_1 + \delta_2)\ \max\{G(x,u,w), G(y,v,z)\}. \end{aligned}$$

Therefore, applying Corollary 2.3 we obtain the desired result. □

Remark 2.5. Taking $\delta_1 = \delta_2 = \frac{k}{2}$ in Corollary 2.4, we can obtain Theorem 3.2 of Choudhury and Maity [10].

3. Uniqueness of Coupled Fixed Point in G-Metric Spaces

In this section, we consider some additional conditions to ensure the uniqueness of a coupled fixed point in the setting of partially ordered G-metric spaces. Furthermore, we study appropriate conditions to ensure that for a coupled fixed point (x,y) we have $x = y$.

Notice that if $(X, \le)$ is a partially ordered set, we endow the product space $X \times X$ with the partial order relation given by

$$(u,v) \le (x,y) \quad \Leftrightarrow \quad x \ge u \ \text{ and } y \ \le v.$$

We say that two pairs (x,y) and (u,v) are *comparable*.

Theorem 3.1. *In addition to the hypotheses of Theorem 2.1, suppose that, for every $(a,b),(c,d) \in X \times X$, there exists a pair $(u,v) \in X \times X$ such that (u,v) is comparable to (a,b) and (c,d). Then F has a unique coupled fixed point.*

Proof. Suppose that (x,y) and (z,t) are coupled fixed point of F, that is, $x = F(x,y)$, $y = F(y,x)$, $z = F(z,t)$ and $t = F(t,z)$.

Let (u,v) be an element of $X \times X$ and comparable to (x,y) and (z,t). Suppose that $(x,y) \ge (u,v)$ (the proof is similar in other cases).

We construct the sequences $\{u_n\}$ and $\{v_n\}$ defined by

$$u_0 = u, \quad v_0 = v, \quad u_{n+1} = F(u_n, v_n), \quad v_{n+1} = F(v_n, u_n).$$

We claim that $(x,y) \ge (u_n, v_n)$ for each $n \in N$.

We will use the induction.

For $n = 0$, as $(x,y) \ge (u,v)$, this means $u_0 = u \le x$ and $v_0 = v \ge y$ and, thus, $(u_0, v_0) \le (x,y)$.

Suppose that $(x, y) \geq (u_n, v_n)$ for some $n \in N$. Then using the mixed monotone property of F, we get

$$u_{n+1} = F(u_n, v_n) \leq F(x, y) = x,$$
$$v_{n+1} = F(v_n, u_n) \geq F(y, x) = y$$

and this proves our claim.

Since $(x, y) \geq (u_n, v_n)$, using the contractive condition we have

$$\begin{aligned}
&\varphi(G(x, u_{n+1}, u_{n+1})) \\
&= \varphi(G(F(x, y), F(u_n, v_n), F(u_n, v_n))) \\
&\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) - \psi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) \\
&\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\})
\end{aligned}$$

and

$$\begin{aligned}
&\varphi(G(y, v_{n+1}, v_{n+1})) \\
&= \varphi(G(F(y, \overline{c}), F(v_n, n_n), F(v_n, u_n))) \\
&\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) - \psi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) \\
&\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}).
\end{aligned}$$

By the last two equation and using the fact that φ is nondecreasing, we obtain (3.1)

$$\begin{aligned}
&\varphi(\max\{G(x, u_{n+1}, u_{n+1}), G(y, v_{n+1}, v_{n+1})\}) \\
&= \max\{\varphi(G(x, u_{n+1}, u_{n+1})), \varphi(G(y, v_{n+1}, v_{n+1}))\} \\
&\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) - \psi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}) \\
&\leq \varphi(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\}).
\end{aligned}$$

This last inequality implies that

$$\max\{G(x, u_{n+1}, u_{n+1}), G(y, v_{n+1}, v_{n+1})\} \leq \max\{G(x, u_n, u_n), G(y, v_n, v_n)\}.$$

Consequently, the sequence $(\max\{G(x, u_n, u_n), G(y, v_n, v_n)\})$ is decreasing and nonnegative, and so, for certain $\alpha \geq 0$

$$\lim_{n\to\infty} \max\{G(x, u_n, u_n), G(y, v_n, v_n)\} = \alpha.$$

Letting $n \to \infty$ in (3.1) we have

$$\varphi(\alpha) \leq \varphi(\alpha) - \psi(\alpha) \leq \varphi(\alpha),$$

and this implies $\psi(\alpha) = 0$ and, thus, $\alpha = 0$.

Finally, as $\lim_{n\to\infty} \max\{G(x, u_n, u_n), G(y, v_n, v_n)\} = 0$, this gives us that the sequences $\{u_n\}$ and $\{v_n\}$ are G-convergent to x and y, respectively. This means that

$$\begin{aligned}
&\lim_{n\to\infty} G(x, u_n, u_n) = \lim_{n\to\infty} G(x, x, u_n) = 0, \\
&\lim_{n\to\infty} G(y, v_n, v_n) = \lim_{n\to\infty} G(y, y, v_n) = 0.
\end{aligned} \tag{3.2}$$

Using a similar argument for a coupled fixed point (z,t), we can obtain $\{u_n\}$ and $\{v_n\}$ are G-convergent to z and t, respectively, that is,

$$\begin{aligned} &\lim_{n\to\infty} G(z,u_n,u_n) = \lim_{n\to\infty} G(z,z,u_n) = 0, \\ &\lim_{n\to\infty} G(t,v_n,v_n) = \lim_{n\to\infty} G(t,t,v_n) = 0. \end{aligned} \tag{3.3}$$

By the rectangle inequality, for any $n \in N$, we have

$$\begin{aligned} G(x,z,z) &\le G(x,u_n,u_n) + G(u_n,z,z), \\ G(y,t,t) &\le G(y,v_n,v_n) + G(v_n,t,t) \end{aligned}$$

Letting $n \to \infty$ in the last inequalities, and using (3.2) and (3.3) we get

$$G(x,z,z) = G(y,t,t) = 0$$

and, consequently, $(x,y) = (z,t)$. □

Theorem 3.2. *In addition to the hypotheses of Theorem 2.1, if x_0 and y_0 are comparable, then the coupled fixed point $(x,y) \in X \times X$ satisfies $x = y$.*

Proof. Assume $x_0 \le y_0$ (a similar argument applies for $y_0 \le x_0$).

Then by using the mathematical induction

$$x_{n+1} = F(x_n,y_n) \le F(y_n,x_n) = y_{n+1}.$$

Taking $n \to \infty$, we have

$$x = \lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n = y.$$

From the contractive condition, we get

$$\begin{aligned} &\varphi\left(G\left(x,y,y\right)\right) \\ &= \varphi\left(G\left(F(x,y),F(y,x),F(y,x)\right)\right) \\ &\le \varphi\left(\max\left\{G(x,y,y),G(x,x,y)\right\}\right) - \psi\left(\max\left\{G(x,y,y),G(x,x,y)\right\}\right) \\ &\le \varphi\left(\max\left\{G(x,y,y),G(x,x,y)\right\}\right) \end{aligned}$$

and

$$\begin{aligned} &\varphi\left(G\left(x,x,y\right)\right) \\ &= \varphi\left(G\left(F(x,y),F(x,y),F(y,x)\right)\right) \\ &\le \varphi\left(\max\left\{G(x,y,y),G(x,x,y)\right\}\right) - \psi\left(\max\left\{G(x,y,y),G(x,x,y)\right\}\right) \\ &\le \varphi\left(\max\left\{G(x,y,y),G(x,x,y)\right\}\right). \end{aligned}$$

Since $\varphi : [0,\infty) \to [0,\infty)$ is nondecreasing, $\varphi(\max\{a,b\}) = \max\{\varphi(a),\varphi(b)\}$ for $a,b \in [0,\infty)$. Taking into account this and the last two inequalities we get

$$\psi\left(\max\left\{G(x,y,y),G(x,x,y)\right\}\right) = 0.$$

Using the fact that ψ is nondecreasing, we have

$$G(x,y,y) = G(x,y,y) = 0$$

and, consequently, $x = y$. □

Example 3.3. Let $X = [0, \frac{1}{2}]$. Then $(X, \leq)$ is a partially ordered set with a natural ordering of real numbers. Let $G(x,y,z) = |x-y|+|y-z|+|z-x|$ for all $x,y,z \in X$. Let $F : X \times X \to X$ be defined as

$$F(x,y) = \begin{cases} \frac{x^2-y^2+1}{3}, & x \leq y \\ \frac{1}{3}, & x > y. \end{cases}$$

Then

(1) (X, G) is a complete G-metric space;

(2) F has the mixed monotone property;

(3) F is continuous;

(4) $0 \leq F(0, \frac{1}{2})$ and $\frac{1}{2} \geq F(\frac{1}{2}, 0)$;

(5) there exist two altering distance functions φ and ψ such that

$$\begin{aligned} &\varphi(G(F(x,y), F(u,v), F(w,z))) \\ &\leq \varphi(\max\{G(x,u,w), G(y,v,z)\}) - \psi(\max\{G(x,u,w), G(y,v,z)\}) \end{aligned}$$

for all $(x,y), (u,v), (w,z) \in X \times X$ with $x \leq u \leq w$ and $y \geq v \geq z$.

Thus by Theorem , F has a coupled fixed point. Moreover, $\left(\frac{1}{3}, \frac{1}{3}\right)$ is the unique coupled fixed point of F.

Proof. The proofs of (1)-(4) are clear.

For any $x \leq u \leq w$ and $y \geq v \geq z$, we have

$$G(x,u,w) = 2(w-x), \quad G(y,v,z) = 2(y-z).$$

The proof of (5) is divided into the following cases.

Case 1. If $w \leq z$. In this case, we have $x \leq u \leq w \leq z \leq v \leq y$, and so

$$F(x,y) = \frac{x^2-y^2+1}{3}, \quad F(u,v) = \frac{u^2-v^2+1}{3}, \quad F(w,z) = \frac{w^2-z^2+1}{3}.$$

Hence, we get

$$\begin{aligned} G(F(x,y), F(u,v), F(w,z)) &= G\left(\tfrac{x^2-y^2+1}{3}, \tfrac{u^2-v^2+1}{3}, \tfrac{w^2-z^2+1}{3}\right) \\ &= \tfrac{2}{3}(y^2-x^2+w^2-z^2) \\ &\leq \tfrac{1}{3}\max\{2(y^2-z^2), 2(w^2-x^2)\} \\ &\leq \tfrac{1}{3}\max\{2(y-z), 2(w-x)\}. \end{aligned}$$

Case 2. $w > z$. We divide the study in two sub-cases:
(a) If $u \leq v$, then $x \leq u \leq v \leq y$. Therefore, we get

$$F(x,y) = \frac{x^2 - y^2 + 1}{3}, \quad F(u,v) = \frac{u^2 - v^2 + 1}{3}, \quad F(w,z) = \frac{1}{3}.$$

Hence, we get

$$\begin{aligned} G(F(x,y), F(u,v), F(w,z)) &= G\left(\tfrac{x^2-y^2+1}{3}, \tfrac{u^2-v^2+1}{3}, \tfrac{1}{3}\right) \\ &= \tfrac{2}{3}(y^2 - x^2) \\ &\leq \tfrac{2}{3}(y^2 - x^2 + w^2 - z^2) \\ &\leq \tfrac{1}{3}\max\{2(y^2 - z^2), 2(w^2 - x^2)\} \\ &\leq \tfrac{1}{3}\max\{2(y - z), 2(w - x)\}. \end{aligned}$$

(b) If $u > v$, hence $F(u,v) = \frac{1}{3} = F(w,z)$; the case where $x > y$ is obvious because we get $F(x,y) = \frac{1}{3}$. If $x \leq y$, we have $F(x,y) = \frac{x^2-y^2+1}{3}$. Therefore

$$\begin{aligned} G(F(x,y), F(u,v), F(w,z)) &= G\left(\tfrac{x^2-y^2+1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) \\ &= \tfrac{2}{3}(y^2 - x^2) \\ &\leq \tfrac{2}{3}(y^2 - x^2 + w^2 - z^2) \\ &\leq \tfrac{1}{3}\max\{2(y^2 - z^2), 2(w^2 - x^2)\} \\ &\leq \tfrac{1}{3}\max\{2(y - z), 2(w - x)\}. \end{aligned}$$

In all the above cases, the condition (5) is satisfied for the altering distance functions $\varphi = I$ and $\psi = \frac{2}{3}I$ (where I is an identity mapping). Since $X = [0, \frac{1}{2}]$ is a totally ordered set, by Theorem 3.2, $\left(\frac{1}{3}, \frac{1}{3}\right)$ is the the unique coupled fixed point of F. □

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