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On the Modified q-Euler Numbers and Polynomials with Weak Weight 0

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Abstract. In this paper, we construct new q-extension of Euler polynomials with weight 0. These modified q-Euler polynomials are useful to study various identities of Carlitz's q-Bernoulli numbers.

1. Introduction

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$.

In this paper, we assume that $q \in \mathbb{C}_p$ with$|1-q|_p < p^{-\frac{1}{p-1}}$. So that $q^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$. The q-number of x is denoted by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let d be a fixed integer bigger than 0 and let p be a fixed prime number. We set

$$X_d = \varprojlim_{N} \mathbb{Z}/dp^N\mathbb{Z},\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$
$$a + dp^N\mathbb{Z}_p = \left\{x \in X | x \equiv a \ (\mathrm{mod} dp^N)\right\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$, (see [1-12]).

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Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the *fermionic p-adic q-integral* on $\mathbb{Z}_p$ is defined by Kim by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [8, 9])}. \tag{1.1}$$

As is well known definition, the *Euler polynomials* are defined by the generating function to be

$$\frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.2}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1-11]). In the special case, $x = 0$, $E_n(0) = E_n$ are called the *n-th Euler numbers numbers*.

Recently, the *q-Euler numbers with weight* α are defined by

$$\tilde{E}_{0,q}^{(\alpha)} = 1, \text{ and } q(q^\alpha \tilde{E}_q^{(\alpha)} + 1)^n + \tilde{E}_{n,q}^{(\alpha)} = 0 \text{ if } n > 0,$$

with the usual convention about replacing $(\tilde{E}_q^{(\alpha)})^n$ by $\tilde{E}_{n,q}^{(\alpha)}$ (see [11]). And in [12], authors defined the modified q-Bernoulli polynomials $\tilde{\beta}_{n,q}(x)$ by generating function, and represent $\tilde{\beta}_{n,q}(x)$ as a p-adic q-integral on $\mathbb{Z}_p$.

The purpose of this paper is to construct new q-extension of Euler numbers and polynomials with weight 0 related to fermionic p-adic q-integral on $\mathbb{Z}_p$, and give new explicit formulas related to these numbers and polynomials.

2. A New Approach of q-Euler Polynomials with Weak Weight 0

The *q-Euler number with weak weight* 0 is defined by

$$\epsilon_{n,q} = \int_{\mathbb{Z}_p} [y]_q^n d\mu_{-1}(y). \tag{2.1}$$

Thus, we defined the *modified q-Euler polynomial with weak weight* 0 as follows:

$$\begin{aligned} \tilde{\epsilon}_{n,q}(x) &= \int_{\mathbb{Z}_p} (x + [y]_q)^n d\mu_{-1}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} \epsilon_{l,q} x^{n-l} \\ &= (E_q + x)^n . \end{aligned} \tag{2.2}$$

By (1.1) and (2.1), we know that

$$\epsilon_{n,q} = \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l}. \tag{2.3}$$

Thus,

$$\tilde{\epsilon}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} \epsilon_{l,q} x^{n-l} \tag{2.4}$$
$$= \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \binom{l}{m} \frac{2}{(1-q)^l} (-1)^m \frac{1}{1+q^m} x^{n-l}.$$

Thus the following theorem is proved.

Theorem 2.1. *For positive integer n,*

$$\tilde{\epsilon}_{n,q}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \binom{l}{m} \frac{2}{(1-q)^l} (-1)^m \frac{1}{1+q^m} x^{n-l}.$$

Consider the following equation.

$$\sum_{n=0}^{\infty} (q-1)^n \epsilon_{n,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} (-1)^{m+n} \frac{2}{1+q^m} \frac{t^n}{n!} \tag{2.5}$$
$$= \left(\sum_{l=0}^{\infty} \frac{(-1)^l t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \frac{2}{1+q^m} \frac{t^m}{m!} \right)$$
$$= e^{-t} \sum_{m=0}^{\infty} \frac{2}{1+q^m} \frac{t^m}{m!}.$$

By (2.3),

$$e^{(q-1)xt} \sum_{n=0}^{\infty} (q-1)^n \epsilon_{n,q} \frac{t^n}{n!} = \left(\sum_{l=0}^{\infty} (q-1)^l x^l \frac{t^l}{l!} \right) \left(\sum_{n=0}^{\infty} (q-1)^n \epsilon_{n,q} \frac{t^n}{n!} \right) \tag{2.6}$$
$$= \sum_{m=0}^{\infty} (q-1)^m \sum_{n=0}^{m} \binom{m}{n} \epsilon_{n,q} x^{m-n} \frac{t^m}{m!}$$
$$= \sum_{m=0}^{\infty} (q-1)^m \tilde{\epsilon}_{m,q}(x) \frac{t^m}{m!}.$$

And

(2.7)

$$e^{(q-1)xt} \left(e^{-t} \sum_{m=0}^{\infty} \frac{2}{1+q^m} \frac{t^m}{m!} \right) = e^{((q-1)x-1)t} \sum_{m=0}^{\infty} \frac{2}{1+q^m} \frac{t^m}{m!}$$
$$= \left(\sum_{l=0}^{\infty} \frac{((q-1)x-1)^l}{l!} t^l \right) \left(\sum_{m=0}^{\infty} \frac{2}{1+q^m} \frac{t^m}{m!} \right)$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} ((q-1)x-1)^{n-m} \binom{n}{m} \frac{2}{1+q^m} \frac{t^n}{n!}.$$

By (2.5), (2.6) and (2.7), we obtain the following equation.

$$(q-1)^m \tilde{\epsilon}_{m,q}(x) = \sum_{n=0}^{m} \binom{m}{n} ((q-1)x-1)^{m-n} \frac{2}{1+q^n}.$$

Therefore, we obtain the following theorem.

Theorem 2.2. *For positive integer m,*

$$\begin{aligned}\tilde{\epsilon}_{m,q}(x) =& \frac{1}{(q-1)^m} \sum_{n=0}^{m} \binom{m}{n} ((q-1)x-1)^{m-n} \frac{2}{1+q^n} \\ =& \sum_{n=0}^{m} \sum_{j=0}^{m-n} \binom{m}{n} \binom{m-n}{j} (-1)^{m-n-j} (q-1)^{j-m} \frac{2}{1+q^n} x^j.\end{aligned}$$

3. Multiple Modified q-Euler Numbers with Weak Weight 0

The *multiple q-Euler number with weak weight* 0 is defined by

$$\epsilon_{n,q}^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{3.1}$$

By (1.1) and (3.1),

$$\begin{aligned}\epsilon_{n,q}^{(k)} =& \lim_{N_1\to\infty} \cdots \lim_{N_k\to\infty} \frac{1}{[p^{N_1}]_{-1}} \cdots \frac{1}{[p^{N_k}]_{-1}} \\ & \sum_{j_1=0}^{p^{N_1}-1} \cdots \sum_{j_k=0}^{p^{N_k}-1} [j_1 + \cdots + j_k]_q^n (-1)^{j_1} \cdots (-1)^{j_k} \\ =& \lim_{N_1\to\infty} \cdots \lim_{N_k\to\infty} \sum_{j_1=0}^{p^{N_1}-1} \cdots \sum_{j_k=0}^{p^{N_k}-1} \left(\frac{1-q^{j_1+\cdots+j_k}}{1-q} \right)^n (-1)^{j_1} \cdots (-1)^{j_k} \\ =& \lim_{N_1\to\infty} \cdots \lim_{N_k\to\infty} \sum_{j_1=0}^{p^{N_1}-1} \cdots \sum_{j_k=0}^{p^{N_k}-1} \frac{1}{(1-q)^n} \\ & \sum_{l=0}^{n} \binom{n}{l} (-1)^l \left(q^{j_1+\cdots+j_k}\right)^l (-1)^{j_1} \cdots (-1)^{j_k}\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&= \lim_{N_1\to\infty}\cdots\lim_{N_{k-1}\to\infty}\sum_{j_1=0}^{p^{N_1}-1}\cdots\sum_{j_{k-1}=0}^{p^{N_{k-1}-1}}\frac{1}{(1-q)^n}\\
&\quad\times\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{(j_1+\cdots+j_{k-1})l}(-1)^{j_1+\cdots+j_{k-1}}\lim_{N_k\to\infty}\sum_{j_k=0}^{p^{N_k}-1}\left(-q^l\right)^{j_k}\\
&= \lim_{N_1\to\infty}\cdots\lim_{N_{k-1}\to\infty}\sum_{j_1=0}^{p^{N_1}-1}\cdots\sum_{j_k=0}^{p^{N_{k-1}-1}}\frac{1}{(1-q)^n}\\
&\quad\times\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{(j_1+\cdots+j_{k-1})l}(-1)^{j_1+\cdots+j_{k-1}}\frac{2}{1+q^l}\\
&=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\left(\frac{2}{1+q^l}\right)^k.
\end{aligned}
$$

Thus, the *multiple modified q-Euler polynomial with weak weight* 0 which is defined by

$$
\tilde{\epsilon}_{n,q}^{(k)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+[x_1+\cdots+x_k]_q)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \tag{3.3}
$$

is represented by

$$
\begin{aligned}
\tilde{\epsilon}_{n,q}^{(k)}(x)&=\sum_{l=0}^{n}\binom{n}{l}\epsilon_{l,q}^{(k)}x^{n-l}\\
&=\sum_{l=0}^{n}\binom{n}{l}\frac{1}{(1-p)^l}\sum_{m=0}^{l}\binom{l}{m}(-1)^m\left(\frac{2}{1+q^m}\right)^k x^{n-l}.
\end{aligned} \tag{3.4}
$$

Therefore, by (3.4), we obtain the following theorem.

Theorem 3.1. *For positive integer n,k,*

$$
\tilde{\epsilon}_{n,q}^{(k)}(x)=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\binom{l}{m}\frac{(-1)^m}{(1-p)^l}\left(\frac{2}{1+q^m}\right)^k x^{n-l}.
$$

Note that, by (3.2),

$$
\epsilon_{n,q}^{(k)}=\frac{1}{(p-1)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\left(\frac{2}{1+p^l}\right)^k. \tag{3.5}
$$

Then

$$
(p-1)^n\epsilon_{n,q}^{(k)}=\sum_{l=0}^{n}\binom{n}{l}(-1)^l\left(\frac{2}{1+p^l}\right)^k, \tag{3.6}
$$

and so

$$\sum_{n=0}^{\infty}(q-1)^n\epsilon_{n,q}^{(k)}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{2}{1+q^l}^k\frac{t^n}{n!} \tag{3.7}$$
$$= \left(\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\left(\frac{2}{1+q^l}\right)^k\frac{t^l}{l!}\right)$$
$$= e^{-t}\sum_{l=0}^{\infty}\left(\frac{2}{1+q^l}\right)^k\frac{t^l}{l!}.$$

By (3.7),

$$e^{(q-1)xt}\sum_{n=0}^{\infty}(q-1)^n\epsilon_{n,q}^{(k)}\frac{t^n}{n!} = \left(\sum_{m=0}^{\infty}(q-1)^m x^m\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}(q-1)^n\epsilon_{n,q}^{(k)}\frac{t^n}{n!}\right) \tag{3.8}$$
$$= \sum_{l=0}^{\infty}(q-1)^l\sum_{n=0}^{l}\binom{l}{n}\epsilon_{n,q}^{(k)}x^{l-n}\frac{t^l}{l!}$$
$$= \sum_{l=0}^{\infty}(q-1)^l\tilde{\epsilon}_{l,q}(x)\frac{t^l}{l!},$$

and

(3.9)

$$e^{(q-1)xt}\left(e^{-t}\sum_{l=0}^{\infty}\left(\frac{2}{1+q^l}\right)\frac{t^l}{l!}\right) = e^{((q-1)x-1)t}\sum_{l=0}^{\infty}\left(\frac{2}{1+q^l}\right)^k\frac{t^l}{l!}$$
$$= \left(\sum_{m=0}^{\infty}\frac{((q-1)x-1)^m}{m!}t^m\right)\left(\sum_{l=0}^{\infty}\left(\frac{2}{1+q^l}\right)^k\frac{t^l}{l!}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}((q-1)x-1)^{n-l}\left(\frac{2}{1+q^l}\right)^k\frac{t^n}{n!}.$$

By (3.7), (3.8) and (3.9), we obtain the following equation.

$$(q-1)^l\tilde{\epsilon}_{l,q}^{(k)}(x) = \sum_{n=0}^{l}\binom{l}{n}((q-1)x-1)^{l-n}\left(\frac{2}{1+q^n}\right)^k$$

Therefore, we obtain the following theorem.

Theorem 3.2. *For positive integer m,*

$$\tilde{\epsilon}_{m,q}(x) = \frac{1}{(q-1)^m}\sum_{n=0}^{m}\binom{m}{n}((q-1)x-1)^{m-n}\left(\frac{2}{1+q^n}\right)^k$$
$$= \sum_{n=0}^{m}\sum_{j=0}^{m-n}\binom{m}{n}\binom{m-n}{j}(-1)^{m-n-j}(q-1)^{j-m}\left(\frac{2}{1+q^n}\right)^k x^j.$$

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