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Bounded Mocanu Variation Properties of Certain Subclass of Meromorphic Functions Involving a Family of Linear Operator

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Abstract. In this paper, we introduce a new subclass of meromorphic functions defined in the punctured unit disc.We derive inclusion relationships, radius problem and some other interesting properties of this class are investigated.

1. Introduction

Let $\mathcal{M}$ denote the class of functions $f(z)$ of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k \, z^k, \tag{1.1}$$

which are analytic in the punctured open unit disc $E^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = E \backslash \{0\}$.

If $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k \ , \tag{1.2}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f \star g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k = (g \star f)(z) \quad (z \in E). \tag{1.3}$$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re_{p(z)} - \rho}{1 - \rho} \right| d\theta \le k\pi, z = re^{i\theta}, \tag{1.4}$$

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where $k \geqslant 2$ and $0 \leq \rho < 1$. This class was introduced by Padmanbhan et. al. in [13]. We note that $P_k(0) = P_k$, see Pinchuk [14], $P_2(\rho) = P(\rho)$, the class of analytic functions with positive real part greater than ρ and $P_2(0) = P$, the class of functions with positive real part. From (1.4) we can easily deduce that $p(z) \in P_k(\rho)$ if, and only if, there exists $p_1(z)$, $p_2(z) \in P(\rho)$ such that for $z \in E$,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z). \tag{1.5}$$

In recent years, several families of integral operators and differential operators were introduced using Hadamard product (or convolution). For example, we choose to mention the Rushcheweyh derivative [15], the Carlson-Shaffer operator [1], the Dzoik-Srivastava operator [4], the Noor integral operator [12] also see, [3, 6, 7, 11]. Motivated by the work of N. E. Cho and K. I. Noor [2, 9], we introduce a family of integral operators defined on the space of meromorphic functions in the class $\mathcal{M}$,see [16]. By using these integral operators, we define a new subclass of meromorphic functions and investigate various inclusion relationships, radius problem and some other properties for the meromorphic function classes introduced here.

For a complex parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ $(\beta_j \in \mathbb{C} \backslash \mathbb{Z}_0^- = \{0, -1, -2, ...\};$ $j = 1, ..., s)$, we now define the function $\phi(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by

$$\begin{aligned} \phi(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) &= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k+1} ... (\alpha_q)_{k+1}}{(\beta_1)_{k+1} ... (\beta_s)_{k+1} \{ \ (k+1)!} z^k, \\ (q &\leq s + 1; s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, ...\}; z \in E), \end{aligned}$$

where $(v)_k$ is the Pochhammer symbol(or shifted factorial) defined in (terms of the Gamma function) by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} 1 & \text{if } k = 0 \text{ and } v \in \mathbb{C} \backslash \{0\} \\ v(v+1)...(v+k-1) & \text{if } k \in \mathbb{N} \text{ and } v \in \mathbb{C}. \end{cases}$$

Now we introduce the following operator

$$I_\mu^p(\alpha_1, ..., \alpha_q, \beta_1, ..., \beta_s) : \mathcal{M} \longrightarrow \mathcal{M}$$

as follows:

Let $F_{\mu,p}(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+\mu+1}{\mu}\right)^p z^k, p \in \mathbb{N}_0, \mu \neq 0$ and let $F_{\mu,p}^{-1}(z)$ be defined such that

$$F_{\mu,p}(z) * F_{\mu,p}^{-1}(z) = \phi(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z).$$

Then

$$I_\mu^p(\alpha_1, ..., \alpha_q, \beta_1, ..., \beta_s) f(z) = F_{\mu,p}^{-1}(z) * f(z). \tag{1.6}$$

From (1.6) it can be easily seen

(1.7)

$$I_\mu^p(\alpha_1, ..., \alpha_q, \beta_1, ..., \beta_s) f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{\mu}{k+\mu+1}\right)^p \frac{(\alpha_1)_{k+1} ... (\alpha_q)_{k+1}}{(\beta_1)_{k+1} ... (\beta_s)_{k+1} \{\} (k+1)!} z^k.$$

For conveniences, we shall henceforth denote

$$I_{\mu}^{p}(\alpha_1,...,\alpha_q,\beta_1,...,\beta_s)f(z)=I_{\mu}^{p}(\alpha_1,\beta_1)f(z). \tag{1.8}$$

For the choices of the parameters $p=0,q=2,\ s=1$, the operator $I_{\mu}^{p}(\alpha_1,\beta_1)f(z)$ is reduced to an operator by N. E. Cho and K. I. Noor [2] and K. I. Noor [9] and when $p=0,\ q=2,\ s=1,\alpha_1=\lambda,\ \alpha_2=1,\beta_1=(n+1)$, the operator $I_{\mu}^{p}(\alpha_1,\beta_1)f(z)$ is reduced to an operator recently introduced by S. -M. Yuan et. al. in [17].

It can be easily verified from the above definition of the operator $I_{\mu}^{p}(\alpha_1,\beta_1)f(z)$ that

$$z(I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z))'=\mu I_{\mu}^{p}(\alpha_1,\beta_1)f(z)-(\mu+1)I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z), \tag{1.9}$$

and

$$z(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))'=\alpha_1 I_{\mu}^{p}(\alpha_1+1,\beta_1)f(z)-(\alpha_1+1)I_{\mu}^{p}(\alpha_1,\beta_1)f(z). \tag{1.10}$$

By using the operator $I_{\mu}^{p}(\alpha_1,\beta_1)f(z)$, we now introduce the following subclass of meromorphic functions:

Definition 1.3. Let $\lambda\in\mathbb{C}$ with $\Re\lambda>0,\ f\in\mathcal{M},\ p\in\mathbb{N}_0, 0\le\rho<1,\ \alpha=\mu>0$ and $k\ge 2$. Then $f\in B_{k,\mu}^{\lambda;p}(\alpha_1,\beta_1,\alpha,\rho)$, if and only if

$$\left\{(1-\lambda)\left(\frac{(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))}{(I_{\mu}^{p}(\alpha_1,\beta_1)g(z))}\right)^{\alpha}+\lambda\left(\frac{(I_{\mu}^{p}(\alpha_1+1,\beta_1)f(z))}{(I_{\mu}^{p}(\alpha_1+1,\beta_1)g(z))}\right)\left(\frac{(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))}{(I_{\mu}^{p}(\alpha_1,\beta_1)g(z))}\right)^{\alpha-1}\right\}$$

$\in P_k(\rho)$,

where $g\in\mathcal{M}$ satisfies the condition:

$$\left(\frac{(I_{\mu}^{p}(\alpha_1+1,\beta_1)g(z))}{(I_{\mu}^{p}(\alpha_1,\beta_1)g(z))}\right)\in P(\eta),\ \ z\in E,\ \text{ with } 0\le\eta<1. \tag{1.11}$$

Unless otherwise mentioned, we assume through this paper that $p\in\mathbb{N}_0, 0\le\rho<1$, $\alpha=\mu>0$.

2. Preliminary Results

In order to establish our main results, we need the following Lemma which is properly known as the Miller-Mocanu Lemma.

Lemma 2.1 [8]**.** Let $u=u_1+iu_2,\ v=v_1+iv_2$ and $\Psi(u,v)$ be a complex valued function satisfying the conditions:
(i) $\Psi(u,v)$ is continuous in a domain $D\subset\mathbb{C}^2$,
(ii) $(1,0)\in D$ and $\Re\Psi(1,0)>0$,
(iii) $\Re\Psi(iu_2,v_1)\le 0$, whenever $(iu_2,v_1)\in D$ and $v_1\le-\frac{1}{2}\left(1+u_2^2\right)$.

If $h(z)=1+c_1z+c_2z^2+\cdots$ is a function analytic in E such that $(h(z),zh'(z))\in D$

and $\Re\Psi\left(h(z), zh'(z)\right) > 0$ for $z \in E$, then $\Re h(z) > 0$ in E.

3. Main Results

Theorem 3.1. *Let $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda > 0$ and $f \in B^{\lambda,p}_{k,\mu}(\alpha_1, \beta_1, \alpha, \rho)$. Then* $\left(\frac{(I^p_\mu(\alpha_1,\beta_1)f(z))}{(I^p_\mu(\alpha_1,\beta_1)g(z))}\right)^\alpha \in P_k(\gamma)$,
where

$$\gamma = \frac{2\mu\alpha_1\rho + \lambda\delta}{2\mu\alpha_1 + \lambda\delta}, \tag{3.1}$$

and $g \in \mathcal{M}$ satisfies the condition (1.11)

$$\delta = \frac{\Re h_0(z)}{|h_0(z)|^2}, \quad h_0(z) = \left(\frac{(I^p_\mu(\alpha_1+1, \beta_1)g(z))}{(I^p_\mu(\alpha_1, \beta_1)g(z))}\right).$$

Proof. Set

$$\left(\frac{(I^p_\mu(\alpha_1, \beta_1)f(z))}{(I^p_\mu(\alpha_1, \beta_1)g(z))}\right)^\alpha = (1-\gamma)h(z) + \gamma, \tag{3.2}$$

$h(0) = 1$, and $h(z)$ is analytic in E and we can write

$$h(z) = (\frac{k}{4} + \frac{1}{2})h_1(z) - (\frac{k}{4} - \frac{1}{2})h_2(z). \tag{3.3}$$

Differentiating (3.2) with respect to z and using the identity (1.10), we have
(3.4)

$$\left\{(1-\lambda)\left(\frac{(I^p_\mu(\alpha_1, \beta_1)f(z))}{(I^p_\mu(\alpha_1, \beta_1)g(z))}\right)^\alpha + \lambda\left(\frac{(I^p_\mu(\alpha_1+1, \beta_1)f(z))}{(I^p_\mu(\alpha_1+1, \beta_1)g(z))}\right)\left(\frac{(I^p_\mu(\alpha_1, \beta_1)f(z))}{(I^p_\mu(\alpha_1, \beta_1)g(z))}\right)^{\alpha-1}\right\}$$

$$= (\frac{k}{4} + \frac{1}{2})\left\{(1-\gamma)h_1(z) + \gamma - \rho + \frac{\lambda(1-\gamma)zh_1'(z)}{\alpha\alpha_1 h_0(z)}\right\}$$

$$-(\frac{k}{4} - \frac{1}{2})\left\{(1-\gamma)h_2(z) + \gamma - \rho + \frac{\lambda(1-\gamma)zh_2'(z)}{\alpha\alpha_1 h_0(z)}\right\}.$$

Now we form the functional $\Psi(u, v)$ by choosing $u = h_i(z) = u_1 + iu_2$ and $v = zh_i'(z) = v_1 + iv_2$. Thus

$$\Psi(u, v) = \left\{(1-\gamma)u + \gamma - \rho + \frac{\lambda(1-\gamma)v}{\alpha\alpha_1 h_0(z)}\right\}.$$

The first two conditions of Lemma 2.1 are clearly satisfied. We verify the condition (iii) as follows:

$$\Psi(iu_2, v_1) = \gamma - \rho + \frac{\lambda(1-\gamma)v_1\Re h_0(z)}{\alpha\alpha_1\left|h_0(z)\right|^2} = \gamma - \rho + \frac{\lambda(1-\gamma)v_1\delta}{\alpha\alpha_1},$$

where $\delta = \frac{\Re h_0(z)}{|h_0(z)|^2}$.
Now, for $v_1 \leq -\frac{1}{2}(1+u_2^2)$, we have

$$\Re\Psi(iu_2, v_1) \leq \gamma - \rho - \frac{1}{2}\frac{\lambda(1-\gamma)(1+u_2^2)\delta}{\alpha\alpha_1}$$

$$= \frac{2\alpha\alpha_1(\gamma-\rho) - \lambda\delta(1-\gamma) - \lambda\delta(1-\gamma)u_2^2}{2\mu\alpha_1} = \frac{A+Bu_2^2}{2C}, \qquad C > 0,$$

$$A = 2\alpha\alpha_1(\gamma-\rho) - \lambda\delta(1-\gamma), B = -\lambda\delta(1-\gamma) \leq 0.$$

Now $\Re\Psi(iu_2, v_1) \leq 0$ if $A \leq 0$ and this gives us γ as defined by (3.1). We now applying Lemma 2.1 to conclude that $h_i \in P$ for $z \in E$ and thus $h \in P_k$which gives us the required result. □

We note that $\gamma = \rho$ when $\eta = 0$.

Theorem 3.2. *For $\lambda \geq 1$, let $f \in B_{k,\mu}^{\lambda,p}(\alpha_1, \beta_1, 1, \rho)$. Then*

$$\left(\frac{(I_\mu^p(\alpha_1+1,\beta_1)f(z))}{(I_\mu^p(\alpha_1+1,\beta_1)g(z))}\right) \in P_k(\rho), \text{ for } z \in E.$$

Proof. We can write, for $\lambda \geq 1$,

$$\lambda\left(\frac{(I_\mu^p(\alpha_1+1,\beta_1)f(z))}{(I_\mu^p(\alpha_1+1,\beta_1)g(z))}\right) = \left\{(1-\lambda)\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right) + \lambda\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right)\right\}$$

$$+ (\lambda-1)\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right).$$

This implies that

$$\left(\frac{(I_\mu^p(\alpha_1+1,\beta_1)f(z))}{(I_\mu^p(\alpha_1+1,\beta_1)g(z))}\right) = \frac{1}{\lambda}\left\{(1-\lambda)\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right) + \lambda\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right)\right\}$$

$$+ (1-\frac{1}{\lambda})\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right)$$

$$= \frac{1}{\lambda}H_1(z) + (1-\frac{1}{\lambda})H_2(z).$$

Since $H_1(z), H_2(z) \in P_k(\rho)$, by Theorem 3.1, Definition 3.1 and $P_k(\rho)$ is a convex set, see [10], we obtain the required result. □

Theorem 3.3. *Let $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda > 0$. If $f \in \mathcal{M}$ satisfies the following condition:*

$\left\{(1-\lambda)\left(z(I_\mu^p(\alpha_1,\beta_1)f(z))\right)^\alpha+\lambda\left(z(I_\mu^p(\alpha_1+1,\beta_1)f(z))\right)\left(z(I_\mu^p(\alpha_1,\beta_1)f(z))\right)^{\alpha-1}\right\}\in P_k(\rho),$ *for* $\alpha>0 (z\in E^*)$, *then*

$$\left(z(I_\mu^p(\alpha_1,\beta_1)f(z))\right)^\alpha\in P_k(\sigma),$$

where

$$\sigma=\rho+(1-\rho)(2\sigma_1-1) \textit{ with } \sigma_1=\int_0^1(1+t^{\Re\frac{\lambda}{\mu\alpha_1}})dt.$$

The value of σ is best possible and cannot be improved.

Proof. We set

$$\left(z(I_\mu^p(\alpha_1,\beta_1)f(z))\right)^\alpha=h(z)=(\frac{k}{4}+\frac{1}{2})h_1(z)-(\frac{k}{4}-\frac{1}{2})h_2(z),$$

where $h(0)=1$ and h is analytic in E. Then by a simple computation together with (1.10), we have

$$\left\{(1-\lambda)\left(z(I_\mu^p(\alpha_1,\beta_1)f(z))\right)^\alpha+\lambda\left(z(I_\mu^p(\alpha_1+1,\beta_1)f(z))\right)\left(z(I_\mu^p(\alpha_1,\beta_1)f(z))\right)^{\alpha-1}\right\}$$

$$=\left\{h(z)+\frac{\lambda zh'(z)}{\mu\alpha_1}\right\}\in P_k(\rho),\quad z\in E.$$

Using Lemma 2.2, we note that $h_i(z)\in P(\sigma)$,

$$\sigma=\rho+(1-\rho)(2\sigma_1-1),$$

$$\sigma_1=\int_0^1(1+t^{\Re\frac{\lambda}{\mu\alpha_1}})dt, \tag{3.5}$$

and consequently $h(z)\in P_k(\sigma)$ and this gives the required result. □

We note that σ_1 given by (3.5) can be expressed in terms of hypergeometric function as

$$\begin{aligned}\sigma_1 &= \int_0^1(1+t^{\Re\frac{\lambda}{\mu\alpha_1}}dt)\\ &= \frac{\mu\alpha_1}{\lambda_1}\int_0^1 u^{\frac{\mu\alpha_1}{\lambda_1}-1}(1+u)^{-1}du,\quad (\lambda_1=\Re\lambda>0)\\ &= {}_2F_1(1,\frac{\mu\alpha_1}{\lambda_1};1+\frac{\mu\alpha_1}{\lambda_1};-1)\\ &= {}_2F_1(1,1;1+\frac{\mu\alpha_1}{\lambda_1};\frac{1}{2}).\end{aligned}$$

□

Consider the operator defined by

$$F_c = \left(\frac{c}{z^c} \int_0^z t^c \, (\, f(t)) \, dt \right) \qquad (c > 0; z \in E^*). \tag{3.6}$$

It is clear that the function $F_c \in \mathcal{M}$ and

$$z((I_\mu^p(\alpha_1, \beta_1) \; F_c(f))'(z) = c(I_\mu^p(\alpha_1, \beta_1) \; f(z) - (c+1)(I_\mu^p(\alpha_1, \beta_1) F_c(f)(z). \tag{3.7}$$

Theorem 3.4. *Let $\lambda \in \mathbb{C} \backslash \{0\}$ with $\Re\lambda > 0$. If $f \in \mathcal{M}$ satisfies the following condition:*

$$\left\{(1-\lambda) \left(z(I_\mu^p(\alpha_1, \beta_1) F_c(f)(z)\right) + \lambda z \left((I_\mu^p(\alpha_1, \beta_1) f(z)\right)\right\} \in P_k(\rho), for (z \in E^*) \tag{3.8}$$

then the function defined by

$$\left(z(I_\mu^p(\alpha_1, \beta_1) \; F_c(f)(z)\right) \in P_k(\rho_1), \tag{3.9}$$

where

$$\rho_1 = \rho + (1-\rho)(2\sigma_2 - 1) \text{ with } \sigma_2 = \int_0^1 (1 + t^{\Re \frac{\lambda}{c}} dt).$$

The value of ρ_1 is best possible and cannot be improved.

Proof. Set

$$\left(z(I_\mu^p(\alpha_1, \beta_1) F_c(f)(z)\right) = h(z) = (\frac{k}{4} + \frac{1}{2}) h_1(z) - (\frac{k}{4} - \frac{1}{2}) h_2(z). \tag{3.10}$$

Then $h(z)$ is analytic in E with $h(0) = 1$.

Differentiating equation (3.10) with respect z and using (3.7) in the resulting equation, we have

$$\begin{aligned} &\left\{(1-\lambda) \left(z(I_\mu^p(\alpha_1, \beta_1) F_c(f)(z)\right) + \lambda z \left((I_\mu^p(\alpha_1, \beta_1) \; f(z)\right)\right\} \\ &= \left\{h(z) + \tfrac{\lambda}{c} z h'(z)\right\} \in P_k(\rho), \quad z \in E. \end{aligned}$$

Using Lemma 2.2, we note that $h_i(z) \in P(\rho_1)$,

$$\rho_1 = \rho + (1-\rho)(2\sigma_2 - 1),$$

$$\sigma_2 = \int_0^1 (1 + t^{\Re \frac{\lambda}{c}} dt), \tag{3.11}$$

and consequently $h(z) \in P_k(\rho_1)$ and this gives the required result. □

In term of hypergeometric function σ_2 can be written as

$$\sigma_2 = {}_2F_1(1, 1; \frac{c}{Re\lambda} + 1; \frac{1}{2})$$

Theorem 3.5. *For* $0 \leq \lambda_2 < \lambda_1$,

$$B_{k,\mu}^{\lambda_1,p}(\alpha_1, \beta_1, \alpha, \rho) \subset B_{k,\mu}^{\lambda_2,p}(\alpha_1, \beta_1, \alpha, \rho).$$

If $\lambda_2 = 0$, then the proof is immediate from Theorem 3.1. Let $\lambda_2 > 0$ and $f \in B_{k,\mu}^{\lambda_1,p}(\alpha_1, \beta_1, \alpha, \rho)$. Then there exist two functions $H_1, H_2 \in P_k(\rho)$ such that

$$\left\{(1-\lambda_1)\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right)^\alpha + \lambda_1\left(\frac{(I_\mu^p(\alpha_1+1,\beta_1)f(z))}{(I_\mu^p(\alpha_1+1,\beta_1)g(z))}\right)\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right)^{\alpha-1}\right\} = H_1(z),$$

and

$$\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right)^\alpha = H_2(z).$$

Then

(3.12)

$$\left\{(1-\lambda_2)\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right)^\alpha + \lambda_2\left(\frac{(I_\mu^p(\alpha_1+1,\beta_1)f(z))}{(I_\mu^p(\alpha_1+1,\beta_1)g(z))}\right)\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right)^{\alpha-1}\right\}$$

$$= \frac{\lambda_2}{\lambda_1}H_1(z) + (1 - \frac{\lambda_2}{\lambda_1})H_2(z),$$

and since $P_k(\rho)$ is a convex set, see [10], it follows that the right hand side of (3.12) belongs to $P_k(\rho)$ and this completes the proof. □

We next take the converse case of Theorem 3.1 as follows:

Theorem 3.6. *Let* $\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right)^\alpha \in P_k(\rho)$ *with* $\left(\frac{(I_\mu^p(\alpha_1+1,\beta_1)f\ g(z)}{(I_\mu^p(\alpha_1,\beta_1)g(z)}\right) \in P(\eta)$, *for* $z \in E$. *Then* $f \in B_{k,\mu}^{\lambda,p}(\alpha_1, \beta_1, \alpha, \rho))$ *for* $|z| < r$, *where* r *is given by*

$$r = \frac{\mu\alpha_1}{\{(1-\eta)\mu\alpha_1 + |\lambda|\} + \sqrt{\eta\mu(\alpha_1)^2 + |\lambda|^2 + 2|\lambda|(1-\eta)\mu\alpha_1}}. \tag{3.13}$$

Proof. Let

$$\left(\frac{(I_\mu^p(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)g(z))}\right)^\alpha = H,$$

$$\left(\frac{(I_\mu^p(\alpha_1+1,\beta_1)f\ g(z)}{(I_\mu^p(\alpha_1,\beta_1)g(z)}\right) = H_0,$$

then $H \in P_k(\rho), H_0 \in P(\eta)$.

Proceeding as in Theorem 3.1, for $\mu > 0, k \geq 2, \lambda \in \mathbb{C}\backslash\{0\}, 0 \leq \rho, \eta < 1$, and

$$H = (1-\rho)h + \rho,$$
$$H_0 = (1-\eta)h_0 + \eta, \quad \text{with } h \in P_k, h_0 \in P,$$

we have

$$\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))}{(I_{\mu}^{p}(\alpha_1,\beta_1)g(z))}\right)^{\alpha}+\lambda\left(\frac{(I_{\mu}^{p}(\alpha_1+1,\beta_1)f(z))}{(I_{\mu}^{p}(\alpha_1+1,\beta_1)g(z))}\right)\left(\frac{(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))}{(I_{\mu}^{p}(\alpha_1,\beta_1)g(z))}\right)^{\alpha-1}-\rho\right\}$$

$$=\left\{h(z)+\frac{\lambda}{\mu\alpha_1}\frac{zh'(z)}{(1-\eta)h_0(z)+\eta}\right\}$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z)+\frac{\lambda}{\mu\alpha_1}\frac{zh_1'(z)}{\{(1-\eta)h_0(z)+\eta\}}\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z)+\frac{\lambda}{\mu\alpha_1}\frac{zh_2'(z)}{\{(1-\eta)h_0(z)+\eta\}}\right].$$

Using well known estimates, see [5], for $h_i \in P$,

$$\begin{aligned} |zh_i'(z)| &\leq \frac{2r\Re h_i(z)}{1-r^2},\\ \frac{1-r}{1+r} \leq |h_i(z)| &\leq \frac{1+r}{1-r}, \end{aligned}$$

we have

(3.15)

$$\Re\left[h_i(z)+\frac{\lambda}{\mu\alpha_1}\frac{zh_i'(z)}{\{(1-\eta)h_0(z)+\eta\}}\right]\geq \Re h_i(z)\left[1-\frac{2|\lambda|r}{\mu\alpha_1}\frac{1}{1-r^2}\left(\frac{1+r}{(1-(1-2\eta)r)}\right)\right]$$

$$\geq \Re h_i(z)\left[1-\frac{2|\lambda|r}{\mu\alpha_1}\frac{1}{1-r}\left(\frac{1+r}{(1-(1-2\eta)r)}\right)\right]$$

$$\geq \Re h_i(z)\left[\frac{\mu\alpha_1[(1-r-(1-2\eta)r+(1-2\eta)r^2]-2|\lambda|r}{\mu\alpha_1(1-r)\{1-(1-(1-2\eta)r\}}\right]$$

$$\geq \Re h_i(z)\left[\frac{\mu\alpha_1(1-2\eta)r^2-2[(1-\eta)\mu\alpha_1+|\lambda|]r+\mu\alpha_1}{\mu}\alpha_1(1-r)1-(1-(1-2\eta)r)\right].$$

Right hand side of (3.14) is positive for $|z|<r$, where r is given by (3.13). □

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