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Exposed Symmetric Bilinear Forms of $\mathcal{L}_s(^2d_*(1,w)^2)$

Sung Guen Kim
Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea
e-mail : sgk317@knu.ac.kr

Abstract. We classify the exposed symmetric bilinear forms of the unit ball of $\mathcal{L}_s(^2d_*(1,w)^2)$.

1. Introduction

We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. $x \in B_E$ is called an *exposed point* of B_E if there is a $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $expB_E$ and $extB_E$ the sets of exposed and extreme of B_E, respectively. For $n \geq 2$, we denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\leq k\leq n} |T(x_1, \cdots, x_n)|$. $\mathcal{L}_s(^nE)$ denotes the subspace of all continuous symmetric n-linear forms on E. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists $T \in \mathcal{L}_s(^nE)$ such that $P(x) = T(x, \cdots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [7]. We will denote by $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)$ and $P(x, y) = ax^2 + by^2 + cxy$ a symmetric bilinear form and a 2-homogeneous polynomial on a real Banach space of dimension 2 respectively. For $1 \leq p \leq \infty$, we let $l_p^2 = \mathbb{R}^2$ with the l_p-norm. Note that in ([6], Theorem 1, remark after Theorem 1, and Theorem 2) the following results are proved:

(i) $expB_{\mathcal{P}(^2l_1^2)} = extB_{\mathcal{P}(^2l_1^2)} \setminus \{\pm(x^2 - y^2 \pm 2xy)\}$;

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(ii) $expB_{\mathcal{P}(^2l^2_\infty)} = extB_{\mathcal{P}(^2l^2_\infty)} \backslash \{\pm(\frac{1}{2}x^2 - \frac{1}{2}y^2 \pm xy)\}$.

The author [11] characterized $expB_{\mathcal{P}(^2l^2_p)}$ as follows:

(i) If $1 < p < 2$, then $expB_{\mathcal{P}(^2l^2_p)} = extB_{\mathcal{P}(^2l^2_p)}$;

(ii) If $2 < p < \infty$, then $expB_{\mathcal{P}(^2l^2_p)} = extB_{\mathcal{P}(^2l^2_p)} \backslash \{\pm x^2, \pm y^2)\}$.

We refer to ([1–6, 8–21] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces.

We denote the 2-dimensional real predual of the Lorentz sequence space with a positive weight $0 < w < 1$ by

$$d_*(1,w)^2 := \{(x,y) \in \mathbb{R}^2 : \|(x,y)\|_{d_*} := \max\{|x|,\ |y|,\ \frac{|x|+|y|}{1+w}\ \}.$$

Very recently, the author [14] characterize the extreme points of the unit ball of $\mathcal{L}_s(^2d_*(1,w)^2)$. Using their results, in this note, we show that $expB_{\mathcal{L}_s(^2d_*(1,w)^2)} = extB_{\mathcal{L}_s(^2d_*(1,w)^2)}$ for every $0 < w < 1$.

2. Main Results

Theorem 2.1. Let $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s(^2d_*(1,w)^2)$. Then the following are equivalent:

(a) $ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in expB_{\mathcal{L}_s(^2d_*(1,w)^2)}$;

(b) $-ax_1x_2 - by_1y_2 - +c(x_1y_2 + x_2y_1) \in expB_{\mathcal{L}_s(^2d_*(1,w)^2)}$;

(c) $ax_1x_2 + by_1y_2 - c(x_1y_2 + x_2y_1) \in expB_{\mathcal{L}_s(^2d_*(1,w)^2)}$;

(d) $bx_1x_2 + ay_1y_2 + c(x_1y_2 + x_2y_1) \in expB_{\mathcal{L}_s(^2d_*(1,w)^2)}$.

Proof. Let $S((x_1,y_1),(x_2,y_2)) := T((u_1,v_1),(u_2,v_2))$ for some $((u_1,v_1),(u_2,v_2)) = ((x_1,y_1),(-x_2,-y_2))$ or $((x_1,-y_1),(x_2,-y_2))$ or $((y_1,x_1),(y_2,x_2))$. Then $S \in \mathcal{L}_s(^2d_*(1,w)^2)$ and T is exposed if and only if S is exposed. □

Theorem 2.2. [14, Theorem 2.3] Let $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s(^2d_*(1,w)^2)$. Then

(a) Let $w < \sqrt{2} - 1$. Then T is extreme if and only if

$$\begin{aligned} T \quad \in \quad & \{\pm x_1x_2, \pm y_1y_2, \pm\frac{1}{1+w^2}(x_1x_2 + y_1y_2), \\ & \pm\frac{1}{(1+w)^2}[x_1x_2 + y_1y_2 \pm (x_1y_2 + x_2y_1)], \\ & \pm\frac{1}{1+2w-w^2}[x_1x_2 - y_1y_2 \pm (x_1y_2 + x_2y_1)], \end{aligned}$$

$$\pm\frac{1}{1+w^2}[x_1x_2 - y_1y_2 \pm w(x_1y_2 + x_2y_1)],$$
$$\pm\frac{1}{1+w^2}[wx_1x_2 - wy_1y_2 \pm (x_1y_2 + x_2y_1)],$$
$$\pm\frac{1}{(1+w)^2(1-w)}[(1-w-w^2)x_1x_2 - wy_1y_2 \pm (x_1y_2 + x_2y_1)],$$
$$\pm\frac{1}{(1+w)^2(1-w)}[wx_1x_2 - (1-w-w^2)y_1y_2 \pm (x_1y_2 + x_2y_1)]\}.$$

(b) Let $w = \sqrt{2} - 1$. Then T is extreme if and only if

$$\begin{aligned} T \in \; & \{\pm x_1x_2, \pm y_1y_2, \pm\frac{2+\sqrt{2}}{4}(x_1x_2 + y_1y_2), \pm\frac{1}{2}[x_1x_2 + y_1y_2 \pm (x_1y_2 + x_2y_1)], \\ & \pm\frac{\sqrt{2}}{4}[x_1x_2 + y_1y_2 \pm (\sqrt{2}+1)(x_1y_2 + x_2y_1)], \\ & \pm\frac{\sqrt{2}}{4}[(\sqrt{2}+1)(x_1y_2 - x_2y_1) \pm (x_1y_2 + x_2y_1)]\}. \end{aligned}$$

(c) Let $w > \sqrt{2} - 1$. Then T is extreme if and only if

$$\begin{aligned} T \in \; & \{\pm x_1x_2, \pm y_1y_2, \pm\frac{1}{1+w^2}(x_1x_2 + y_1y_2), \\ & \pm\frac{1}{(1+w)^2}[x_1x_2 + y_1y_2 \pm (x_1y_2 + x_2y_1)], \\ & \pm\frac{1}{1+2w-w^2}[x_1x_2 - y_1y_2 \pm (x_1y_2 + x_2y_1)], \\ & \pm\frac{1}{1+w^2}[x_1x_2 - y_1y_2 \pm \frac{1-w}{1+w}(x_1y_2 + x_2y_1)], \\ & \pm\frac{1}{1+w^2}[\frac{1-w}{1+w}(x_1x_2 - y_1y_2) \pm (x_1y_2 + x_2y_1)], \\ & \pm\frac{1}{2+2w}[(2+w)x_1x_2 - \frac{1}{w}y_1y_2 \pm (x_1y_2 + x_2y_1)], \\ & \pm\frac{1}{2+2w}[\frac{1}{w}x_1x_2 - (2+w)y_1y_2 \pm (x_1y_2 + x_2y_1)]\}. \end{aligned}$$

Theorem 2.3. Let E be a real Banach space such that $extB_E$ is finite. Suppose that $x \in extB_E$ satisfies that there exists an $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in extB_E \backslash \{x\}$. Then $x \in expB_E$.

Proof. Let $extB_E = \{x_1, \ldots, x_m\}$. By the Krein-Milman theorem, B_E is the closed convex hull of $extB_E$. Let $z \in B_E$ such that $f(z) = 1$. We will show that $z = x$. Let $x = x_{i_0}$ for some $1 \leq i_0 \leq m$. By the Krein-Milman theorem, $z = \lim_{j\to\infty} \lambda_1^{(j)} x_1 + \cdots + \lambda_m^{(j)} x_m$ for some $\sum_{1\leq k\leq m} |\lambda_k^{(j)}| \leq 1$ for every $j \in \mathbb{N}$. Since $(\lambda_1^{(j)}), \ldots, (\lambda_m^{(j)})$ are sequences in $[-1, 1]$, there exist subsequences $(\beta_1^{(j)}), \ldots, (\beta_m^{(j)})$

of $(\lambda_1^{(j)}),\ldots,(\lambda_m^{(j)})$, respectively such that $\lim_{j\to\infty}\beta_k^{(j)} = \beta_k \in [-1,1]$ for each $k=1,\ldots,m$. Thus $z=\beta_1 x_1+\cdots+\beta_m x_m$ and $\sum_{1\le k\le m}|\beta_k|\le 1$.

Claim: $\beta_k = 0$ for every $1\le k\ne i_0\le m$

Otherwise. Let $\beta_{k_0}\ne 0$ for some $1\le k_0\ne i_0\le m$ and $\delta := \max\{|f(x_k)| : 1\le k\ne i_0\le m\} < 1$. Then

$$\begin{aligned}
1 &= f(z)=\beta_{i_0}f(x)+\sum_{1\le k\ne i_0\le m}\beta_k f(x_k)\\
&\le |\beta_{i_0}|\,|f(x)|+|\beta_{k_0}|\,|f(x_{k_0})|+\sum_{1\le k\ne i_0,k\ne k_0\le m}|\beta_k|\,|f(x_k)|\\
&\le |\beta_{i_0}|\,|f(x)|+|\beta_{k_0}|\delta+\sum_{1\le k\ne i_0,k\ne k_0\le m}|\beta_k|\,|f(x_k)|\\
&< |\beta_{i_0}|\,|f(x)|+|\beta_{k_0}|+\sum_{1\le k\ne i_0,k\ne k_0\le m}|\beta_k|\,|f(x_k)|\\
&\le |\beta_{i_0}|+|\beta_{k_0}|+\sum_{1\le k\ne i_0,k\ne k_0\le m}|\beta_k|\\
&\le 1,
\end{aligned}$$

which is impossible. Therefore, $1=f(z)=\beta_{i_0}f(x)=\beta_{i_0}$, so $z=\beta_1x_1+\cdots+\beta_m x_m = x$. □

Theorem 2.4. Let $f\in\mathcal{L}_s({}^2d_*(1,w)^2)^*$ and $\alpha=f(x_1x_2),\beta=f(y_1y_2),\gamma=f(x_1y_2+x_2y_1)$.

(a) Let $w<\sqrt{2}-1$. Then

$$\begin{aligned}
\|f\| &= \max\Big\{|\alpha|,|\beta|,\frac{1}{1+w^2}|\alpha+\beta|,\frac{1}{(1+w)^2}(|\alpha+\beta|+|\gamma|),\\
&\quad \frac{1}{1+2w-w^2}(|\alpha-\beta|+|\gamma|),\frac{1}{1+w^2}(|\alpha-\beta|+w|\gamma|),\\
&\quad \frac{1}{1+w^2}(w|\alpha-\beta|+|\gamma|),\frac{1}{(1+w)^2(1-w)}(|(1-w-w^2)\alpha-w\beta|+|\gamma|),\\
&\quad \frac{1}{(1+w)^2(1-w)}(|w\alpha-(1-w-w^2)\beta|+|\gamma|)\Big\}.
\end{aligned}$$

(b) Let $w=\sqrt{2}-1$. Then

$$\begin{aligned}
\|f\| &= \max\Big\{|\alpha|,|\beta|,\frac{2+\sqrt{2}}{4}|\alpha+\beta|,\frac{1}{2}(|\alpha+\beta|+|\gamma|),\frac{\sqrt{2}}{4}(|\alpha-\beta|+(\sqrt{2}+1)|\gamma|),\\
&\quad \frac{\sqrt{2}}{4}((\sqrt{2}+1)|\alpha-\beta|+|\gamma|)\Big\}.
\end{aligned}$$

(c) Let $\sqrt{2}-1 < w$. Then

$$\begin{aligned}\|f\| &= \max\{|\alpha|, |\beta|, \frac{1}{1+w^2}|\alpha+\beta|, \frac{1}{(1+w)^2}(|\alpha+\beta|+|\gamma|),\\ &\quad \frac{1}{1+2w-w^2}(|\alpha-\beta|+|\gamma|), \frac{1}{1+w^2}(|\alpha-\beta|+\frac{1-w}{1+w}|\gamma|),\\ &\quad \frac{1}{1+w^2}(\frac{1-w}{1+w}|\alpha-\beta|+|\gamma|), \frac{1}{2+2w}(|(2+w)\alpha-\frac{1}{w}\beta|+|\gamma|),\\ &\quad \frac{1}{2+2w}(|\frac{1}{w}\alpha-(2+w)\beta|+|\gamma|)\}.\end{aligned}$$

Proof. It follows from Theorem 2.2 since

$$\|f\| = \sup\{|f(T)| : T \in extB_{\mathcal{L}_s({}^2d_*(1,w)^2)}\}.$$ □

Using Theorems 2.1–4, we classify the exposed symmetric bilinear forms of the unit ball of $\mathcal{L}_s({}^2d_*(1,w)^2)$.

Theorem 2.5. $expB_{\mathcal{L}_s({}^2d_*(1,w)^2)} = extB_{\mathcal{L}_s({}^2d_*(1,w)^2)}$.

Proof. Case 1: $w < \sqrt{2}-1$

Claim: x_1x_2 is exposed.

Let $\alpha = 1, \beta = 0 = \gamma$. By Theorem 2.4(a), $f(x_1x_2) = 1 = \|f\|$ and $|f(T)| < 1$ for every $T \in extB_{\mathcal{L}_s({}^2d_*(1,w)^2)}$ with $T \neq x_1x_2$. By Theorem 2.3, it is exposed. Similarly, $-x_1x_2, \pm y_1y_2$ are exposed.

Claim: $\frac{1}{1+w^2}(x_1x_2+y_1y_2)$ is exposed.

Let $\alpha = \frac{1+w^2}{2} = \beta, \gamma = 0$. By Theorem 2.4(a), $f(\frac{1}{1+w^2}(x_1x_2+y_1y_2)) = 1 = \|f\|$ and $|f(T)| < 1$ for every $T \in extB_{\mathcal{L}_s({}^2d_*(1,w)^2)}$ with $T \neq \frac{1}{1+w^2}(x_1x_2+y_1y_2)$. By Theorem 2.3, it is exposed. Similarly, $-\frac{1}{1+w^2}(x_1x_2+y_1y_2)$ is exposed.

Claim: $\frac{1}{(1+w)^2}(x_1x_2+y_1y_2+x_1y_2+x_2y_1)$ is exposed.

Let $\alpha = \frac{1+w^2}{2}, \beta = \frac{1+w^2}{2}-\epsilon, \gamma = 2w+\epsilon$ for a sufficiently small $\epsilon > 0$. By Theorem 2.4(a), $f(\frac{1}{(1+w)^2}(x_1x_2+y_1y_2+x_1y_2+x_2y_1)) = 1 = \|f\|$ and $|f(T)| < 1$ for every $T \in extB_{\mathcal{L}_s({}^2d_*(1,w)^2)}$ with $T \neq \frac{1}{(1+w)^2}(x_1x_2+y_1y_2+x_1y_2+x_2y_1)$. By Theorem 2.3, it is exposed. Similarly, $-\frac{1}{(1+w)^2}(x_1x_2+y_1y_2+x_1y_2+x_2y_1)$ is exposed.

Claim: $\frac{1}{1+w^2}(x_1x_2-y_1y_2+w(x_1y_2+x_2y_1))$ is exposed.

Let $\alpha = \frac{1}{2} = -\beta, \gamma = w$. By Theorem 2.4(a), $f(\frac{1}{1+w^2}(x_1x_2-y_1y_2+w(x_1y_2+x_2y_1))) = 1 = \|f\|$ and $|f(T)| < 1$ for every $T \in extB_{\mathcal{L}_s({}^2d_*(1,w)^2)}$ with $T \neq \frac{1}{1+w^2}(x_1x_2-y_1y_2+w(x_1y_2+x_2y_1))$. By Theorem 2.3, it is exposed. By Theorem 2.2, $\pm\frac{1}{1+w^2}(wx_1x_2-wy_1y_2\pm(x_1y_2+x_2y_1))$ are exposed.

Claim: $\frac{1}{1+2w-w^2}(x_1x_2-y_1y_2+(x_1y_2+x_2y_1))$ is exposed.

Let $2w < \gamma < 1-w^2$ and $\alpha = \frac{1+2w-w^2-\gamma}{2}, \beta = -\alpha$. By Theorem 2.4(a), $f(\frac{1}{1+2w-w^2}(x_1x_2-y_1y_2+(x_1y_2+x_2y_1))) = 1 = \|f\|$ and $|f(T)| < 1$ for every

$T \in extB_{\mathcal{L}_s(^2d_*(1,w)^2)}$ with $T \neq \frac{1}{1+2w-w^2}(x_1x_2 - y_1y_2 + (x_1y_2 + x_2y_1))$. By Theorem 2.3, it is exposed.

Claim: $\frac{1}{(1+w)^2(1-w)}((1-w-w^2)x_1x_2 - wy_1y_2 + (x_1y_2 + x_2y_1))$ is exposed.

Let $\alpha = w + \epsilon, \beta = 0, \gamma = 1 + \epsilon(-1 + w + w^2)$ for a sufficiently small $\epsilon > 0$. By Theorem 2.4(a), $f(\frac{1}{(1+w)^2(1-w)}((1-w-w^2)x_1x_2 - wy_1y_2 + (x_1y_2 + x_2y_1))) = 1 = \|f\|$ and $|f(T)| < 1$ for every $T \in extB_{\mathcal{L}_s(^2d_*(1,w)^2)}$ with $T \neq \frac{1}{(1+w)^2(1-w)}((1-w-w^2)x_1x_2 - wy_1y_2 + (x_1y_2 + x_2y_1))$. By Theorem 2.3, it is exposed. By Theorem 2.1, $\pm\frac{1}{(1+w)^2(1-w)}(wx_1x_2 - (1-w-w^2)y_1y_2 \pm (x_1y_2 + x_2y_1))$ are exposed.

Case 2: $w = \sqrt{2} - 1$

By the similar argument as Case 1, $\pm x_1x_2, \pm y_1y_2, \pm\frac{2+\sqrt{2}}{4}(x_1x_2 + y_1y_2), \pm\frac{1}{2}[x_1x_2 + y_1y_2 \pm (x_1y_2 + x_2y_1)]$ are exposed. It is enough to show that $\frac{\sqrt{2}}{4}[x_1x_2 - y_1y_2 + (\sqrt{2}+1)(x_1y_2 + x_2y_1)]$ is exposed. Let $\alpha = 0 = \beta, \gamma = 2(2-\sqrt{2})$. By Theorem 2.4(b), $f(\frac{\sqrt{2}}{4}[x_1x_2 - y_1y_2 + (\sqrt{2}+1)(x_1y_2 + x_2y_1)]) = 1 = \|f\|$ and $|f(T)| < 1$ for every $T \in extB_{\mathcal{L}_s(^2d_*(1,w)^2)}$ with $T \neq \frac{\sqrt{2}}{4}[x_1x_2 - y_1y_2 + (\sqrt{2}+1)(x_1y_2 + x_2y_1)]$. By Theorem 2.3, it is exposed.

Case 3: $\sqrt{2} - 1 < w$

By the similar argument as Case 1, $\pm x_1x_2, \pm y_1y_2, \pm\frac{1}{1+w^2}(x_1x_2 + y_1y_2), \pm\frac{1}{(1+w)^2}[x_1x_2 + y_1y_2 \pm (x_1y_2 + x_2y_1)], \pm\frac{1}{1+2w-w^2}[x_1x_2 - y_1y_2 \pm (x_1y_2 + x_2y_1)]$ are exposed.

Claim: $\frac{1}{1+w^2}[x_1x_2 - y_1y_2 + \frac{1-w}{1+w}(x_1y_2 + x_2y_1)]$ is exposed.

Let $\alpha = \frac{1+w^2}{2} = -\beta, \gamma = 0$. By Theorem 2.4(c), $f(\frac{1}{1+w^2}[x_1x_2 - y_1y_2 + \frac{1-w}{1+w}(x_1y_2 + x_2y_1)]) = 1 = \|f\|$ and $|f(T)| < 1$ for every $T \in extB_{\mathcal{L}_s(^2d_*(1,w)^2)}$ with $T \neq \frac{1}{1+w^2}[x_1x_2 - y_1y_2 + \frac{1-w}{1+w}(x_1y_2 + x_2y_1)]$. By Theorem 2.3, it is exposed. By Theorem 2.1,

$\pm\frac{1}{1+w^2}[\frac{1-w}{1+w}(x_1x_2 - y_1y_2) \pm (x_1y_2 + x_2y_1)]$ are exposed.

Claim: $\frac{1}{2+2w}[(2+w)x_1x_2 - \frac{1}{w}y_1y_2 + (x_1y_2 + x_2y_1)]$ is exposed.

Let $\alpha = 1 - \epsilon, \beta = -w^2, \gamma = \epsilon(2+w)$ for a sufficiently small $\epsilon > 0$. By Theorem 2.4(c), $f(\frac{1}{2+2w}[(2+w)x_1x_2 - \frac{1}{w}y_1y_2 + (x_1y_2 + x_2y_1)]) = 1 = \|f\|$ and $|f(T)| < 1$ for every $T \in extB_{\mathcal{L}_s(^2d_*(1,w)^2)}$ with $T \neq \frac{1}{2+2w}[(2+w)x_1x_2 - \frac{1}{w}y_1y_2 + (x_1y_2 + x_2y_1)]$. By Theorem 2.3, it is exposed. By Theorem 2.1, $\pm\frac{1}{2+2w}[\frac{1}{w}x_1x_2 - (2+w)y_1y_2 \pm (x_1y_2 + x_2y_1)]$ are exposed. □

References

[1] R. M. Aron, Y. S. Choi, S. G. Kim and M. Maestre, *Local properties of polynomials on a Banach space*, Illinois J. Math., **45(1)**(2001), 25–39.

[2] Y. S. Choi, H. Ki and S. G. Kim, *Extreme polynomials and multilinear forms on l_1*, J. Math. Anal. Appl., **228(2)**(1998), 467–482.

[3] Y. S. Choi and S. G. Kim, *The unit ball of* $\mathcal{P}(^2l_2^2)$, Arch. Math. (Basel), **71(6)**(1998), 472–480.

[4] Y. S. Choi and S. G. Kim, *Extreme polynomials on* c_0, Indian J. Pure Appl. Math., **29(10)**(1998), 983–989.

[5] Y. S. Choi and S. G. Kim, *Smooth points of the unit ball of the space* $\mathcal{P}(^2l_1)$, Results Math., **36(1-2)**(1999), 26–33.

[6] Y. S. Choi and S. G. Kim, *Exposed points of the unit balls of the spaces* $\mathcal{P}(^2l_p^2)$ $(p = 1, 2, \infty)$, Indian J. Pure Appl. Math., **35(1)**(2004), 37–41.

[7] S. Dineen, Complex Analysis on Infinite Dimensional Spaces, Springer-Verlag, London (1999).

[8] S. Dineen, *Extreme integral polynomials on a complex Banach space*, Math. Scand., **92(1)**(2003), 129–140.

[9] B. C. Grecu, *Geometry of 2-homogeneous polynomials on* l_p *spaces*, $1 < p < \infty$, J. Math. Anal. Appl., **273(2)**(2002), 262–282 .

[10] B. C. Grecu, G. A. Munoz-Fernandez and J.B. Seoane-Sepulveda, *Unconditional constants and polynomial inequalities*, J. Approx. Theory, **161(2)**(2009), 706–722.

[11] S. G. Kim, *Exposed 2-homogeneous polynomials on* $\mathcal{P}(^2l_p^2)$ $(1 \leq p \leq \infty)$, Math. Proc. Royal Irish Acad., **107(2)**(2007), 123–129.

[12] S. G. Kim, The unit ball of $\mathcal{L}_s(^2l_\infty^2)$, Extracta Math., **24(1)**(2009), 17–29.

[13] S. G. Kim, *The unit ball of* $\mathcal{P}(^2d_*(1,w)^2)$, Math. Proc. Royal Irish Acad., **111**(2011), 77–92.

[14] S. G. Kim, *The unit ball of* $\mathcal{L}_s(^2d_*(1,w)^2)$, Kyungpook Math. J., **53(2)**(2013), 295–306.

[15] S. G. Kim, *Smooth polynomials of* $\mathcal{P}(^2d_*(1,w)^2)$, Math. Proc. Royal Irish Acad., **113(1)**(2013), 45–58.

[16] S. G. Kim, *Extreme bilinear forms of* $\mathcal{L}(^2d_*(1,w)^2)$, Kyungpook Math. J., **53(4)**(2013), 625–638.

[17] S. G. Kim and S. H. Lee, *Exposed 2-homogeneous polynomials on Hilbert spaces*, Proc. Amer. Math. Soc., **131(2)**(2003), 449–453.

[18] J. Lee and K. S. Rim, *Properties of symmetric matrices*, J. Math. Anal. Appl., **305(1)**(2005), 219–226.

[19] G. A. Munoz-Fernandez, S. Revesz and J. B. Seoane-Sepulveda, *Geometry of homogeneous polynomials on non symmetric convex bodies*, Math. Scand., **105(1)**(2009), 147–160.

[20] G. A. Munoz-Fernandez and J. B. Seoane-Sepulveda, *Geometry of Banach spaces of trinomials*, J. Math. Anal. Appl., **340(2)**(2008), 1069–1087.

[21] R. A. Ryan and B. Turett, *Geometry of spaces of polynomials*, J. Math. Anal. Appl., **221(2)**(1998), 698–711.