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Closure, Interior and Compactness in Ordinary Smooth Topological Spaces

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Abstract

It presents the concepts of ordinary smooth interior and ordinary smooth closure of an ordinary subset and their structural properties. It also introduces the notion of ordinary smooth (open) preserving mapping and addresses some their properties. In addition, it develops the notions of ordinary smooth compactness, ordinary smooth almost compactness, and ordinary near compactness and discusses them in the general framework of ordinary smooth topological spaces.

Keywords: Ordinary smooth topological space, Ordinary smooth closure (resp. interior), Ordinary smooth continuity, Ordinary smooth preserving, Ordinary smooth (resp. almost and near) compactness

1. Introduction

By considering the degree of openness of fuzzy sets, Badard [1] introduced the concept of a smooth topological space as a generalization of a classical topology as well as a Chang's fuzzy topology [2]. Hazra et al. [3], Chattopadhyay et al. [4], Demirci [5], and Ramadan [6] have investigated the smooth topological spaces in the various aspects. El Gayyar et al. [7] showed how the concepts of closure, interior, subspace, almost and near compactness can be smoothed while obtaining more general structures with even nice properties. Ying [8, 9] presented the notion of a fuzzifying topology (called an ordinary smooth topology (OST) by Lim et al. [10]) with the consideration of the degree of openness of ordinary subsets, and discussed some of its properties. Cheong et al. [11] constructed the collection OST(X) of all ordinary smooth topologies on a set X and investigated it in the perspective of a lattice.

Here we first introduce the concepts of ordinary smooth interior and ordinary smooth closure of an ordinary subset and we investigate some of their structural properties. We also present the notions of ordinary smooth (open) preserving mapping and some their properties. In addition, we develop the notions of ordinary smooth compactness, ordinary smooth almost compactness, and ordinary near compactness and examine them in the general framework of ordinary smooth topological spaces.

2. Preliminaries

Let $2 = \{0, 1\}$ and let 2^X denote the set of all ordinary subsets of a set X.

Definition 2.1 [10]. Let X be a nonempty set. Then a mapping $\tau : 2^X \to I$ is called an

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ordinary smooth topology (in short, *ost*) on X or a *gradation of openness of ordinary subsets* of X if τ

satisfies the following axioms:

(OST_1) $\tau(\emptyset) = \tau(X) = 1$.

(OST_2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$, $\forall A, B \in 2^X$.

(OST_3) $\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha)$, $\forall\{A_\alpha\} \subset 2^X$.

The pair (X, τ) is called an *ordinary smooth topological space* (in short, *osts*). We will denote the set of all ost's on X as OST(X).

Remark 2.2. Ying [8] called the mapping $\tau : 2^X \to I$ [resp. $\tau : I^X \to 2$ and $\tau : I^X \to I$] satisfying the axioms in Definition 2.1 as a *fuzzyfying topology* [resp. *fuzzy topology* and *bifuzzy topology*] on X.

Remark 2.3. If $I = 2$, then Definition 2.1 coincides with the known definition of classical topology.

Definition 2.4 [10]. Let X be a nonempty set. Then a mapping $\mathcal{C} : 2^X \to I$ is called an *ordinary smooth cotopology* (in short, *osct*) on X or a *gradation of closedness of ordinary subsets* of X if $\mathcal{C}$ satisfies the following axioms :

(OSCT_1) $\mathcal{C}(\emptyset) = \mathcal{C}(X) = 1$.

(OSCT_2) $\mathcal{C}(A \cup B) \geq \mathcal{C}(A) \wedge \mathcal{C}(B)$, $\forall A, B \in 2^X$.

(OSCT_3) $\mathcal{C}(\bigcap_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mathcal{C}(A_\alpha)$, $\forall\{A_\alpha\} \subset 2^X$.

The pair $(X, \mathcal{C})$ is called an *ordinary smooth cotopological space* (in short, *oscts*). We will denote the set of all osct's on X as OSCT(X).

Remark 2.5. If $I = 2$, then Definition 2.4 also coincides with the known definition of classical topology.

Result 2.A [10, Proposition 2.7]. Let X be a nonempty set. We define two mappings f : OST(X) $\to$ OSCT(X) and g : OSCT(X) $\to$ OST(X) as follows, respectively :

$$[f(\tau)](A) = \tau(A^c), \forall\tau \in \mathrm{OST}(X), \forall A \in 2^X$$

and

$$[g(\mathcal{C})](A) = \mathcal{C}(A^c), \forall\mathcal{C} \in \mathrm{OSCT}(X), \forall A \in 2^X.$$

Then f and g are well-defined. Furthermore $g \circ f = id_{\mathrm{OST(X)}}$ and $f \circ g = id_{\mathrm{OSCT(X)}}$.

Remark 2.6. Let $f(\tau) = \mathcal{C}_\tau$ and $g(\mathcal{C}) = \tau_{\mathcal{C}}$. Then, by Result 2.A, we can easily see that $\tau_{\mathcal{C}_\tau} = \tau$ and $\mathcal{C}_{\tau_{\mathcal{C}}} = \mathcal{C}$.

Definition 2.7 [10]. Let (X, τ) be an osts and let $r \in I$. Then we define two ordinary subsets of X as follows :

$$[\tau]_r = \{A \in 2^X : \tau(A) \geq r\}$$

and

$$[\tau]^*_r = \{A \in 2^X : \tau(A) > r\}.$$

We call these the $r-$*level* set and the *strong r-level* set of τ, respectively.

It is clear that $[\tau]_0 = 2^X$, the classical discrete topology on X and $[\tau]^*_1 = \emptyset$. Also it can be easily seen that $[\tau]^*_r \subset [\tau]_r$ for each $r \in I$.

Result 2.B [10, Proposition 2.10]. Let (X, τ) be an osts and let $T(X)$ be the set of all classical topologies on X. Then :

(a) $[\tau]_r \in \mathrm{T}(X)$, $\forall r \in I$.

(a)′ $[\tau]^*_r \in \mathrm{T}(X)$, $\forall r \in I_1$.

(b) For any $r, s \in I$, if $r \leq s$, then $[\tau]_s \subset [\tau]_r$ and $[\tau]^*_s \subset [\tau]^*_r$.

(c) $[\tau]_r = \bigcap_{s<r} [\tau]_s$, $\forall r \in I_0$.

(c)′ $[\tau]^*_r = \bigcup_{s>r} [\tau]^*_s$, $\forall r \in I_1$, where $I_1 = [0, 1)$ and $I_0 = (0, 1]$.

3. Closures and Interiors in Ordinary Smooth Topological Spaces

Definition 3.1. Let (X, τ) be an ordinary smooth topological space and let $A \in 2^X$. Then the *ordinary smooth closure* of A in (X, τ), denoted by $\bar{A}$, is defined by

$$\bar{A} = \begin{cases} 0, & \text{if } \mathcal{C}_\tau(\mathrm{A}) = 1, \\ \bigcap\{F \in 2^X : A \subset F \\ \quad \text{and } \mathcal{C}_\tau(\mathrm{A}) < \mathcal{C}_\tau(\mathrm{F})\}, & \text{if } \mathcal{C}_\tau(\mathrm{A}) \neq 1. \end{cases}$$

Proposition 3.2. Let (X, τ) be an ordinary smooth topological space and let $A, B \in 2^X$. Then :

(a) $\mathcal{C}_\tau(A) \leq \mathcal{C}_\tau(\bar{A})$.

(b) If $B \subset A$ and $\mathcal{C}_\tau(B) \leq \mathcal{C}_\tau(A)$, then $\bar{B} \subset \bar{A}$.

Proof. (a) From Definition 3.1 and the condition (OSCT_1), it is obvious.

(b) Let $A, B \in 2^X$. Suppose $B \subset A$ and $\mathcal{C}_\tau(B) \leq \mathcal{C}_\tau(A)$.

Case (i): Suppose $\mathcal{C}_\tau(B) = 1$. Then $\bar{B} = B$. Since $\mathcal{C}_\tau(B) \leq \mathcal{C}_\tau(A)$, $\mathcal{C}_\tau(A) = 1$. Thus $\bar{A} = A$. Since $B \subset A$, $\bar{B} \subset \bar{A}$.

Case (ii): Suppose $\mathcal{C}_\tau(B) \neq 1$ and $\mathcal{C}_\tau(A) = 1$. Then $\bar{A} = A$ and

$$\bar{B} = \bigcap\{F \in 2^X : B \subset F \quad \text{and} \quad \mathcal{C}_\tau(\mathrm{B}) < \mathcal{C}_\tau(\mathrm{F})\}.$$

Since $\mathcal{C}_\tau(A) = 1$, $A \in \{F \in 2^X : B \subset F \quad \text{and} \quad \mathcal{C}_\tau(\text{B}) < \mathcal{C}_\tau(\text{F})\}$. Thus

$$\bar{B} = \bigcap\{F \in 2^X : B \subset F \quad \text{and} \quad \mathcal{C}_\tau(\text{B}) < \mathcal{C}_\tau(\text{F})\} \subset \text{A}.$$

So $\bar{B} \subset \bar{A}$.

Case (ii): $\mathcal{C}_\tau(B) \neq 1$ and $\mathcal{C}_\tau(A) \neq 1$. Then

$$\bar{B} = \bigcap\{F \in 2^X : B \subset F \quad \text{and} \quad \mathcal{C}_\tau(\text{B}) < \mathcal{C}_\tau(\text{F})\}$$

and

$$\bar{A} = \bigcap\{F \in 2^X : A \subset F \quad \text{and} \quad \mathcal{C}_\tau(\text{A}) < \mathcal{C}_\tau(\text{F})\}.$$

Thus, by the hypothesis, $\bar{V} \subset \bar{A}$. This completes the proof.

Proposition 3.3. Let (X, τ) be an ordinary smooth topological space, and let $A, B \in 2^X$. Then:

(a) $\bar{\emptyset} = \emptyset$.

(b) $A \subset \bar{A}$.

(c) $\bar{A} \subset \bar{\bar{A}}$.

(d) $\bar{A} \cap \bar{B} \subset \overline{A \cup B}$.

Proof. (a) From Definition 3.1, it is obvious.

(b) Let $A \in 2^X$.

Case (i): Suppose $\mathcal{C}_\tau(A) = 1$. Then $A = \bar{A}$.

Case (ii): Suppose $\mathcal{C}_\tau(A) \neq 1$. Then, by Definition 3.1, $A \subset \bar{A}$. Thus $A \subset \bar{A}$.

(c) By (b), $A \subset \bar{A}$. Moreover, $\mathcal{C}_\tau(A) \leq \mathcal{C}_\tau(\bar{A})$, by Proposition 3.2 (a). Thus, by Proposition 3.2 (b), $\bar{A} \subset \bar{\bar{A}}$.

(d) Let $A, B \in 2^X$.

Case (i): Suppose $\mathcal{C}_\tau(A) = \mathcal{C}_\tau(B) = 1$. Then, by the condition (OSCT_2), $\mathcal{C}_\tau(A \cup B) = 1$. Thus $\overline{A \cup B} = A \cup B$. By the hypothesis, $\bar{A} = A$ and $\bar{B} = B$. So $\bar{A} \cap \bar{B} \subset \overline{A \cup B}$.

Case (ii): Suppose $\mathcal{C}_\tau(A) = 1, \mathcal{C}_\tau(B) \neq 1$ and $\mathcal{C}_\tau(A \cup B) \neq 1$. Then, by the condition (OSCT_2), $\mathcal{C}_\tau(A \cup B) \geq \mathcal{C}_\tau(B)$. Thus, by Proposition 3.2 (b), $\bar{B} \subset \overline{A \cup B}$. So $\bar{A} \cap \bar{B} \subset \overline{A \cup B}$.

Case (iii): Suppose $\mathcal{C}_\tau(A) = 1, \mathcal{C}_\tau(B) \neq 1$ and $\mathcal{C}_\tau(A \cup B) = 1$. Then $\bar{A} \subset \overline{A \cup B}$. Thus $\bar{A} \cap \bar{B} \subset \overline{A \cup B}$.

Case (iv): Suppose $\mathcal{C}_\tau(A) \neq 1, \mathcal{C}_\tau(B) = 1$ and $\mathcal{C}_\tau(A \cup B) \neq 1$. Then it is similar to Case (ii).

Case (v): Suppose $\mathcal{C}_\tau(A) \neq 1, \mathcal{C}_\tau(B) = 1$ and $\mathcal{C}_\tau(A \cup B) = 1$. Then it is similar to Case (iii).

Case (vi): Suppose $\mathcal{C}_\tau(A) \neq 1, \mathcal{C}_\tau(B) \neq 1$ and $\mathcal{C}_\tau(A \cup B) = 1$. Then, by Proposition 3.2 (b), $\bar{A} \subset \overline{A \cup B}$ and $\bar{B} \subset \overline{A \cup B}$. Thus $\bar{A} \cap \bar{B} \subset \overline{A \cup B}$.

Case (vii): Suppose $\mathcal{C}_\tau(A) \neq 1, \mathcal{C}_\tau(B) \neq 1$ and $\mathcal{C}_\tau(A \cup B) \neq 1$. Then

$$\begin{aligned}
&\overline{A \cup B} \\
&= \bigcap\{F \in 2^X : A \cup B \subset F \text{ and } \mathcal{C}_\tau(\text{A} \cup \text{B}) < \mathcal{C}_\tau(\text{F})\} \\
&\supset \bigcap\{F \in 2^X : A \cup B \subset F \text{ and } \mathcal{C}_\tau(\text{A}) \wedge \mathcal{C}_\tau(\text{B}) < \mathcal{C}_\tau(\text{F})\} \\
&\qquad \text{[By the condition (OSCT2)]} \\
&= \bigcap\{F \in 2^X : A \subset F, B \subset F \text{ and } \mathcal{C}_\tau(\text{A}) < \mathcal{C}_\tau(\text{F}) \text{ or } \\
&\qquad \mathcal{C}_\tau(B) < \mathcal{C}_\tau(F)\} \\
&= \bigcap\{F \in 2^X : (A \subset F, B \subset F \text{ and } \mathcal{C}_\tau(\text{A}) < \mathcal{C}_\tau(\text{F})) \text{ or } \\
&\qquad (A \subset F, B \subset F \text{ and } \mathcal{C}_\tau(\text{B}) < \mathcal{C}_\tau(\text{F}))\} \\
&= \bigcap[\{F \in 2^X : A \subset F \text{ and } \mathcal{C}_\tau(\text{A}) < \mathcal{C}_\tau(\text{F})\} \\
&\qquad \cup \{F \in 2^X : B \subset F \text{ and } \mathcal{C}_\tau(\text{B}) < \mathcal{C}_\tau(\text{F})\}] \\
&\supset [\bigcap\{F \in 2^X : A \subset F \text{ and } \mathcal{C}_\tau(\text{A}) < \mathcal{C}_\tau(\text{F})\}] \\
&\qquad \cap [\{F \in 2^X : B \subset F \text{ and } \mathcal{C}_\tau(\text{B}) < \mathcal{C}_\tau(\text{F})\}] \\
&= \bar{A} \cap \bar{B}.
\end{aligned}$$

Hence, in any cases, $\bar{A} \cap \bar{B} \subset \overline{A \cup B}$.

Definition 3.4 Let (X, τ) be an ordinary smooth topological space, let $r \in I$ and let $A \in 2^X$. Then the $[\tau]_r$ - *closure* of A, denoted by $\mathrm{cl_r(A)}$, is defined by

$$\mathrm{cl_r(A)} = \bigcap\{\mathrm{F} \in 2^\mathrm{X} : \mathrm{F} \in \mathrm{F}_{[\tau]_r} \quad \text{and} \quad \mathrm{A} \subset \mathrm{F}\},$$

where $F_{[\tau]_r}$ denotes the set of all $[\tau]_r$-closed sets in X.

Remark 3.5. Let (X, τ) be a classical topological space. Then:

(a) T can be identified with an ordinary smooth topology τ_T on X defined as follows : $\tau_T : 2^X \to I$ is the mapping given by for each $A \in 2^X$

$$\tau_T(A) = \begin{cases} 1, & \text{if} \quad \mathrm{A} \in \mathrm{T}, \\ 0, & \text{otherwise}. \end{cases}$$

In fact, $[\tau_T]_0^* = T$.

(b) Also T can be identified with an ordinary smooth cotopology $\mathcal{C}_T$ on X defined as follows : $\mathcal{C}_T : 2^X \to I$ is the mapping given by for each $A \in 2^X$,

$$\mathcal{C}_T(A) = \begin{cases} 1, & \text{if} \quad \mathrm{A^c} \in \mathrm{T}, \\ 0, & \text{otherwise}. \end{cases}$$

(c) We can calculate the ordinary smooth closure of A w.r.t

τ_T, denoted by $\mathrm{cl}_{\tau_\mathrm{T}}(\mathrm{A})$, defined by Definition 3.1:

$$\mathrm{cl}_{\tau_\mathrm{T}}(\mathrm{A}) = \begin{cases} A, & \text{if } \mathcal{C}_{\tau_\mathrm{T}}(\mathrm{A}) = 1, \\ \bigcap\{F \in 2^X : A \subset F \\ \quad \text{and } \mathcal{C}_{\tau_\mathrm{T}}(\mathrm{A}) < \mathcal{C}_{\tau_\mathrm{T}}(\mathrm{F})\} & \text{if } \mathcal{C}_{\tau_\mathrm{T}}(\mathrm{A}) \neq 1. \end{cases}$$

Proposition 3.6. Let (X, τ) be an ordinary smooth topological space and let $A \in 2^X$.

(a) $\mathrm{cl}_{[\tau]_\mathrm{r}}(\mathrm{A}) = \mathrm{cl}_\mathrm{r}(\mathrm{A}), \forall \mathrm{r} \in \mathrm{I}_0$.

(b) If $\mathcal{C}_\tau(A) = 1$, then $\bar{A} = \mathrm{cl}_\mathrm{r}(\mathrm{A}), \forall \mathrm{r} \in \mathrm{I}_0$.

(c) If $\mathcal{C}_\tau(A) \neq 1$, then $\bar{A} = \bigcap_{r > \mathcal{F}_\tau(A)} \mathrm{cl}_\mathrm{r}(\mathrm{A})$.

Proof. (a) From Result 2.B and Remark 3.5 (a), it is obvious that $\tau_{[\tau]_r} \in \mathrm{OST(X)}$ for each $r \in I_0$.

Case (i) : Suppose $\mathcal{C}_{\tau_{[\tau]_r}}(A) = 1$. Then, by Remark 3.5(b), $A^c \in [\tau]_r$. Thus $A \in F_{[\tau]_r}$. So $\mathrm{cl}_\mathrm{r}(\mathrm{A}) = \bigcap\{\mathrm{F} \in 2^\mathrm{X} : \mathrm{F} \in \mathrm{F}_{[\tau]_\mathrm{r}}$ and $\mathrm{A} \subset \mathrm{F}\} = \mathrm{A}$. On the other hand, by the hypothesis and Remark 3.5 (c), $\mathrm{cl}_{\tau_{[\tau]_\mathrm{r}}}(\mathrm{A}) = \mathrm{A}$. Hence $\mathrm{cl}_{\tau_{[\tau]_\mathrm{r}}}(\mathrm{A}) = \mathrm{cl}_\mathrm{r}(\mathrm{A})$.

Case (ii): Suppose $\mathcal{C}_{\tau_{[\tau]_r}}(A) \neq 1$. Then

$$\begin{aligned} &\mathrm{cl}_{\tau_{[\tau]_\mathrm{r}}}(\mathrm{A}) \\ &= \bigcap\{F \in 2^X : A \subset F \text{ and } \mathcal{C}_{\tau_{[\tau]_\mathrm{r}}}(\mathrm{A}) < \mathcal{C}_{\tau_{[\tau]_\mathrm{r}}}(\mathrm{F})\} \\ &= \bigcap\{F \in 2^X : A \subset F \text{ and } \mathcal{C}_{\tau_{[\tau]_\mathrm{r}}}(\mathrm{F}) = 1\} \\ &\qquad \text{[By Remark 3.5(b)]} \\ &= \bigcap\{F \in 2^X : A \subset F \text{ and } \mathrm{F} \in \mathrm{F}_{[\tau]_\mathrm{r}}\} \\ &\qquad \text{[By Remark 3.5(b)]} \\ &= \mathrm{cl}_\mathrm{r}(\mathrm{A}). \qquad \text{[By Definition 3.4]} \end{aligned}$$

(b) Suppose $\mathcal{C}_\tau(A) = 1$. Then, by Definition 3.1, $\bar{A} = A$. Let $r \in I_0$. Then, by the hypothesis, $\tau(A^c) = \mathcal{C}_\tau(A) = 1 \geq r$. Thus $A^c \in [\tau]_r$. So $A \in F_{\tau_{[\tau]_r}}$. Hence, by Definition 3.4, $\mathrm{cl}_\mathrm{r}(\mathrm{A}) = \mathrm{A}$. Therefore $\bar{A} = \mathrm{cl}_\mathrm{r}(\mathrm{A})$ for each $r \in I_0$.

(c) Suppose $\mathcal{C}_\tau(A) \neq 1$. Then

$$\begin{aligned} &\bigcap_{r > \mathcal{C}_\tau(A)} \mathrm{cl}_\mathrm{r}(\mathrm{A}) \\ &= \bigcap_{r > \mathcal{C}_\tau(A)} [\bigcap\{F \in 2^X : F \in F_{[\tau]_r} \text{ and } \mathrm{A} \subset \mathrm{F}\}] \\ &= \bigcap_{r > \mathcal{C}_\tau(A)} [\bigcap\{F \in 2^X : A \subset F \text{ and } \mathrm{F^c} \in [\tau]_\mathrm{r}\}] \\ &= \bigcap_{r > \mathcal{C}_\tau(A)} [\bigcap\{F \in 2^X : A \subset F \text{ and } \mathcal{C}_\tau(\mathrm{F}) = \tau(\mathrm{F^c}) \geq \mathrm{r}\}] \\ &= \bigcap[\bigcap_{r > \mathcal{C}_\tau(A)} \{F \in 2^X : A \subset F \text{ and } \mathcal{C}_\tau(\mathrm{F}) \geq \mathrm{r}\}] \\ &= \bigcap\{F \in 2^X : A \subset F \text{ and } \mathcal{C}_\tau(\mathrm{A}) < \mathcal{C}_\tau(\mathrm{F})\} \\ &= \bar{A}. \qquad \text{[By Definition 3.1]} \end{aligned}$$

This completes the proof.

Definition 3.7. Let (X, τ) be an ordinary smooth topological space and let $A \in 2^X$. Then the *ordinary smooth interior* of A in (X, τ),denoted by $\overset{\circ}{A}$, is defined as follows:

$$\overset{\circ}{A} = \begin{cases} A, & \text{if } \quad \tau(\mathrm{A}) = 1, \\ \bigcup\{U \in 2^X : U \subset A \\ \quad \text{and } \tau(\mathrm{U}) > \tau(\mathrm{A})\} & \text{if } \quad \tau(\mathrm{A}) \neq 1. \end{cases}$$

Proposition 3.8. Let (X, τ) be an ordinary smooth topological space and let $A, B \in 2^X$. Then:

(a) $\tau(A) \leq \tau(\overset{\circ}{A})$.

(b) If $B \subset A$ and $\tau(B) > \tau(A)$, then $\overset{\circ}{B} \subset \overset{\circ}{A}$. **Proof.** (a) from Definition 3.7 and the condition ($\mathrm{OST_1}$), it is obvious.

(b) The proof is similar to that of Proposition 3.2 (b).

Proposition 3.9. Let (X, τ) be an ordinary smooth topological space and let $A, B \in 2^X$. Then:

(a) $\overset{\circ}{X} = X$.

(b) $\overset{\circ}{A} \subset A$.

(c) $(\overset{\circ}{A})^\circ \subset \overset{\circ}{A}$.

(d) $(A \cap B)^\circ \subset \overset{\circ}{A} \cup \overset{\circ}{B}$. **Proof.** The proofs are similar to these of Proposition 3.3.

Definition 3.10. Let (X, τ) be an ordinary smooth topological space, let $r \in I$ and let $A \in 2^X$. Then the $[\tau]_r$*-interior* of A, denoted by $\mathrm{int_r(A)}$, is defined by

$$\mathrm{int_r(A)} = \bigcup\{\mathrm{U} \in 2^\mathrm{X} : \mathrm{U} \in [\tau]_\mathrm{r} \quad \text{and} \quad \mathrm{U} \subset \mathrm{A}\}.$$

Proposition 3.11. Let (X, τ) be an ordinary smooth topological space and let $A \in 2^X$.

(a) $\mathrm{int}_{\tau_{[\tau]_\mathrm{r}}}(\mathrm{A}) = \mathrm{int_r(A)}, \forall \mathrm{r} \in \mathrm{I}_0$, where $\mathrm{int}_{\tau_{[\tau]_\mathrm{r}}}(\mathrm{A})$ is the ordinary smooth interior of A in $(X, \tau_{[\tau]_r})$, defined by

$$\mathrm{int}_{\tau_{[\tau]_\mathrm{r}}}(\mathrm{A}) = \begin{cases} A, & \text{if } \tau_{[\tau]_\mathrm{r}}(\mathrm{A}) = 1, \\ \bigcup\{U \in 2^X : U \subset A \\ \text{and } \tau_{[\tau]_\mathrm{r}}(\mathrm{U}) > \tau_{[\tau]_\mathrm{r}}(\mathrm{A})\} & \text{if } \tau_{[\tau]_\mathrm{r}}(\mathrm{A}). \end{cases}$$

(b) If $\tau(A) = 1$, then $\overset{\circ}{A} = \mathrm{int_r(A)}, \quad \forall \mathrm{r} \in \mathrm{I}_0$.

(c) If $\tau(A) \neq 1$, then $\overset{\circ}{A} = \bigcup_{r > \tau(A)} \mathrm{int_r(A)}$.

Proof. The proofs are similar to these of Proposition 3.6.

4. Ordinary Smooth Open Preserving Mappings

Definition 4.1. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological spaces. Then a mapping $f : X \to Y$ is said to be *ordinary smooth preserving*[resp. *ordinary strict smooth preserving*] if for any $A, B \in 2^Y$,

$$\tau_2(B) \leq \tau_2(A) \Leftrightarrow \tau_1(f^{-1}(B)) \leq \tau_1(f^{-1}(A))$$

$$[\text{resp.}\tau_2(\text{B}) < \tau_2(\text{A}) \Leftrightarrow \tau_1(\text{f}^{-1}(\text{B})) < \tau_1(\text{f}^{-1}(\text{A}))].$$

The following is the characterization of Definition 4.1.

Theorem 4.2. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological spaces: Suppose $f : X \to Y$ is an ordinary smooth preserving [resp. an ordinary strict smooth preserving] mapping. Then for any $A, B \in 2^Y$,

$$\mathcal{C}_{\tau_2}(B) \leq \mathcal{C}_{\tau_2}(A) \Leftrightarrow \mathcal{C}_{\tau_1}(f^{-1}(B)) \leq \mathcal{C}_{\tau_1}(f^{-1}(A))$$

$$[\text{resp.} \quad \mathcal{C}_{\tau_2}(\text{B}) < \mathcal{C}_{\tau_2}(\text{A}) \Leftrightarrow \mathcal{C}_{\tau_1}(\text{f}^{-1}(\text{B})) < \mathcal{C}_{\tau_1}(\text{f}^{-1}(\text{A}))].$$

Proof. (i) Suppose f is ordinary smooth preserving and let $A, B \in 2^X$. Then

$$\begin{aligned} & \mathcal{C}_{\tau_2}(B) \leq \mathcal{C}_{\tau_2}(A) \\ \Leftrightarrow & \tau_2(B^c) \leq \tau_2(A^c) \\ & \quad [\text{By the definition of } \mathcal{C}_\tau \text{ for each } \tau \in \text{OST(X)}] \\ \Leftrightarrow & \tau_1(f^{-1}(B^c)) \leq \tau_1(f^{-1}(A^c)) \quad [\text{By the hypothesis}] \\ \Leftrightarrow & \tau_1((f^{-1}(B))^c) \leq \tau_1((f^{-1}(A))^c) \\ \Leftrightarrow & \mathcal{C}_{\tau_1}(f^{-1}(B)) \leq \mathcal{C}_{\tau_1}(f^{-1}(A)). \end{aligned}$$

(ii) Suppose f is ordinary strict smooth preserving. Then the proof is similar to that of (i).

Definition 4.3 [9]. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological spaces. Then a mapping $f : X \to Y$ is said to be *ordinary smooth continuous* if $\tau_2(A) \leq \tau_1(f^{-1}(A)), \quad \forall A \in 2^Y$.

Theorem 4.4. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological spaces and let $f : X \to Y$ be a mapping. Then f is ordinary smooth continuous if and only if $\mathcal{C}_{\tau_2}(A) \leq \mathcal{C}_{\tau_1}(f^{-1}(A)), \quad \forall A \in 2^Y$.

The following is the immediate result of Result 2.A and Definition 4.3.

Proposition 4.5. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological space. Suppose $f : X \to Y$ is an injective, ordinary strict smooth preserving and ordinary smooth continuous mapping. Then $f(\bar{A}) \subset \overline{f(A)}, \quad \forall A \in 2^X$.

Proof. Let $A \in 2^X$.

Case (i): Suppose $\mathcal{C}_{\tau_2}(f(A)) = 1$. Then

$$\begin{aligned} 1 = \mathcal{C}_{\tau_2}(f(A)) &\leq \mathcal{C}_{\tau_1}(f^{-1}(f(A))) \\ &= \mathcal{C}_{\tau_1}(A). \quad [\text{Since } f \text{ is injective}] \end{aligned}$$

Thus $\mathcal{C}_{\tau_1}(A) = 1$. So, by Definition 3.1, $\bar{A} = A$. Hence, by the hypothesis,

$$\overline{f(A)} = f(A) = f(\bar{A})$$

Case (ii): Suppose $\mathcal{C}_{\tau_2}(f(A)) \neq 1$ and $\mathcal{C}_{\tau_1}(A) = 1$. Then clearly $\bar{A} = A$. Thus, by Proposition 3.3 (b),

$$f(\bar{A}) = f(A) \subset \overline{f(A)}.$$

Case (iii): Suppose $\mathcal{C}_{\tau_2}(f(A)) \neq 1$ and $\mathcal{C}_{\tau_1}(A) \neq 1$. Then

$$\begin{aligned} & f^{-1}\overline{(f(A))} \\ &= f^{-1}[\bigcap\{F \in 2^Y : f(A) \subset F \\ &\qquad \text{and } \mathcal{C}_{\tau_2}(\text{f(A)}) < \mathcal{C}_{\tau_2}(\text{F})\}] \text{ [By Definition 3.1]} \\ &\subset f^{-1}[\bigcap\{F \in 2^Y : A \subset f^{-1}(F) \\ &\qquad \text{and } \mathcal{C}_{\tau_1}(\text{A}) < \mathcal{C}_{\tau_1}(\text{f}^{-1}(\text{F}))] \\ &\quad [\text{Since } f \text{ is injective and ordinary smooth preserving}] \\ &= \bigcap\{f^{-1}(F) \in 2^X : A \subset f^{-1}(F) \\ &\qquad \text{and } \mathcal{C}_{\tau_1}(\text{A}) < \mathcal{C}_{\tau_1}(\text{f}^{-1}(\text{F}))\} \\ &= \bigcap\{B \in 2^X : A \subset B \\ &\qquad \text{and } \mathcal{C}_{\tau_1}(\text{A}) < \mathcal{C}_{\tau_1}(\text{B})\} \\ &= \bar{A} \end{aligned}$$

Hence, in any cases

$$\overline{f(A)} \subset \overline{f(A)}, \quad \forall A \in 2^X.$$

Proposition 4.6. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological space. Suppose $f : X \to Y$ is an ordinary strict smooth preserving and ordinary smooth continuous mapping. Then $\overline{f^{-1}(A)} \subset f^{-1}(\bar{A}), \quad \forall A \in 2^Y$. **Proof.** Let $A \in 2^Y$.

Case (i): Suppose $\mathcal{C}_{\tau_2}(A) = 1$. Since f is ordinary smooth continuous, $\mathcal{C}_{\tau_1}(f^{-1}(A)) = 1$. Then $\overline{f^{-1}(A)} = f^{-1}(A)$. By the hypothesis, $A = \bar{A}$. Thus $f^{-1}(A) = f^{-1}(\bar{A})$. So

$\overline{f^{-1}(A)} = f^{-1}(\bar{A})$.

Case (ii): Suppose $\mathcal{C}_{\tau_2}(A) \neq 1$ and $\mathcal{C}_{\tau_1}(f^{-1}(A)) = 1$. By Proposition 3.3 (b), it is clear that $A \subset \bar{A}$. Then $f^{-1}(A) \subset f^{-1}(\bar{A})$. Since $\mathcal{C}_{\tau_1}(f^{-1}(A)) = 1, \overline{f^{-1}(A)} = f^{-1}(A)$. Thus $\overline{f^{-1}(A)} \subset f^{-1}(\bar{A})$.

Case (iii): Suppose $\mathcal{C}_{\tau_2}(A) \neq 1$ and $\mathcal{C}_{\tau_1}(f^{-1}(A)) \neq 1$. Then

$$\begin{aligned}
&f^{-1}(\bar{A}) \\
&= f^{-1}(\bigcap\{F \in 2^Y : A \subset F \\
&\qquad \text{and} \quad \mathcal{C}_{\tau_2}(\mathrm{A}) < \mathcal{C}_{\tau_2}(\mathrm{F})\}) \\
&\supset f^{-1}(\bigcap\{F \in 2^Y : f^{-1}(A) \subset f^{-1}(F) \\
&\qquad \text{and} \quad \mathcal{C}_{\tau_1}(\mathrm{f}^{-1}(\mathrm{A})) < \mathcal{C}_{\tau_1}(\mathrm{f}^{-1}(\mathrm{F}))\}) \\
&\qquad [\text{Since } f \text{ is ordinary strict smooth preserving}] \\
&= \bigcap\{f^{-1}(F) \in 2^X : f^{-1}(A) \subset f^{-1}(F) \\
&\qquad \text{and} \quad \mathcal{C}_{\tau_1}(\mathrm{f}^{-1}(\mathrm{A})) < \mathcal{C}_{\tau_1}(\mathrm{f}^{-1}(\mathrm{F}))\} \\
&= \bigcap\{B \in 2^X : f^{-1}(A) \subset B \\
&\qquad \text{and} \quad \mathcal{C}_{\tau_1}(\mathrm{f}^{-1}(\mathrm{A})) < \mathcal{C}_{\tau_1}(\mathrm{B})\} \\
&= \overline{f^{-1}(A)}.
\end{aligned}$$

Hence, in any cases, $\overline{f^{-1}(A)} \subset f^{-1}(\bar{A}), \quad \forall A \in 2^Y$.

Definition 4.7 [9]. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological spaces. Then a mapping $f : X \to Y$ is said to be *ordinary smooth open* [resp. *closed*] if for each $A \in 2^X$,

$$\tau_1(A) \leq \tau_2(f(A)) \quad [\text{resp.}\mathcal{C}_1(\mathrm{A}) \leq \mathcal{C}_2(\mathrm{f(A)})],$$

where $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ are ordinary smooth cotopological spaces.

Definition 4.8. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological spaces. Then a mapping $f : X \to Y$ is said to be *ordinary smooth open preserving* [resp. *ordinary strict smooth open preserving*] if for any $A, B \in 2^X$,

$$\tau_1(B) \leq \tau_1(A) \Rightarrow \tau_2(f(B)) \leq \tau_2(f(A))$$

$$[\text{resp.} \quad \tau_1(B) < \tau_1(A) \Rightarrow \tau_2(f(B)) < \tau_2(f(A))].$$

Notice that the concept of an ordinary smooth open preserving mapping differs from the concept of an ordinary smooth open mapping.

Example 4.9. Let $X = \{a, b\}$, let $A = \{a\}$ and let $B = \{b\}$. For each $i = 1, 2, 3$, we define a mapping $\tau_i : 2^X \to I$ as follows : For each $C \in 2^X$,

$$\begin{aligned}
\tau_i(C) &= 1, && \text{if } C = \emptyset \quad \text{or} \quad C = X, \\
\tau_1(A) &= 0.15, && \tau_1(B) = 0.80, \\
\tau_2(A) &= 0.30, && \tau_2(B) = 0.50, \\
\tau_3(A) &= 0.90, && \tau_3(B) = 0.80, \\
\tau_i(C) &= 0.10, && \text{if} \quad C \notin \{\emptyset, X, A, B\}.
\end{aligned}$$

Then it is obvious that $\tau_i \in \mathrm{OST(X)}$ for each $i = 1, 2, 3$. Moreover, we can easily see that the identity mapping $\mathrm{id} : (\mathrm{X}, \tau_1) \to (\mathrm{X}, \tau_2)$ is ordinary smooth open preserving but not ordinary smooth open. However, the identity mapping $\mathrm{id} : (\mathrm{X}, \tau_2) \to (\mathrm{X}, \tau_3)$ is ordinary smooth open but not ordinary smooth open preserving.

Proposition 4.10. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological spaces. Suppose $f : X \to Y$ is an ordinary strict smooth open preserving and ordinary smooth open mapping. Then $f(\overset{\circ}{A}) \subset (f(A))^\circ, \quad \forall A \in 2^X$.

Proof. Let $A \in 2^X$.

Case (i): Suppose $\tau_1(A) = 1$. Since f is ordinary smooth open, $\tau_2(f(A)) = 1$. Then $(f(A))^\circ = f(A)$. Since $\tau_1(A) = 1, \overset{\circ}{A} = A$. Thus $(f(A))^\circ = f(\overset{\circ}{A})$.

Case (ii): Suppose $\tau_1(A) \neq 1$ and $\tau_2(f(A)) = 1$. By Proposition 3.9 (b), $\overset{\circ}{A} \subset A$. Thus $f(\overset{\circ}{A}) \subset f(A)$. Since $\tau_2(f(A)) = 1, (f(A))^\circ = f(A)$. Thus $f(\overset{\circ}{A}) \subset (f(A))^\circ$.

Case (iii): Suppose $\tau_1(A) \neq 1$ and $\tau_2(f(A)) \neq 1$. Then

$$\begin{aligned}
f(\overset{\circ}{A}) &= f[\cup\{U \in 2^X : U \subset A \quad \text{and} \quad \tau_1(\mathrm{U}) > \tau_1(\mathrm{A})\}] \\
&\subset f[\cup\{U \in 2^X : f(U) \subset f(A) \\
&\qquad \text{and} \quad \tau_2(\mathrm{f(U)}) > \tau_2(\mathrm{f(A)})\}] \\
&\quad [\text{Since } f \text{ is ordinary strict smooth open preserving}] \\
&= \cup\{f(U) \in 2^Y : f(U) \subset f(A) \\
&\qquad \text{and} \quad \tau_2(\mathrm{f(U)}) > \tau_2(\mathrm{f(A)})\} \\
&= \cup\{V \in 2^Y : V \subset f(A) \\
&\qquad \text{and} \quad \tau_2(\mathrm{V}) > \tau_2(\mathrm{f(A)})\} \\
&= (f(A))^\circ.
\end{aligned}$$

Hence, in any cases, $f(\overset{\circ}{A}) \subset (f(A))^\circ$.

Proposition 4.11. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological spaces. Suppose $f : X \to Y$ is an ordinary strict smooth preserving and ordinary smooth continuous. Then for each $A \in 2^Y$, $f^{-1}(\overset{\circ}{A}) \subset (f^{-1}(A))^{\circ}$. **Proof.** By using Definition 3.7, the proof is similar to that of Proposition 4.5.

5. Some Types of Ordinary Smooth Compactness

For an ordinary smooth topological spaces (X, τ), let us define $S(\tau) = \{A \in 2^X : \tau(A) > 0\}$ and $S(\tau)$ will be called the *support* of τ.

Definition 5.1. An ordinary smooth topological space (X, τ) is said to be:

(i) *ordinary smooth compact* if for every family $\{A_\alpha\}_{\alpha\in\Gamma}$ in $S(\tau)$ covering X, there is a finite subset Γ_0 of Γ such that $\bigcup_{\alpha\in\Gamma_0} A_\alpha = X$.

(ii) *ordinary smooth almost compact* if for every family $\{A_\alpha\}_{\alpha\in\Gamma}$ in $S(\tau)$ covering X, there is a finite subset Γ_0 of Γ such that $\bigcup_{\alpha\in\Gamma_0} \bar{A}_\alpha = X$.

(iii) *ordinary smooth nearly compact* if for every family $\{A_\alpha\}_{\alpha\in\Gamma}$ in $S(\tau)$ covering X, there is a finite subset Γ_0 of Γ such that $\bigcup_{\alpha\in\Gamma_0} (\bar{A}_\alpha)^\circ = X$, or equivalently for every family $\{A_\alpha\}_{\alpha\in\Gamma}$ in $\{A \in 2^X : \tau(A) > 0 \quad \text{and} \quad \mathrm{A} = (\bar{\mathrm{A}})^\circ\}$, there is a finite subset Γ_0 of Γ such that $\bigcup_{\alpha\in\Gamma_0} A_\alpha = X$.

The following is the characterization of Definition 5.1 (i).

Theorem 5.2. Let (X, τ) be an ordinary smooth topological space. Then (X, τ) is ordinary smooth compact if and only if every family in $S(\tau)$ having the finite intersection property (in short, F.I.P.) has a nonempty intersection.

Proof. $(\Rightarrow)$: Suppose (X, τ) is ordinary smooth compact. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be the family in $S(\mathcal{C}_\tau)$ having the F.I.P., i.e., for any finite subset $\Gamma_0 \subset \Gamma$, $\bigcap_{\alpha\in\Gamma_0} A_\alpha \neq \emptyset$. Assume that $\bigcap_{\alpha\in\Gamma} A_\alpha = \emptyset$. Then clearly $\bigcup_{\alpha\in\Gamma} {A_\alpha}^c = X$. Since $A_\alpha \in S(\mathcal{C}_\tau)$ for each $\alpha \in \Gamma$, $\mathcal{C}_\tau(A_\alpha) = \tau({A_\alpha}^c) > 0$ for each $\alpha \in \Gamma$. Thus $\{{A_\alpha}^c\}_{\alpha\in\Gamma}$ is a covering of X. So, by the hypothesis, there is a finite subset $\Gamma_0 \subset \Gamma$ such that $\bigcup_{\alpha\in\Gamma_0} {A_\alpha}^c = X$. Hence $\bigcap_{\alpha\in\Gamma_0} A_\alpha = \emptyset$. This is a contradiction.

$(\Leftarrow)$: Suppose the necessary condition holds. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a family in $S(\tau)$ covering X. Then $\bigcup_{\alpha\in\Gamma} A_\alpha = X$. Assume that for any subset $\Gamma_0 \subset \Gamma$, $\bigcup_{\alpha\in\Gamma_0} A_\alpha \neq X$. Then $\bigcap_{\alpha\in\Gamma_0} {A_\alpha}^c \neq \emptyset$. Since $\{A_\alpha\}_{\alpha\in\Gamma}$ is a family in $S(\tau), \mathcal{C}_\tau({A_\alpha}^c) = \tau(A_\alpha) > 0$. Thus $\{{A_\alpha}^c\}_{\alpha\in\Gamma}$ is the family in $S(\mathcal{C}_\tau)$ having the F.I.P. So, by the hypothesis, $\bigcap_{\alpha\in\Gamma} {A_\alpha}^c \neq \emptyset$. Hence $\bigcup_{\alpha\in\Gamma} A_\alpha \neq X$. This is a contradiction.

Definition 5.3. An ordinary smooth topological space (X, τ) is said to be *ordinary smooth regular* if for each $A \in S(\tau)$,

$$A = \bigcup\{B \in 2^X : \tau(A) \leq \tau(B) \quad \text{and} \quad \bar{\mathrm{B}} \subset \mathrm{A}\}$$

Proposition 5.4. Let (X, τ) be an ordinary smooth topological space.

(a) If (X, τ) is ordinary smooth almost compact and ordinary smooth regular, then so is it.

(b) If (X, τ) is ordinary smooth nearly compact and ordinary smooth regular, then it is ordinary smooth compact.

Proof. (a) Suppose (X, τ) is ordinary smooth almost compact and ordinary smooth regular. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a family in $S(\tau)$ covering X, i.e., $\bigcup_{\alpha\in\Gamma} A_\alpha = X$. Since (X, τ) is ordinary smooth regular, for each $\alpha \in \Gamma$,

$$A_\alpha = \bigcup\{B_\alpha \in 2^X : \tau(A_\alpha) \leq \tau(B_\alpha) \quad \text{and} \quad \bar{\mathrm{B}}_\alpha \subset \mathrm{A}_\alpha\}.$$

Then clearly $\bigcup_{\alpha\in\Gamma} B_\alpha = X$ and $\{B_\alpha\}_{\alpha\in\Gamma}$ is a family in $S(\tau)$. Since (X, τ) is ordinary smooth almost compact, there is a finite subset $\Gamma_0 \subset \Gamma$ such that $\bigcup_{\alpha\in\Gamma_0} \bar{B}_\alpha = X$. Since $\bar{B}_\alpha \subset A_\alpha$ for each $\alpha \in \Gamma$, $\bigcup_{\alpha\in\Gamma_0} A_\alpha = X$. Hence (X, τ) is ordinary smooth compact.

(b) The proof is quite to that of (a) taking into account that the ordinary smooth interior of an ordinary subset remains always smaller then itself.

Proposition 5.5. Let (X, τ_1) and (Y, τ_2) be two ordinary smooth topological spaces and let $f : X \to Y$ be a surjective ordinary smooth continuous and ordinary strict smooth preserving mapping.

(a) If (X, τ_1) is ordinary smooth almost compact, then so is (Y, τ_2).

(b) If (X, τ_1) is ordinary smooth nearly compact, then (Y, τ_2) is ordinary smooth almost compact.

Proof. (a) Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a family in $S(\tau_2)$ covering Y, i.e., $\bigcup_{\alpha\in\Gamma} A_\alpha = Y$. Since f is ordinary smooth continuous,

$$\tau_2(A_\alpha) \leq \tau_1(f^{-1}(A_\alpha)) \quad \text{for each } \alpha \in \Gamma.$$

Since $\bigcup_{\alpha\in\Gamma} A_\alpha = Y$, $\bigcup_{\alpha\in\Gamma} f^{-1}(A_\alpha) = X$. Since $\tau_2(A_\alpha) > 0$ for each $\alpha \in \Gamma$, $\tau_1(f^{-1}(A_\alpha)) > 0$ for each $\alpha \in \Gamma$. Thus $\{f^{-1}(A_\alpha)\}_{\alpha\in\Gamma}$ is a family in $S(\tau)$ covering X, since (X, τ_1) is ordinary smooth almost compact, there is a finite subset $\Gamma_0 \subset \Gamma$ such that $\bigcup_{\alpha\in\Gamma_0} \overline{f^{-1}(A_\alpha)} = X$. Since f is surjective,

$$f(\bigcup_{\alpha\in\Gamma_0} \overline{f^{-1}(A_\alpha)}) = \bigcup_{\alpha\in\Gamma_0} \overline{f(f^{-1}(A_\alpha))} = X.$$

Since f is ordinary smooth continuous and ordinary strict smooth preserving, by Proposition 4.5,

$$\overline{f^{-1}(A_\alpha)} \subset f^{-1}(\bar{A}_\alpha), \quad \text{for each } \alpha \in \Gamma.$$

So $f(\overline{f^{-1}(A_\alpha)}) \subset f(f^{-1}(\bar{A}_\alpha)) = \bar{A}_\alpha$, for each $\alpha \in \Gamma$. Hence $\bigcup_{\alpha \in \Gamma_0} \bar{A}_\alpha = X$. Therefore (Y, τ_2) is ordinary smooth almost compact.

(b) The proof is similar to that of (a).

6. Conclusions

It is difficult to investigate the compactness using the notions of the closure and the interior introduced by Ying [8, 9]. To handle the difficulty, we introduced the new definitions of the closure and the interior different from the Ying's definitions. We discussed the topological properties based on the definitions. The new definitions help to naturally study some compactness in an ordinary smooth topological space.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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