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L-upper Approximation Operators and Join Preserving Maps

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Abstract

In this paper, we investigate the properties of join and meet preserving maps in complete residuated lattice using Zhang's the fuzzy complete lattice which is defined by join and meet on fuzzy posets. We define L-upper (resp. L-lower) approximation operators as a generalization of fuzzy rough sets in complete residuated lattices. Moreover, we investigate the relations between L-upper (resp. L-lower) approximation operators and L-fuzzy preorders. We study various L-fuzzy preorders on L^X. They are considered as an important mathematical tool for algebraic structure of fuzzy contexts.

Keywords: Complete residuated lattices, Join and meet preserving maps, L-upper (lower) approximation operators, L-fuzzy preorder

1. Introduction

Pawlak [1, 2] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. Hájek [3] introduced a complete residuated lattice which is an algebraic structure for many valued logic. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [3-6]. Hájek [3] and Bělohlávek [4] developed the notion of fuzzy contexts using Galois connections with $R \in L^{X \times Y}$ on a complete residuated lattice. Zhang et al. [7, 8] introduced the fuzzy complete lattice which is defined by join and meet on fuzzy posets. It is an important mathematical tool for algebraic structure of fuzzy contexts [1, 2, 5-11]. Kim [12] showed that join (resp. meet, meet join, join meet) preserving maps and upper (resp. lower, meet join, join meet) approximation maps are equivalent in complete residuated lattices.

In this paper, we investigate the properties of join and meet preserving maps in complete residuated lattice. We define an L-upper (resp. L-lower) approximation operator as a generalization of fuzzy rough set in complete residuated lattices. Moreover, we investigate the relations between L-upper (resp. L-lower) approximation operators and L-fuzzy preorders. We give their examples.

Definition 1.1. [3] A triple $(L, \vee, \wedge, \odot, \rightarrow, \perp, \top)$ is called a *complete residuated lattice* if it satisfies the following properties:
(L1) $(L, \vee, \wedge, \perp, \top)$ is a complete lattice where $\perp$ is the bottom element and $\top$ is the top element;
(L2) $(L, \odot, \top)$ is a monoid;

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(L3) adjointness properties,i.e.

$$x \leq y \to z \text{ iff } x \odot y \leq z.$$

A map $^*: L \to L$ defined by $a^* = a \to \bot$ is called a *strong negation* if $a^{**} = a$.

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise.} \end{cases} \quad \top_x^*(y) = \begin{cases} \bot, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *.

Definition 1.2. [7, 8] Let X be a set. A function $e_X : X \times X \to L$ is called:

(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,

(E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

If e satisfies (E1) and (E2), (X, e_X) is a fuzzy preorder set. If e satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially order set (simply, fuzzy poset).

Example 1.3. (1) We define a function $e_L : L \times L \to L$ as $e_L(x, y) = x \to y$. Then (L, e_L) is a fuzzy poset.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as

$$e_{L^X}(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)).$$

Then (L^X, e_{L^X}) is a fuzzy poset from Lemma 2.10 (9).

Definition 1.4. [7, 8] Let (X, e_X) be a fuzzy poset and $A \in L^X$.

(1) A point x_0 is called a join of A, denoted by $x_0 = \sqcup A$, if it satisfies

(J1) $A(x) \leq e_X(x, x_0)$,

(J2) $\bigwedge_{x \in X}(A(x) \to e_X(x, y)) \leq e_X(x_0, y)$.

A point x_1 is called a meet of A, denoted by $x_1 = \sqcap A$, if it satisfies

(M1) $A(x) \leq e_X(x_1, x)$,

(M2) $\bigwedge_{x \in X}(A(x) \to e_X(y, x)) \leq e_X(y, x_1)$.

Remark 1.5. Let (X, e_X) be a fuzzy poset and $A \in L^X$.

(1) If x_0 is a join of A, then it is unique because $e_X(x_0, y) = e_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e_X(x_0, y_0) = e_X(y_0, x_0) = \top$ implies $x_0 = y_0$. Similarly, if a meet of A exist, then it is unique.

(2) x_0 is a join of A iff

$$\bigwedge_{x \in X} (A(x) \to e_X(x, y)) = e_X(x_0, y).$$

(3) x_1 is a meet of A iff

$$\bigwedge_{x \in X} (A(x) \to e_X(y, x)) = e_X(y, x_1).$$

Remark 1.6. Let (L, e_L) be a fuzzy poset and $A \in L^L$.

(1) Since x_0 is a join of A iff $\bigwedge_{x \in L}(A(x) \to e_L(x, y)) = \bigwedge_{x \in L}(A(x) \to (x \to y)) = \bigvee_{x \in L}(x \odot A(x)) \to y = e_L(x_0, y) = x_0 \to y$, then $x_0 = \sqcup A = \bigvee_{x \in L}(x \odot A(x))$.

(2) Since x_0 is a join of A iff $\bigwedge_{x \in L}(A(x) \to e_L(x, y) = \bigwedge_{x \in L}(A(x) \to (y \to x)) = \bigwedge_{x \in L}(y \to (A(x) \to x)) = y \to \bigwedge_{x \in L}(A(x) \to x) = y \to \sqcap A$, then

$$\sqcap A = \bigwedge_{x \in L} (A(x) \to x).$$

Remark 1.7. Let (L^X, e_{L^X}) be a fuzzy poset and $\Phi \in L^{L^X}$.

(1) We have $\sqcup \Phi = \bigvee_{A \in L^X}(\Phi(A) \odot A)$ from:

$$\begin{aligned} &\bigwedge_{A \in L^X}(\Phi(A) \to e_{L^X}(A, B)) \\ &= e_{L^X}(\bigvee_{A \in L^X}(\Phi(A) \odot A), B) = e_{L^X}(\sqcup \Phi, B). \end{aligned}$$

(2) We have $\sqcap \Phi = \bigwedge_{A \in L^X}(\Phi(A) \to A)$ from:

$$\begin{aligned} &\bigwedge_{A \in L^X}(\Phi(A) \to e_{L^X}(B, A) \\ &= \bigwedge_{A \in L^X} e_{L^X}(B, (\Phi(A) \to A)) \\ &= e_{L^X}(B, \bigwedge_{A \in L^X}(\Phi(A) \to A)). \end{aligned}$$

Definition 1.8. [7, 8] Let (L^X, e_{L^X}) and (L^Y, e_{L^Y}) be fuzzy posets.

(1) $\mathcal{H} : L^X \to L^Y$ is a join preserving map if

$$\mathcal{H}(\sqcup \Phi) = \sqcup \mathcal{H}^{\to}(\Phi)$$

for all $\Phi \in L^{L^X}$, where $\mathcal{H}^{\to}(\Phi)(B) = \bigvee_{\mathcal{H}(A) = B} \Phi(A)$.

(2) $\mathcal{J} : L^X \to L^Y$ is a meet preserving map if

$$\mathcal{J}(\sqcap \Phi) = \sqcap \mathcal{J}^{\to}(\Phi)$$

for all $\Phi \in L^{L^X}$.

Theorem 1.9. [12] Let X and Y be two sets. Let (L^X, e_{L^X}) and (L^Y, e_{L^Y}) be fuzzy posets. Then the following statements are equivalent:

(1) $\mathcal{H} : L^X \to L^Y$ is a join preserving map iff $\mathcal{H}(\alpha \odot A) =$

$\alpha \odot \mathcal{H}(A)$ and $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$ for all $A, A_i \in L^X$, and $\alpha \in L$.

(2) $\mathcal{J} : L^X \to L^Y$ is a meet preserving map iff $\mathcal{J}(\alpha \to A) = \alpha \to \mathcal{J}(A)$ and $\mathcal{J}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{J}(A_i)$ for all $A, A_i \in L^X$, and $\alpha \in L$.

Lemma 1.10. [3, 4, 9] Let $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *. For each $x, y, z, x_i, y_i \in L$, the following properties hold.

(1) $\odot$ is isotone in both arguments.

(2) $\to$ is antitone in the first and isotone in the second argument.

(3) $x \to y = \top$ iff $x \le y$.

(4) $x \to \top = \top$ and $\top \to x = x$.

(5) $x \odot y \le x \wedge y$.

(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.

(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i) \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(9) $(x \to y) \odot x \le y$ and $(y \to z) \odot (x \to y) \le (x \to z)$.

(10) $x \to y \le (y \to z) \to (x \to z)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.

(13) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.

2. L-upper Approximation Operators and Join Preserving Maps

Theorem 2.1. Let (L^X, e_{L^X}) be a fuzzy poset. Let $\mathcal{H}, \mathcal{H}^{-1} : L^X \to L^X$ be join preserving maps such that $\mathcal{H}^{-1}(\top_x)(y) = \mathcal{H}(\top_y)(x)$. Let $\mathcal{J}, \mathcal{J}^{-1} : L^X \to L^X$ be meet preserving maps such that $\mathcal{J}^{-1}(\top_x^*)(y) = \mathcal{J}(\top_y^*)(x)$ and $\mathcal{H}(\top_x)(y) = \mathcal{J}^*(\top_x^*)(y)$. Define mappings $H^{\to}, J^{\to}, H^{\leftarrow}, J^{\leftarrow} : L^X \to L^X$ as follows:

$$H^{\to}(A, B) = e_{L^X}(A, \mathcal{H}(B)),$$
$$H^{\leftarrow}(A, B) = e_{L^X}(A, \mathcal{H}^{-1}(B))$$

$$J^{\to}(A, B) = e_{L^X}(\mathcal{J}(A), B),$$
$$J^{\leftarrow}(A, B) = e_{L^X}(\mathcal{J}^{-1}(A), B).$$

Then we have the following properties.

(1) $\mathcal{H}(A)(y) = \bigvee_x (A(x) \odot \mathcal{H}(\top_x)(y)$ and $\mathcal{J}(A)(y) = \bigwedge_x (A^*(x) \to \mathcal{J}(\top_x^*)(y))$.

(2) $H^{\to}(\top_x, B) = \mathcal{H}(B)(x),\ H^{\leftarrow}(\top_x, B) = \mathcal{H}^{-1}(B)(x)$ $H^{\to}(\top_x, \top_y) = \mathcal{H}(\top_y)(x),\ H^{\leftarrow}(\top_x, \top_y) = \mathcal{H}^{-1}(\top_y)(x)$.

(3) $J^{\to}(A, \top_x^*) = \mathcal{J}^*(B)(x), J^{\leftarrow}(A, \top_x^*) = \mathcal{J}^{-1*}(A)(x)$, $J^{\to}(\top_y^*, \top_x^*) = \mathcal{J}^*(\top_y^*)(x),\ J^{\leftarrow}(\top_y^*, \top_x^*) = \mathcal{J}^{-1*}(\top_y^*)(x)$.

(4) $\mathcal{H}(\top_x)(y) = \mathcal{J}^*(\top_x^*)(y)$ iff

$$H^{\to}(B, A) = J^{\to}(A^*, B^*)$$

iff

$$H^{\leftarrow}(A, B) = J^{\leftarrow}(B^*, A^*).$$

(5) $e_{L^X}(\mathcal{H}(A), B) = e_{L^X}(A, \mathcal{J}^{-1}(B))$, and

$$e_{L^X}(\mathcal{H}^{-1}(A), B) = e_{L^X}(A, \mathcal{J}(B)).$$

(6) $e_{L^X}(\mathcal{H}(\mathcal{J}^{-1}(B)), B) = e_{L^X}(A, \mathcal{J}^{-1}(\mathcal{H}(A))) = \top$, and

$$e_{L^X}(\mathcal{H}^{-1}(\mathcal{J}(B)), B) = e_{L^X}(A, \mathcal{J}(\mathcal{H}^{-1}(A))) = \top.$$

(7) $e_{L^X}(A, B) \le e_{L^X}(\mathcal{H}(A), \mathcal{H}(B))$ and $e_{L^X}(A, B) \le e_{L^X}(\mathcal{J}(A), \mathcal{J}(B))$.

(8) $e_{L^X}(A, B) \le e_{L^{L^X}}(H^{\to})_A$, and

$$(H^{\to})_B) = e_{L^X}(\mathcal{H}(A), \mathcal{H}(B)),$$

where $(H^{\to})_A(C) = H^{\to}(C, A)$.

(9) $e_{L^X}(B, A) \le e_{L^{L^X}}((J^{\to})^A$, and

$$(J^{\to})^B) = e_{L^X}(\mathcal{J}(B), \mathcal{J}(A)),$$

where $(J^{\to})^A(C) = J^{\to}(A, C)$.

(10) $(H_p)^{\to}(A, B) \le e_{L^{L^X}}((H^{\to})_A, (H^{\to})_B)$, where

$$\mathcal{H}_p(\top_x)(y) = e_{L^X}(\mathcal{H}(\top_y), \mathcal{H}(\top_x)).$$

(11) $(H_s)^{\leftarrow}(A, B) \le e_{L^{L^X}}((H^{\leftarrow})_A, (H^{\leftarrow})_B)$, where

$$\mathcal{H}_s(\top_x)(y) = e_{L^X}(\mathcal{H}^{-1}(\top_x), \mathcal{H}^{-1}(\top_y)).$$

(12) $(H_r)^{\to}(A, B) \le e_{L^{L^X}}((H^{\leftarrow})_A, (H^{\leftarrow})_B)$, where

$$\mathcal{H}_r(\top_x)(y) = e_{L^X}(\mathcal{H}^{-1}(\top_y), \mathcal{H}^{-1}(\top_x)).$$

(13) $(H_t)^{\leftarrow}(A, B) \le e_{L^{L^X}}((H^{\to})_A, (H^{\to})_B)$, where

$$\mathcal{H}_t(\top_x)(y) = e_{L^X}(\mathcal{H}(\top_x), \mathcal{H}(\top_y)).$$

Proof. (1) For $A = \bigvee_{x\in X}(A(x)\odot \top_x)$,

$$\begin{aligned}\mathcal{H}(A)(y) &= \mathcal{H}(\bigvee_{x\in X}(A(x)\odot \top_x))(y)\\ &= \bigvee_{x\in X}(A(x)\odot \mathcal{H}(\top_x)(y)).\end{aligned}$$

For $A = \bigwedge_{x\in X}(A^*(x)\to \top^*_x)$,

$$\begin{aligned}\mathcal{J}(A)(y) &= \mathcal{J}(\bigwedge_{x\in X}(A^*(x)\to \top^*_x))(y)\\ &= \bigwedge_{x\in X}(A^*(x)\to \mathcal{H}(\top^*_x)(y)).\end{aligned}$$

(2) $H^{\to}(\top_x, B) = \bigwedge_{y\in X}(\top_x(y)\to \mathcal{H}(B)(y)) = \mathcal{H}(B)(x)$. Other cases are similarly proved.

(3) $J^{\to}(B, \top^*_x) = \bigwedge_{y\in X}(\mathcal{J}(B)(y)\to \top^*_x(y)) = \mathcal{J}^*(B)(x)$. Other cases are similarly proved.

(4) Let $\mathcal{H}(\top_x)(y) = \mathcal{J}^*(\top^*_x)(y)$. Then $H^{\to}(B, A) = J^{\to}(A^*, B^*)$ from:

$$\begin{aligned}&H^{\to}(B, A) = e_{L^X}(B, \mathcal{H}(A)) = e_{L^X}(\mathcal{H}^*(A), B^*)\\ &= \bigwedge_{y\in X}(\mathcal{H}^*(A)(y)\to B^*(y))\\ &= \bigwedge_{y\in X}(\bigwedge_{x\in X}(A(x)\to \mathcal{H}^*(\top_x)(y))\to B^*(y))\\ &= \bigwedge_{y\in X}(\bigwedge_{x\in X}(A(x)\to \mathcal{J}(\top^*_x)(y))\to B^*(y))\\ &= \bigwedge_{y\in X}(\mathcal{J}(\bigwedge_{x\in X}(A(x)\to \top^*_x))(y)\to B^*(y))\\ &= \bigwedge_{y\in X}(\mathcal{J}(A^*)(y)\to B^*(y))\\ &= e_{L^X}(\mathcal{J}(A^*), B^*) = J^{\to}(A^*, B^*)\end{aligned}$$

Let $H^{\to}(A, B) = J^{\to}(B^*, A^*)$. Put $A = \top_y$ and $B = \top_x$.

$$\mathcal{H}(\top_x)(y) = H^{\to}(\top_y, \top_x) = J^{\to}(\top^*_x, \top^*_y) = \mathcal{J}^*(\top^*_x)(y).$$

Other case follows from: $\mathcal{H}^{-1}(\top_x)(y) = \mathcal{J}^{-1*}(\top^*_x)(y)$ iff $H^{\leftarrow}(A, B) = J^{\leftarrow}(B^*, A^*)$.

(5)

$$\begin{aligned}&e_{L^X}(\mathcal{H}(A), B)\\ &= \bigwedge_{y\in X}(\mathcal{H}(A)(y)\to B(y))\\ &= \bigwedge_{y\in X}(\bigvee_{x\in X}(A(x)\odot \mathcal{H}(\top_x)(y))\to B(y))\\ &= \bigwedge_{y\in X}\bigwedge_{x\in X}(A(x)\to (\mathcal{J}^{-1*}(\top^*_y)(x))\to B^*(y))\\ &= \bigwedge_{x\in X}(A(x)\to \bigwedge_{y\in X}(B(y)\to \mathcal{J}^{-1}(\top^*_y)(x)))\\ &= \bigwedge_{x\in X}(A(x)\to \mathcal{J}^{-1}(\bigwedge_{y\in X}(B(y)\to \top^*_y)(x)))\\ &= \bigwedge_{x\in X}(A(x)\to \mathcal{J}^{-1}(B)(x))\\ &= e_{L^X}(A, \mathcal{J}^{-1}(B))\end{aligned}$$

(6) By (5), $e_{L^X}(\mathcal{H}(A), \mathcal{H}(A)) = e_{L^X}(A, \mathcal{J}^{-1}(\mathcal{H}(A))) = \top$ and $e_{L^X}(\mathcal{H}^{-1}(A), B) = e_{L^X}(A, \mathcal{J}(\mathcal{H}^{-1}(A))) = \top$.

(7) Since $e_{L^X}(A, B)\odot A(x)\odot \mathcal{H}(\top_x)(y) \le B(x)\odot \mathcal{H}(\top_x)(y)$, $e_{L^X}(A, B) \le e_{L^X}(\mathcal{H}(A), \mathcal{H}(B))$.

Since $(B^*(x)\to A^*(x))\odot(A^*(x)\to \mathcal{J}(\top^*_x)(y)) \le A^*(x)\to \mathcal{J}(\top^*_x)(y)$, $e_{L^X}(A, B) \le e_{L^X}(\mathcal{J}(A), \mathcal{J}(B))$.

(8) Since $(A(x)\to B(x))\odot(C(y)\to A(x)\odot \mathcal{H}(\top_x)(y))\odot C(y) \le B(x)\odot \mathcal{H}(\top_x)(y)$ iff $(A(x)\to B(x)) \le \Big((C(y)\to A(x)\odot \mathcal{H}(\top_x)(y))\Big)\to \Big(C(y)\to B(x)\odot \mathcal{H}(\top_x)(y)\Big)$, we have $e_{L^X}(A, B) \le e_{L^{L^X}}((H^{\to})_A, (H^{\to})_B)$.

$$\begin{aligned}&e_{L^{L^X}}((H^{\to})^A, (H^{\to})^B)\\ &= \bigwedge_{C\in L^X}((H^{\to})^A(C)\to (H^{\to})^B(C))\\ &= \bigwedge_{C\in L^X}(H^{\to}(A, C)\to H^{\to}(B, C))\\ &\le \bigwedge_{x\in X}(H^{\to}(A, \top^*_x)\to H^{\to}(B, \top^*_x))\\ &= \bigwedge_{x\in X}(\mathcal{H}^*(A)(x)\to \mathcal{H}^*(B)(x))\\ &= \bigwedge_{x\in X}(\mathcal{H}(B)(x)\to \mathcal{H}(A)(x))\end{aligned}$$

$$\begin{aligned}&\bigwedge_{x\in X}(\mathcal{H}^*(A)(x)\to \mathcal{H}^*(B)(x))\\ &\odot \bigwedge_{x\in X}(C^*(x)\to \mathcal{H}^*(A)(x))\odot C^*(x)\\ &\le (\mathcal{H}^*(A)(x)\to \mathcal{H}^*(B)(x))\\ &\odot(C^*(x)\to \mathcal{H}^*(A)(x))\odot C^*(x)\\ &\le (\mathcal{H}^*(A)(x)\to \mathcal{H}^*(B)(x))\odot \mathcal{H}^*(A)(x) \le \mathcal{H}^*(B)(x)\end{aligned}$$

$$\begin{aligned}&\bigwedge_{x\in X}(\mathcal{H}^*(A)(x)\to \mathcal{H}^*(B)(x))\\ &\le \bigwedge_{x\in X}(C^*(x)\to \mathcal{H}^*(A)(x))\\ &\to \bigwedge_{x\in X}(C^*(x)\to \mathcal{H}^*(A)(x))\\ &= H^{\to}(A, C)\to H^{\to}(B, C)\end{aligned}$$

(9) Since $(B^*(x)\to A^*(x))\odot(C^*(y)\to B^*(x)\odot \mathcal{J}^*(\top^*_x)(y))\odot C^*(y) \le A^*(x)\odot \mathcal{J}^*(\top^*_x)(y)$ iff $(A(x)\to B(x)) \le \Big((C^*(y)\to B^*(x)\odot \mathcal{J}^*(\top^*_x)(y))\Big)\to \Big(C^*(y)\to A^*(x)\odot \mathcal{J}^*(\top^*_x)(y)\Big)$, we have $e_{L^X}(A, B) \le e_{L^{L^X}}((J^{\to})^B, (J^{\to})^A)$.

$$\begin{aligned}&e_{L^{L^X}}((J^{\to})^A, (J^{\to})^B)\\ &= \bigwedge_{C\in L^X}((J^{\to})^A(C)\to (J^{\to})^B(C))\\ &= \bigwedge_{C\in L^X}(J^{\to}(A, C)\to J^{\to}(B, C))\\ &= \bigwedge_{C\in L^X}(e_{L^X}(\mathcal{J}_R(A), C)\to e_{L^X}(\mathcal{J}_R(B), C))\\ &\ge e_{L^X}(\mathcal{J}_R(B), \mathcal{J}_R(A)) \ge e_{L^X}(B, A).\end{aligned}$$

$$\begin{aligned}&e_{L^{L^X}}((J^{\to})^A, (J^{\to})^B)\\ &= \bigwedge_{C\in L^X}((J^{\to})^A(C)\to (J^{\to})^B(C))\\ &= \bigwedge_{C\in L^X}(J^{\to}(A, C)\to J^{\to}(B, C))\\ &\le \bigwedge_{x\in X}(J^{\to}(A, \top^*_x)\to J^{\to}(B, \top^*_x))\\ &= \bigwedge_{x\in X}(\mathcal{J}^*_R(A)(x)\to \mathcal{J}^*_R(B)(x))\\ &= \bigwedge_{x\in X}(\mathcal{J}_R(B)(x)\to \mathcal{J}_R(A)(y))\\ &= e_{L^X}(\mathcal{J}_R(B), \mathcal{J}_R(A)).\end{aligned}$$

(10)

$$
\begin{aligned}
&(H_p)^{\rightarrow}(A,B)\odot(H^{\rightarrow})_A(C)\odot C(z)\\
&=\bigwedge_{x\in X}(A(x)\rightarrow\bigvee_y(B(y)\odot\mathcal{H}_p(\top_y)(x)))\\
&\odot(H^{\rightarrow})(C,A)\odot C(z)\\
&\le(A(x)\rightarrow\bigvee_y(B(y)\odot\mathcal{H}_p(\top_y)(x)))\odot\\
&(C(z)\rightarrow\bigvee_x(A(x)\odot\mathcal{H}(\top_x)(z)))\odot C(z)\\
&\le\bigvee_x(A(x)\rightarrow\bigvee_y(B(y)\odot\mathcal{H}_p(\top_y)(x)))\\
&\odot(A(x)\odot\mathcal{H}(\top_x)(z)))
\end{aligned}
$$

$$
\begin{aligned}
&\le\bigvee_x\bigvee_y(B(y)\odot\mathcal{H}_p(\top_y)(x)))\odot\mathcal{H}(\top_x)(z)))\\
&\le\bigvee_x\bigvee_y(B(y)\odot\bigwedge_w(\mathcal{H}(\top_x)(w)\rightarrow\mathcal{H}(\top_y)(w)))\\
&\odot\mathcal{H}(\top_x)(z)))\\
&\le\bigvee_x\bigvee_y(B(y)\odot(\mathcal{H}(\top_x)(z)\rightarrow\mathcal{H}(\top_y)(z)))\\
&\odot\mathcal{H}(\top_x)(z)))\\
&\le\bigvee_y(B(y)\odot\mathcal{H}(\top_y)(z)))=\mathcal{H}(B)(z).
\end{aligned}
$$

(11)

$$
\begin{aligned}
&(H_s)^{\leftarrow}(A,B)\odot(H^{\leftarrow})_A(C)\odot C(z)\\
&=\bigwedge_{x\in X}(A(x)\rightarrow\bigvee_y(B(y)\odot\mathcal{H}_s(\top_x)(y)))\\
&\odot(H^{\leftarrow})(C,A)\odot C(z)\\
&\le(A(x)\rightarrow\bigvee_y(B(y)\odot\mathcal{H}_s(\top_x)(y)))\odot\\
&(C(z)\rightarrow\bigvee_x(A(x)\odot\mathcal{H}(\top_z)(x)))\odot C(z)\\
&\le\bigvee_x(A(x)\rightarrow\bigvee_y(B(y)\odot\mathcal{H}_s(\top_x)(y)))\odot\\
&(A(x)\odot\mathcal{H}(\top_z)(x)))\\
&\le\bigvee_x\bigvee_y(B(y)\odot\mathcal{H}_s(\top_x)(y)))\odot\mathcal{H}(\top_z)(x)))\\
&\le\bigvee_x\bigvee_y(B(y)\odot\bigwedge_w(\mathcal{H}(\top_w)(x)\rightarrow\mathcal{H}(\top_w)(y)))\\
&\odot\mathcal{H}(\top_z)(x)))\\
&\le\bigvee_x\bigvee_y(B(y)\odot(\mathcal{H}(\top_z)(x)\rightarrow\mathcal{H}(\top_z)(y)))\\
&\odot\mathcal{H}(\top_z)(x)))\\
&\le\bigvee_y(B(y)\odot\mathcal{H}(\top_z)(y)))=\mathcal{H}(B)(z).
\end{aligned}
$$

(12) and (13) are similarly proved.

Definition 2.2. [12] (1) A join preserving map $\mathcal{H}:L^X\rightarrow L^X$ is called an *L-upper approximation operator* iff it satisfies the following conditions

(H1) $A\le\mathcal{H}(A)$,

(H2) $\mathcal{H}(\mathcal{H}(A))\le\mathcal{H}(A)$.

(2) A meet preserving map $\mathcal{J}:L^X\rightarrow L^X$ is called an *L-lower approximation operator* iff it satisfies the following conditions

(J1) $\mathcal{J}(A)\le A$,

(J2) $\mathcal{J}(\mathcal{J}(A))\ge\mathcal{J}(A)$.

Example 2.3. Let $R\in L^{X\times X}$ be a fuzzy relation. Define $\mathcal{H},\mathcal{J}:L^X\rightarrow L^X$ as follows

$$\mathcal{H}(A)(y)=\bigvee_{x\in X}(A(x)\odot R(x,y))$$

$$\mathcal{J}(A)(y)=\bigwedge_{x\in X}(R(x,y)\rightarrow A(x)).$$

(1) Since $\mathcal{H}(\alpha\odot A)=\alpha\odot\mathcal{H}(A)$ and $\mathcal{H}(\bigvee_{i\in\Gamma}A_i)=\bigvee_{i\in\Gamma}\mathcal{H}(A_i)$, $\mathcal{H}$ is a join preserving map.

(2) Since $\mathcal{J}(\alpha\rightarrow A)=\alpha\rightarrow\mathcal{J}(A)$ and $\mathcal{J}(\bigwedge_{i\in\Gamma}A_i)=\bigwedge_{i\in\Gamma}\mathcal{J}(A_i)$, $\mathcal{J}$ is a meet preserving map.

(3) [5, 9, 12] If R is an L-fuzzy preorder, then $\mathcal{H}$ and $\mathcal{J}$ are L-upper and L-lower approximation operators, respectively.

Theorem 2.4. Let $\mathcal{H},\mathcal{J}:L^X\rightarrow L^X$ be L-upper and L-lower approximation operators, respectively. The following statements hold.

(1) $A(x)=e_{L^X}(\mathcal{H}(\top_x),A)$ for each $A=\mathcal{H}(A)$.

(2) $\mathcal{H}(\top_y)(x)=e_{L^X}(\mathcal{H}(\top_x),\mathcal{H}(\top_y))$.

(3) $A(x)=e_{L^X}(A^*,\mathcal{J}(\top_x^*))$ for each $A^*=\mathcal{J}(A^*)$.

(4) $\mathcal{J}^*(\top_y^*)(x)=e_{L^X}(\mathcal{J}(\top_y^*),\mathcal{J}(\top_x^*))$.

Proof. (1)

$$
\begin{aligned}
&e_{L^X}(\mathcal{H}(\top_x),A)=\bigwedge_{y\in X}(\mathcal{H}(\top_x)(y)\rightarrow A(y))\\
&\le\mathcal{H}(\top_x)(x)\rightarrow A(x))=A(x).
\end{aligned}
$$

$$
\begin{aligned}
&e_{L^X}(\mathcal{H}(\top_x),\mathcal{H}(A))=\bigwedge_{y\in X}(\mathcal{H}(\top_x)(y)\rightarrow\mathcal{H}(A)(y))\\
&=\bigwedge_{y\in X}(\mathcal{H}(\top_x)(y)\rightarrow\bigvee_{z\in X}(A(z)\odot\mathcal{H}(\top_z)(y))\\
&\ge\bigwedge_{y\in X}(\mathcal{H}(\top_x)(y)\rightarrow(A(x)\odot\mathcal{H}(\top_x)(y))\ge A(x).
\end{aligned}
$$

(2) Since $\mathcal{H}(\top_y)=\mathcal{H}(\mathcal{H}(\top_y))$, by (1),

$$\mathcal{H}(\top_y)(x)=e_{L^X}(\mathcal{H}(\top_x),\mathcal{H}(\top_y)).$$

(3)

$$
\begin{aligned}
&e_{L^X}(A^*,\mathcal{J}(\top_x^*))=\bigwedge_{y\in X}(A^*(y)\rightarrow\mathcal{J}(\top_x^*)(y))\\
&\le A^*(x)\rightarrow\top_x^*(x)=A(x).
\end{aligned}
$$

$$
\begin{aligned}
&e_{L^X}(\mathcal{J}(A^*),\mathcal{J}(\top_x^*))\\
&=\bigwedge_{y\in X}(\bigwedge(A(z)\rightarrow\mathcal{J}(\top_z^*)(y))\rightarrow\mathcal{J}(\top_x^*)(y))\\
&\ge\bigwedge_{y\in X}((A(x)\rightarrow\mathcal{J}(\top_x^*)(y))\rightarrow\mathcal{J}(\top_x^*)(y))\\
&\ge A(x).
\end{aligned}
$$

(4) Since $\mathcal{J}(\top_y^*)=\mathcal{J}(\mathcal{J}(\top_y^*))$, by (3),

$$\mathcal{J}^*(\top_y^*)(x)=e_{L^X}(\mathcal{J}(\top_y^*),\mathcal{J}(\top_x^*)).$$

Theorem 2.5. Let $\mathcal{H}, \mathcal{H}^{-1} : L^X \to L^X$ be L-join preserving operators. The following statements are equivalent.

(1) $\top_x \leq \mathcal{H}(\top_x)$ and $\mathcal{H}(\mathcal{H}(\top_x)) \leq \mathcal{H}(\top_x)$ for all $x \in X$.

(2) $\top_x \leq \mathcal{H}(\top_x)$ and $\mathcal{H}(\top_x)(y) = e_{L^X}(\mathcal{H}(\top_y), \mathcal{H}(\top_x))$.

(3) There exists an L-fuzzy preorder $R_H \in L^{X\times X}$ such that

$$\mathcal{H}(A)(y) = \bigvee_{x\in X} (A(x) \odot R_H(x,y)).$$

(4) $\top_x \leq \mathcal{H}^{-1}(\top_x)$ and $\mathcal{H}^{-1}(\mathcal{H}^{-1}(\top_x)) = \mathcal{H}^{-1}(\top_x)$ for all $x \in X$.

(5) $\top_x \leq \mathcal{H}(\top_x)$ and

$$\mathcal{H}^{-1}(\top_y)(x) = e_{L^X}(\mathcal{H}^{-1}(\top_x), \mathcal{H}^{-1}(\top_y)).$$

(6) There exists an L-fuzzy preorder $R_{H^{-1}} \in L^{X\times X}$ such that

$$\mathcal{H}(A)(y) = \bigvee_{x\in X} (A(x) \odot R_{H^{-1}}(x,y)).$$

(7) $\mathcal{H}$ is an L-upper approximation operator.

(8) $\mathcal{H}^{-1}$ is an L-upper approximation operator.

(9) $H^{\to}(A,B) = e_{L^X}(\mathcal{H}(A), \mathcal{H}(B))$, for $A, B \in L^X$.

(10) $H^{\leftarrow}(A,B) = e_{L^X}(\mathcal{H}^{-1}(A), \mathcal{H}^{-1}(B))$, for $A, B \in L^X$.

(11) $H^{\to}$ is an L-fuzzy preorder on L^X.

(12) $H^{\leftarrow}$ is an L-fuzzy preorder on L^X.

Proof. (1) ⇔ (2). For $\mathcal{H}(\top_x) = \bigvee_{y\in X}(\mathcal{H}(\top_x)(y) \odot \top_y)$, we have

$$\begin{aligned}&\mathcal{H}(\mathcal{H}(\top_x))(z)\\ &= \mathcal{H}(\textstyle\bigvee_{y\in X}(\mathcal{H}(\top_x)(y) \odot \top_y))(z)\\ &= \textstyle\bigvee_{y\in X}(\mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_y)(z)) \leq \mathcal{H}(\top_x)(z).\end{aligned}$$

$$\begin{aligned}&\mathcal{H}(\top_x)(y) \leq \textstyle\bigwedge_{z\in X} \mathcal{H}(\top_y)(z)) \to \mathcal{H}(\top_x)(z)\\ &\leq \mathcal{H}(\top_y)(y)) \to \mathcal{H}(\top_x)(y) = \mathcal{H}(\top_x)(y).\end{aligned}$$

Hence $\mathcal{H}(\top_x)(y) = e_{L^X}(\mathcal{H}(\top_y), \mathcal{H}(\top_x))$. Conversely, it is similarly proved.

(1) ⇔ (3). Put $R_H(x,y) = \mathcal{H}(\top_x)(y)$. Since $\top = \top_x(x) \leq \mathcal{H}(\top_x)(x) = R_H(x,x)$ and

$$\begin{aligned}\mathcal{H}(\mathcal{H}(\top_x))(z) &= \bigvee_{y\in X} (\mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_y)(z))\\ &= \bigvee_{y\in X} (R_H(x,y) \odot R_H(y,z)) \leq \mathcal{H}(\top_x)(z)\\ &= R_H(x,z)\end{aligned}$$

for all $x, y \in X$.

$$\begin{aligned}&\mathcal{H}(\mathcal{H}(\top_x))(z) = \textstyle\bigvee_{y\in X}(\mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_y)(z))\\ &= \textstyle\bigvee_{y\in X}(R_H(x,y) \odot R_H(y,z))\\ &\leq \mathcal{H}(\top_x)(z) = R_H(x,z).\end{aligned}$$

Conversely, since $R_H(x,y) = \mathcal{H}(\top_x)(y)$, it is similarly proved.

(1) ⇔ (4). It follows from:

$$\begin{aligned}&\mathcal{H}(\mathcal{H}(\top_x))(z)\\ &= \mathcal{H}(\textstyle\bigvee_{y\in X}(\mathcal{H}(\top_x)(y) \odot \top_y))(z)\\ &= \textstyle\bigvee_{y\in X}(\mathcal{H}^{-1}(\top_y)(x) \odot \mathcal{H}^{-1}(\top_z)(y))\\ &= \mathcal{H}^{-1}(\mathcal{H}^{-1}(\top_z))(x)\\ &\leq \mathcal{H}(\top_x)(z) = \mathcal{H}^{-1}(\top_z)(x).\end{aligned}$$

(4) ⇔ (5) and (4) ⇔ (6) are similarly proved (1) ⇔ (2) and (1) ⇔ (3), respectively.

(1) ⇔ (7).

$$\begin{aligned}&\mathcal{H}(\mathcal{H}(A))(z)\\ &= \textstyle\bigvee_{y\in X}(\mathcal{H}(A)(y) \odot \mathcal{H}(\top_y)(z))\\ &= \textstyle\bigvee_{y\in X}(\bigvee_{x\in X}(A(x) \odot \mathcal{H}(\top_x)(y)) \odot \mathcal{H}(\top_y)(z))\\ &= \textstyle\bigvee_{x\in X}(A(x) \odot \bigvee_{y\in X}(\mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_y)(z)))\\ &\leq \textstyle\bigvee_{x\in X}(A(x) \odot \mathcal{H}(\top_x)(z)) = \mathcal{H}(A)(z))\end{aligned}$$

(7) ⇔ (9).

$$\begin{aligned}&e_{L^X}(\mathcal{H}(A), \mathcal{H}(B))\\ &= e_{L^X}(A, \mathcal{H}(A)) \odot e_{L^X}(\mathcal{H}(A), \mathcal{H}(B))\\ &\leq e_{L^X}(A, \mathcal{H}(B)) = H^{\to}(A,B)\\ &\leq e_{L^X}(\mathcal{H}(A), \mathcal{H}(\mathcal{H}(B)))\\ &= e_{L^X}(\mathcal{H}(A), \mathcal{H}(B)\end{aligned}$$

(9) ⇒ (11).

$$\begin{aligned}&H^{\to}(A,B) \odot H^{\to}(B,C)\\ &= e_{L^X}(\mathcal{H}(A), \mathcal{H}(B)) \odot e_{L^X}(\mathcal{H}(B), \mathcal{H}(C))\\ &\leq e_{L^X}(\mathcal{H}(A), \mathcal{H}(C)) = H^{\to}(A,C).\end{aligned}$$

(11) ⇒ (1). Since $\top = H^{\to}(\top_x, \top_x)$ for all $x \in X$, then $\top_x \leq \mathcal{H}(\top_x)$ for all $x \in X$.

$$\begin{aligned}&H^{\to}(\top_z, \top_y) \odot H^{\to}(\top_y, \top_x)\\ &= e_{L^X}(\top_z, \mathcal{H}(\top_y)) \odot e_{L^X}(\top_y, \mathcal{H}(\top_x))\\ &= \mathcal{H}(\top_y)(z) \odot \mathcal{H}(\top_x)(y)\\ &\leq e_{L^X}(\top_z, \mathcal{H}(\top_x)) = \mathcal{H}(\top_x)(z).\end{aligned}$$

Other cases are similarly proved.

Theorem 2.6. Let $\mathcal{J}, \mathcal{J}^{-1} : L^X \to L^X$ be L-meet preserving operators. The following statements are equivalent.

(1) $\mathcal{J}(\top_x^*) \le \top_x^*$ and $\mathcal{J}(\top_x^*)(z) \le \mathcal{J}(\mathcal{J}(\top_x^*))(z)$ for all $x, z \in X$.

(2) $\mathcal{J}^*(\top_y^*)(x) = e_{L^X}(\mathcal{J}(\top_y^*), \mathcal{J}(\top_x^*))$.

(3) There exists an L-fuzzy preorder $R_J \in L^{X\times X}$ such that

$$\mathcal{J}(A)(y) = \bigwedge_{x\in X}(R_J(x,y) \to A(x)).$$

(4) $\mathcal{J}^{-1}(\top_x^*) \le \top_x^*$ and $\mathcal{J}^{-1}(\top_x^*)(z) \le \mathcal{J}^{-1}(\mathcal{J}^{-1}(\top_x^*))(z))$ for all $x, z \in X$.

(5) $\mathcal{J}^{-1*}(\top_y^*)(x) = e_{L^X}(\mathcal{J}^{-1}(\top_y^*), \mathcal{J}^{-1}(\top_x^*))$.

(6) There exists an L-fuzzy preorder $R_{J^{-1}} \in L^{X\times X}$ such that

$$\mathcal{J}^{-1}(A)(y) = \bigwedge_{x\in X}(R_{J^{-1}}(x,y) \to A(x)).$$

(7) $\mathcal{J}$ is an L-lower approximation operator.

(8) $\mathcal{J}^{-1}$ is an L-lower approximation operator.

(9) $J^{\to}(A,B) = e_{L^X}(\mathcal{J}(A), \mathcal{J}(B))$, for $A, B \in L^X$.

(10) $J^{\leftarrow}(A,B) = e_{L^X}(\mathcal{J}^{-1}(A), \mathcal{J}^{-1}(B))$, for $A, B \in L^X$.

(11) $J^{\to}$ is an L-fuzzy preorder on L^X.

(12) $J^{\leftarrow}$ is an L-fuzzy preorder on L^X.

Proof. (1) $\Leftrightarrow$ (2). For $\mathcal{J}(\top_x^*) = \bigwedge_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \to \top_y^*)$, we have

$$\begin{aligned}
&\mathcal{J}(\mathcal{J}(\top_x^*))(z)\\
&= \mathcal{J}(\textstyle\bigwedge_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \to \top_y^*))(z)\\
&= \textstyle\bigwedge_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \to \mathcal{J}(\top_y^*)(z)) \ge \mathcal{J}(\top_x^*)(z).
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}^*(\top_x^*)(y) &\le \textstyle\bigwedge_{z\in X} \mathcal{J}(\top_x^*)(z) \to \mathcal{J}(\top_y^*)(z)\\
&\le \mathcal{J}(\top_x^*)(y) \to \mathcal{J}(\top_y^*)(y) = \mathcal{J}^*(\top_x^*)(y).
\end{aligned}$$

Hence $\mathcal{J}^*(\top_y^*)(x) = e_{L^X}(\mathcal{J}(\top_y^*), \mathcal{J}(\top_x^*))$. Conversely, it is similarly proved.

(1) $\Leftrightarrow$ (3). Put $R_J(x,y) = \mathcal{J}^*(\top_x^*)(y)$. Since $\top = \top_x(x) \le \mathcal{J}^*(\top_x^*)(x) = R_J(x,x)$ and

$$\mathcal{J}(\mathcal{J}(\top_x^*))(z) = \bigwedge_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \to \mathcal{J}(\top_y^*)(z)) \ge \mathcal{J}(\top_x^*)(z)$$

iff $\bigvee_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \odot \mathcal{J}^*(\top_y^*)(z)) = \bigvee_{y\in X}(R_J(x,y) \odot R_J(y,z)) \le \mathcal{J}^*(\top_x^*)(z) = R_J(x,z)$ for all $x, y \in X$.

Conversely, since $R_J(x,y) = \mathcal{J}^*(\top_x^*)(y)$, it is similarly proved.

(1) $\Leftrightarrow$ (4).

$$\begin{aligned}
&\mathcal{J}(\mathcal{J}(\top_x^*))(z)\\
&= \textstyle\bigwedge_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \to \mathcal{J}(\top_y^*)(z))\\
&= \textstyle\bigwedge_{y\in X}(\mathcal{J}^{-1*}(\top_y^*)(x) \to \mathcal{J}^{-1}(\top_z^*)(y))\\
&= \textstyle\bigwedge_{y\in X}(\mathcal{J}^{-1*}(\top_z^*)(y) \to \mathcal{J}^{-1}(\top_y^*)(x))\\
&= \mathcal{J}^{-1}(\mathcal{J}^{-1}(\top_x^*))(z) \ge \mathcal{J}(\top_x^*)(z) = \mathcal{J}^{-1}(\top_z^*)(x).
\end{aligned}$$

(4) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6) are similarly proved (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3), respectively.

(1) $\Leftrightarrow$ (7). Since $\mathcal{J}(A)^*(y) = (\bigwedge_{x\in X}(A^*(x) \to \mathcal{J}(\top_x^*)(y)))^* = \bigvee_{x\in X}(A^*(x) \odot \mathcal{J}^*(\top_x^*)(y))$, we have

$$\begin{aligned}
&\mathcal{J}(\mathcal{J}(A))(z)\\
&= \textstyle\bigwedge_{y\in X}(\mathcal{J}^*(A)(y) \to \mathcal{J}(\top_y^*)(z))\\
&= \textstyle\bigwedge_{y\in X}(\bigvee_{x\in X}(A^*(x) \odot \mathcal{J}^*(\top_x^*)(y)) \to \mathcal{J}(\top_y^*)(z))\\
&= \textstyle\bigwedge_{y\in X}\bigwedge_{x\in X}(A^*(x) \to (\mathcal{J}^*(\top_x^*)(y) \to \mathcal{J}(\top_y^*)(z))\\
&= \textstyle\bigwedge_{x\in X}(A^*(x) \to \bigwedge_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \to \mathcal{J}(\top_y^*)(z))\\
&\ge \textstyle\bigwedge_{x\in X}(A^*(x) \to \mathcal{J}(\top_x^*)(z)) = \mathcal{J}(A)(z).
\end{aligned}$$

(7) $\Rightarrow$ (9).

$$\begin{aligned}
&e_{L^X}(\mathcal{J}(A), \mathcal{J}(B))\\
&= e_{L^X}(\mathcal{J}(A), \mathcal{J}(B)) \odot e_{L^X}(\mathcal{J}(B), B)\\
&\le e_{L^X}(\mathcal{J}(A), B) = J^{\to}(A,B)\\
&\le e_{L^X}(\mathcal{J}(A), \mathcal{J}(\mathcal{J}(B)))\\
&= e_{L^X}(\mathcal{J}(A), \mathcal{J}(B))
\end{aligned}$$

(9) $\Rightarrow$ (11).

$$\begin{aligned}
&J^{\to}(A,B) \odot J^{\to}(B,C)\\
&= e_{L^X}(\mathcal{J}(A), \mathcal{J}(B)) \odot e_{L^X}(\mathcal{J}(B), \mathcal{J}(C))\\
&\le e_{L^X}(\mathcal{J}(A), \mathcal{J}(C)) = J^{\to}(A,C).
\end{aligned}$$

(11) $\Rightarrow$ (1). Since $\top = J^{\to}(\top_x^*, \top_x^*)$ for all $x \in X$, then $\mathcal{J}(\top_x^*) \le \top_x^*$ for all $x \in X$.

$$\begin{aligned}
&J^{\to}(\top_x^*, \top_y^*) \odot J^{\to}(\top_y^*, \top_z^*)\\
&= e_{L^X}(\mathcal{J}(\top_x^*), \top_y^*) \odot e_{L^X}(\mathcal{J}(\top_y^*), \top_z^*)\\
&= \mathcal{J}^*(\top_x^*)(y) \odot \mathcal{J}^*(\top_y^*)(z)\\
&\le e_{L^X}(\mathcal{J}(\top_x^*), \top_z^*) = \mathcal{J}^*(\top_x^*)(z).
\end{aligned}$$

Other cases are similarly proved.

Example 2.7. Let $(L = [0,1], \odot, \to, ^*)$ be a complete residuated lattice with the law of double negation defined by

$$x \odot y = (x+y-1) \vee 0, \quad x \to y = (1-x+y) \wedge 1, \quad x^* = 1-x.$$

Let $X = \{x, y, z\}$ and $A, B \in L^X$ as follows:

$$A(x) = 0.9, A(y) = 0.8, A(z) = 0.3,$$

$$B(x) = 0.3, A(y) = 0.7, A(z) = 0.8$$

Define $\mathcal{H}(1_x)(y) = \mathcal{J}^*(1_x^*)(y)$ as follows

$$\begin{array}{ccc} \mathcal{H}(1_x)(x) = 1 & \mathcal{H}(1_x)(y) = 0.8 & \mathcal{H}(1_x)(z) = 0.6 \\ \mathcal{H}(1_y)(x) = 0.7 & \mathcal{H}(1_y)(y) = 1 & \mathcal{H}(1_y)(z) = 0.3 \\ \mathcal{H}(1_z)(x) = 0.5 & \mathcal{H}(1_x)(y) = 0.6 & \mathcal{H}(1_x)(y) = 1 \end{array}$$

(1) Since $\mathcal{H}(\mathcal{H}(1_x))(z) = \bigvee_{y\in X}(\mathcal{H}(1_x)(y) \odot \mathcal{H}(1_y)(z) = \mathcal{H}(1_x)(z)$ and $1_x \leq \mathcal{H}(1_x)$ for all $x, y \in X$, then $\mathcal{H}$ is an L-upper approximation operator. Since $\mathcal{H}(A)(y) = \bigvee_{x\in X}(A(x)\odot (\mathcal{H}(1_x)(y))$, we have

$$\mathcal{H}(A) = (0.9, 0.8, 0.5),\ \mathcal{H}(B) = (0.4, 0.7, 0.8).$$

Moreover, by Theorem 2.12, $e_{L^X}(A, B) = 0.4$ and

$$\begin{aligned} H^{\rightarrow}(A, B) &= e_{L^X}(A, \mathcal{H}(B)) \\ &= e_{L^X}(\mathcal{H}(A), \mathcal{H}(B)) = 0.5. \end{aligned}$$

(2) Since $\mathcal{J}(\mathcal{J}(1_x^*))(z) = \bigwedge_{y\in X}(\mathcal{J}^*(1_x^*)(y) \rightarrow \mathcal{J}(1_y^*)(z) = \mathcal{J}(1_x^*)(z)$ and $1_x \leq \mathcal{J}^*(1_x^*)$ for all $x, y \in X$, then $\mathcal{J}$ is an L-lower approximation operator. Since

$$\mathcal{J}(A)(y) = \bigwedge_{x\in X}(A^*(x) \rightarrow (\mathcal{J}(1_x^*)(y)),$$

we have

$$\mathcal{J}(A) = (0.8, 0.7, 0.3),\ \mathcal{J}(B) = (0.3, 0.5, 0.7),$$

$$\begin{aligned} J^{\rightarrow}(A, B) &= e_{L^X}(\mathcal{J}(A), B) \\ &= e_{L^X}(\mathcal{J}(A), \mathcal{J}(B)) = 0.5. \end{aligned}$$

(3) We obtain $\mathcal{H}^{-1}(1_x)(y) = \mathcal{H}(1_y)(x) = \mathcal{J}^{-1*}(1_x^*)(y)$ as follows

$$\begin{array}{ccc} \mathcal{H}^{-1}(1_x)(x) = 1 & \mathcal{H}^{-1}(1_x)(y) = 0.7 & \mathcal{H}^{-1}(1_x)(z) = 0.5 \\ \mathcal{H}^{-1}(1_y)(x) = 0.8 & \mathcal{H}^{-1}(1_y)(y) = 1 & \mathcal{H}^{-1}(1_y)(z) = 0.6 \\ \mathcal{H}^{-1}(1_z)(x) = 0.6 & \mathcal{H}^{-1}(1_x)(y) = 0.3 & \mathcal{H}^{-1}(1_x)(y) = 1 \end{array}$$

Since $\bigvee_{y\in X}(\mathcal{H}^{-1}(1_x)(y) \odot \mathcal{H}^{-1}(1_y)(z) = \mathcal{H}^{-1}(1_x)(z)$ and $1_x \leq \mathcal{H}^{-1}(1_x)$ for all $x, y \in X$, then $\mathcal{H}^{-1}$ is an L-upper approximation operator. Since $\mathcal{H}^{-1}(A)(y) = \bigvee_{x\in X}(A(x) \odot (\mathcal{H}^{-1}(1_x)(y))$, we have

$$\mathcal{H}^{-1}(A) = (0.9, 0.8, 0.4),\ \mathcal{H}^{-1}(B) = (0.5, 0.7, 0.8).$$

$$\begin{aligned} H^{\leftarrow}(A, B) &= e_{L^X}(A, \mathcal{H}^{-1}(B)) \\ &= e_{L^X}(\mathcal{H}{-}1(A), \mathcal{H}^{-1}(B)) = 0.6. \end{aligned}$$

(4) Since $\mathcal{J}^{-1}(\mathcal{J}^{-1}(1_x^*))(z) = \bigwedge_{y\in X}(\mathcal{J}^{-1*}(1_x^*)(y) \rightarrow \mathcal{J}^{-1}(1_y^*)(z) = \mathcal{J}^{-1}(1_x^*)(z)$ and $1_x \leq \mathcal{J}^*(1_x^*)$ for all $x, y \in X$, then $\mathcal{J}^{-1}$ is an L-lower approximation operator. Since $\mathcal{J}^{-1}(A)(y) = \bigwedge_{x\in X}(A^*(x) \rightarrow (\mathcal{J}^{-1}(1_x^*)(y))$, we have

$$\mathcal{J}^{-1}(A) = (0.7, 0.8, 0.3),\ \mathcal{J}^{-1}(B) = (0.3, 0.6, 0.8),$$

$$\begin{aligned} J^{-1\rightarrow}(A, B) &= e_{L^X}(\mathcal{J}^{-1}(A), B) \\ &= e_{L^X}(\mathcal{J}^{-1}(A), \mathcal{J}^{-1}(B)) = 0.6. \end{aligned}$$

3. Conclusions

In this paper, by using the concepts of fuzzy complete lattices [7, 8], we generalized lower and upper approximation operators without fuzzy relations in complete residuated lattices. The relations between L-upper (resp. L-lower) approximation operators and L-fuzzy preorders are also analyzed. The studied L-fuzzy preorders on L^X are an important mathematical tool for algebraic structure of fuzzy contexts.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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