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EXISTENCE AND REGULARITY FOR SEMILINEAR NEUTRAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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Abstract. In this paper, we construct some results on the existence and regularity for solutions of neutral functional differential equations with unbounded principal operators in Hilbert spaces. In order to establish the existence and regularity for solutions of the neutral system by using fractional power of operators and the local Lipschtiz continuity of nonlinear term without using many of the strong restrictions considering in the previous literature.

1. Introduction

Let H and V be real Hilbert spaces such that V is a dense subspace in H. In this paper, we are concerned with the global existence of solution and the approximate controllability for the following abstract neutral functional differential system in a Hilbert space H:

$$\begin{cases} \frac{d}{dt}[(x(t)+(Bx)(t)] = Ax(t)+f(t,x(t))+k(t), \quad t\in(0,T], \\ x(0)=x_0, \quad (Bx)(0)=y_0, \end{cases} \tag{1.1}$$

where A is an operator associated with a sesquilinear form on $V\times V$ satisfying Gårding's inequality, f is a nonlinear mapping of $[0,T]\times V$ into H satisfying the local Lipschitz continuity, $B : L^2(0,T;V)\to L^2(0,T;H)$ is a bounded linear mapping.

Recently, the existence of solutions for mild solutions for neutral differential equations with state-dependence delay has been studied in the literature in [1] and references therein. As for partial neutral integro-differential equations, we refer to [2]. However there are few papers treating the regularity for neutral systems with local Lipschipz continuity, we can just find a recent article Wang [3] in case semilinear systems.

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In this paper, we construct some results on the regularity of solutions for neutral functional differential equations with unbounded principal operators in Hilbert spaces. In order to establish the existence and regularity of solutions of the neutral system by using fractional power of operators and the local Lipschitz continuity of nonlinear term without using many of the strong restrictions considering in the previous literature.

2. preliminaries

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. For brevity, we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \geq \delta ||u||^2, \quad \delta > 0. \tag{2.2}$$

Let A be the operator associated with this sesquilinear form: $(Au, v) = a(u, v)$ for any $u,\ v \in V$. Then A is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to $D(A) = \{u \in V : Au \in H\}$ is also denoted by A. From (2.2) we may think that there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||^{1/2}_{D(A)} |u|^{1/2}. \tag{2.3}$$

Thus we have the following sequence:

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.4}$$

where each space is dense in the next one and continuous injection.

Lemma 2.1. *With the notations (2.3), (2.4), we have*

$$(V, V^*)_{1/2,2} = H, \quad (D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Sec. 1.3 of [4]).

It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. By virtue of (2.2), we have that $0 \in \rho(A)$ the closed half plane $\{\lambda : \text{Re}\,\lambda \geq 0\}$ is contained in the resolvent set of A. In this case, $A^{-\alpha}$ is a bounded operator. So we can assume that there is a constant $M_0 > 0$ such that

$$||A^{-\alpha}||_{\mathcal{L}(H)} \leq M_0, \quad ||A^{-\alpha}||_{\mathcal{L}(V^*,V)} \leq M_0. \tag{2.5}$$

For each $\alpha \geq 0$ we can define the fractional power $A^\alpha(\alpha > 0)$ of A and collect some simple properties of the fractional power of A.

Lemma 2.2. *(a) A^α is a closed operator with its domain dense.*
(b) If $0 < \alpha < \beta$, then $D(A^\alpha) \supset D(A^\beta)$.
(c) For any $T > 0$, there exists a positive constant C_α such that the following inequalities hold for all $t > 0$([5, Lemma 3.6.2]*):*

$$||A^\alpha S(t)||_{\mathcal{L}(H)} \le \frac{C_\alpha}{t^\alpha}, \quad ||A^\alpha S(t)||_{\mathcal{L}(V,H)} \le \frac{C_\alpha}{t^{3\alpha/2}}. \tag{2.6}$$

By a simple calculation, we obtain the following.

Lemma 2.3. *For every $k \in L^2(0,T;H)$, let $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that such that*

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.7}$$

3. Neutral differential equations

In this section, we will show that the initial value problem (1.1) has a solution by solving the integral equation:

$$\begin{aligned} x(t) =& S(t)[x_0 + y_0] - (Bx)(t) + \int_0^t AS(t-s)Bx(s)ds \\ &+ \int_0^t S(t-s)\{f(s,x(s)) + k(s)\}ds. \end{aligned}$$

Now we give the basic assumptions on the system (1.1)

Assumption (B). Let $B : L^2(0,T;V) \to L^2(0,T;H)$ be a bounded linear mapping such that there exists constants $\beta > 2/3$ and $L > 0$ such that

$$||A^\beta Bx||_{L^2(0,T;H)} \le L||x||_{L^2(0,T;V)}, \quad \forall x \in L^2(0,T;V).$$

Assumption (F). f is a nonlinear mapping of $[0,T] \times V$ into H satisfying following:

(i) There exists a function $L_1 : \mathbb{R}_+ \to \mathbb{R}$ such that for $||x|| \le r$ and $||y|| \le r$,

$$|f(t,x) - f(t,y)| \le L_1(r)||x-y||, \quad t \in [0,T].$$

(ii) The inequality

$$|f(t,x)| \le L_1(r)(||x|| + 1)$$

holds For every $t \in [0,T]$ and $x \in V$.

From now on, we establish the following results on the solvability of the equation (1.1).

Theorem 3.1. *Let Assumptions (B) and (F) be satisfied. Assume that $x_0 \in H$, $k \in L^2(0,T;V^*)$ for $T > 0$. Then, there exists a solution x of the equation (1.1) such that*

$$x \in \mathcal{W}_1(T) \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

Moreover, there is a constant C_3 independent of x_0 and the forcing term k such that

$$||x||_{\mathcal{W}_1(T)} \le C_3(1+|x_0|+||k||_{L^2(0,T;V^*)}). \tag{3.1}$$

One of the main useful tools is the following Sadvoskii's fixed point theorem.

Lemma 3.2. *Suppose that Σ is a closed convex subset of a Banach space X. Assume that K_1 and K_2 are mappings from Σ into X such that the following conditions are satisfied:*
(i) $(K_1+K_2)(\Sigma)\subset\Sigma$,
(ii) K_1 is a completely continuous mapping,
(iii) K_2 is a contraction mapping.
Then the operator K_1+K_2 has a fixed point in Σ.

Proof of Theorem.
Let $r_0 = 2(C_1|x_0+y_0|+r_0M_0L)$, where C_1 is constant satisfying

$$||x||_{\mathcal{W}(T)} \le C_1(||x_0||+||k||_{L^2(0,T;H)}). \tag{3.2}$$

Let $\gamma=\max\{1/2,(3\beta-2)^{1/2}\}$, choose $0<T_1<T$ such that

$$T_1^\gamma\big[\{C_2L_1(r_0)(r_0+1)+C_2||k||_{L^2(0,T_1;V)}\}+(3\beta-2)^{-1/2}r_0LC_{1-\beta}\big] \tag{3.3}$$
$$\le C_1|x_0+y_0|+r_0M_0L,$$

where C_2 is constant in (2.7) and

$$\hat{M}\equiv T_1^\gamma\big\{C_2L_1(r_0)+(3\beta-2)^{-1/2}C_{1-\beta}L\big\}<1. \tag{3.4}$$

Define a mapping $J: L^2(0,T_1;V)\to L^2(0,T_1;V)$ as

$$\begin{aligned}(Jx)(t) =& S(t)(x_0+y_0)-(Bx)(t)\\ &+\int_0^t AS(t-s)(Bx)(s)ds+\int_0^t S(t-s)\{f(s,x(s))+k(s)\}ds.\end{aligned}$$

It will be shown that the operator J has a fixed point in the space $L^2(0,T_1;V)$. By assumptions (B) and (F), we know that J is continuous from $C([0,T_1];H)$ into itself. Let

$$\Sigma=\{x\in L^2(0,T_1;V): ||x||_{L^2(0,T_1;V)}\le r_0,\ x(0)=x_0\},$$

which is a bounded closed subset of $L^2(0,T_1;V)$. By (2.5) and Assumption (B) we have

$$||Bx||_{L^2(0,T_1;V)} \le ||A^{-\beta}||_{\mathcal{L}(H,V)}||A^\beta Bx||_{L^2(0,T_1;H)} \le r_0M_0L. \tag{3.5}$$

By virtue of (2.7), for $0<t<T_1$, it holds

$$||\int_0^t S(t-s)\{f(s,x(s))+k(s)\}ds||_{L^2(0,T_1;V)} \tag{3.6}$$
$$\le C_2\sqrt{T_1}||f(\cdot,x)+k||_{L^2(0,T_1;H)}$$
$$\le C_2\sqrt{T_1}\{L_1(r_0)(r_0+1)+||k||_{L^2(0,T_1;V)}\}.$$

Since (2.6) and Assumption (F) the following inequality holds:

$$||AS(t-s)Bx(s)|| = ||A^{1-\beta}S(t-s)A^{\beta}Bx(s)|| \le \frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}r_0L,$$

there holds

$$||\int_0^t AS(t-s)Bx(s)ds||_{L^2(0,T_1;V)} \le (3\beta-2)^{-1/2}r_0LC_{1-\beta}T_1^{\sqrt{3\beta-2}}. \tag{3.7}$$

Therefore, from (3.2), (3.4)-(3.7) it follows that

$$\begin{aligned}&||Jx||_{L^2(0,T_1;V)} \le C_1|x_0+y_0| + r_0M_0L\\ &+T_1^{\gamma}\big[\{C_2L_1(r_0)(r_0+1)+C_2||k||_{L^2(0,T_1;V)}\} + (3\beta-2)^{-1/2}r_0LC_{1-\beta}\big] \le r_0,\end{aligned}$$

and hence J maps Σ into Σ. Define mapping $J = K_1 + K_2$ on $L^2(0,T_1;V)$ by the formula

$$\begin{aligned}(K_1x)(t) =& -(Bx)(t)\\ (K_2x)(t) =& S(t)(x_0+y_0) + \int_0^t AS(t-s)(Bx)(s)ds\\ &+\int_0^t S(t-s)\{f(s,x(s))+k(s)\}ds.\end{aligned}$$

We can now employ Lemma 3.1 with Σ. Assume that a sequence $\{x_n\}$ of $L^2(0,T_1;V)$ converges weakly to an element $x_\infty \in L^2(0,T_1;V)$, i.e., $w-\lim_{n\to\infty} x_n = x_\infty$. Then we will show that

$$\lim_{n\to\infty} ||K_1x_n - K_1x_\infty|| = 0, \tag{3.8}$$

which is equivalent to the completely continuity of K_1 since $L^2(0,T_1;V)$ is reflexive. For a fixed $t \in [0,T_1]$, let $x_t^*(x) = (K_1x)(t)$ for every $x \in L^2(0,T_1;V)$. Then $x_t^* \in L^2(0,T_1;V^*)$ and we have $\lim_{n\to\infty} x_t^*(x_n) = x_t^*(x_\infty)$ since $w-\lim_{n\to\infty} x_n = x_\infty$. Hence,

$$\lim_{n\to\infty}(K_1x_n)(t) = (K_1x_\infty)(t), \quad t\in[0,T_1].$$

By (2.5) and Assumption (B) we have

$$||(K_1x)(t)|| \le ||A^{-\beta}||_{\mathcal{L}(H,V)}||A^{\beta}Bx||_{L^2(0,T_1;H)} \le \infty.$$

Therefore, by Lebesgue's dominated convergence theorem it holds

$$\lim_{n\to\infty}||K_1x_n||_{L^2(0,T_1;V)} = ||K_1x_\infty||_{L^2(0,T_1;V)}.$$

Since $L^2(0,T_1;V)$ is a Hilbert space, it holds (3.8).

Next, we prove that K_2 is a contraction mapping on Σ. Indeed, for every x_1 and $x_2 \in \Sigma$, by similar to (3.7) and (3.8), we have

$$||K_2x_1-K_2x_2||_{L^2(0,T_1;V)} \le T_1^{\gamma}\{C_2L_1(r_0)+(3\beta-2)^{-1/2}C_{1-\beta}L\}||x_1-x_2||_{L^2(0,T_1;V)}.$$

So by virtue of the condition (3.4) the contraction mapping principle gives that the solution of (1.1) exists uniquely in $[0,T_1]$. So by virtue of the condition

(3.4), K_2 is contractive. Thus, Lemma 3.1 gives that the equation of (1.1) has a solution in $\mathcal{W}_1(T_1)$.

From now on we establish a variation of constant formula (3.1) of solution of (1.1). Let x be a solution of (1.1) and $x_0 \in H$. Then we have that from (3.5)-(3.8) it follows that

$$||x||_{L^2(0,T_1;V)} \leq C_1|x_0 + y_0| + r_0M_0L + T_1^\gamma\big[\{C_2L_1(r_0)(||x||_{L^2(0,T_1;V^*)} + 1) \\ + C_2||k||_{L^2(0,T_1;V^*)}\} + (3\beta - 2)^{-1/2}C_{1-\beta}L||x||_{L^2(0,T_1;V)}\big]$$

Taking into account (3.4), there exists a constant C_3 such that

$$||x||_{L^2(0,T_1;V)} \leq (1 - \hat{M})^{-1}\big[C_1|x_0 + y_0| + r_0M_0L \\ + T_1^\gamma\{C_2L_1(r_0) + C_2||k||_{L^2(0,T_1;V^*)}\}\big] \leq C_3(1 + |x_0| + ||k||_{L^2(0,T_1;V^*)})$$

which obtain the inequality (3.1). Since the conditions (3.3) and (3.4) are independent of initial value, we know

$$|x(T_1)| \leq ||x||_{C([0,T_1;H])} \leq M_1||x||_{\mathcal{W}_1(T)}.$$

Here, we used the relation $W_1(T) \hookrightarrow C([0,T_1;H])$, which is an easy consequence of the definition of real interpolation spaces by the trace method. So, by repeating the above process, the solution can be extended to the interval $[0,T]$. □

From the following result, we obtain that the solution mapping is continuous, which is useful for physical applications of the given equation. The proof is immediately obtained from Theorem 3.1.

Theorem 3.3. *Let Assumptions (B) and (F) be satisfied and* $(x_0, y_0, k) \in H \times H \times L^2(0,T;V^*)$. *Then the solution x of the equation (1.1) belongs to* $x \in \mathcal{W}_1(T) \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ *and the mapping*

$$H \times H \times L^2(0,T;V^*) \ni (x_0, y_0, k) \mapsto x \in \mathcal{W}_1(T)$$

is continuous.

For $k \in L^2(0,T;V^*)$ let x_k be the solution of equation (1.1) with k instead of Bu. Here, we remark that if V is compactly embedded in H by assumption, the embedding $\mathcal{W}_1(T) \subset L^2(0,T;H)$ is compact in view of Theorem 2 of Aubin [6]. So we can prove the following result from Theorem 3.1.

Corollary 3.4. *Let us assume that the embedding $V \subset H$ is compact. For $k \in L^2(0,T;V^*)$ let x_k be the solution of equation (1.1). Then the mapping $k \mapsto x_k$ is compact from $L^2(0,T;V^*)$ to $L^2(0,T;H)$. Moreover, if we define the operator $\mathcal{F}$ by $\mathcal{F}(k) = f(\cdot, x_k)$, then $\mathcal{F}$ is also a compact mapping from $L^2(0,T;V^*)$ to $L^2(0,T;H)$.*

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