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A CONTINUOUS HÖLDER INEQUALITY ON MORREY SPACES

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Abstract. A continuous form of Hölder inequality is established on the settings of Morrey spaces.

1. Introduction

1.1. Hölder's inequality

Let $Y = (Y, \nu)$ be a measure spaces with positive measure ν. Classical Hölder inequality (see [5], for example) says that

$$\int_Y f_1(y)^p f_2(y)^{1-p} d\nu(y) \le \left(\int_Y f_1(y)\ d\nu(y) \right)^p \left(\int_Y f_2(y)\ d\nu(y) \right)^{1-p}, \quad (1.1)$$

where f_1 and f_2 are positive functions of $L^1(Y)$ and $0 \le p \le 1$.

It is well-known fact that (1.1) can be extended to the case of a multiple product of functions (see [1], [2], etc.), and even to a continuous version (see [3]) as the following

Theorem A. Let $X = (X, \mu)$ and $Y = (Y, \nu)$ be σ-finite measure spaces with positive measures μ and ν, and denote $\mu \times \nu$ the product measure of μ and ν. If $\mu(X) = 1$ and if $f(x, y)$ is a positive measurable function defined on $X \times Y$, then

$$\int_Y \exp\left(\int_X \log f\ d\mu \right) d\nu \le \exp\left\{ \int_X \log\left(\int_Y f\ d\nu \right) d\mu \right\}. \quad (1.2)$$

Remark. Theorem A may be regarded as a generalization of (1.1). In fact, for $0 \le p \le 1$ if we take

$$X = \{1, 2\} \quad \text{and} \quad d\mu = \left\{ p\chi_{\{1\}} + (1-p)\chi_{\{2\}} \right\} dm,$$

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dm the counting measure, $\chi_{\{.\}}$ the corresponding characteristic functions, and $f(k,y) = f_k(y),\ k \in X$, then (1.2) reduces to (1.1):

$$\begin{aligned}\int_Y f_1(y)^p f_2(y)^{1-p}\ d\nu(y) &= \int_Y \exp\left(\int_X \log f_k(y)\ d\mu(k)\right) d\nu(y)\\ &\le \exp\left\{\int_X \log\left(\int_Y f_k(y)\ d\nu(y)\right) d\mu(k)\right\}\\ &= \left(\int_Y f_1(y)\ d\nu(y)\right)^p \left(\int_Y f_2(y)\ d\nu(y)\right)^{1-p}.\end{aligned}$$

1.2. Morrey spaces

Morrey space was introduced in the course of estimating the solution of partial differential equations in [4].

Definition 1. For $0 < p < \infty$ and $0 \le \lambda \le n$, Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}} < +\infty\},$$

where

$$\|f\|_{L^{p,\lambda}} = \sup_B \left(\frac{1}{r^\lambda}\int_B |f(x)|^p dx\right)^{\frac{1}{p}}$$

and the supremum is taken over all ball $B = B(a,r) = \{x \in \mathbb{R}^n : |x-a| < r\}$ with center a and radius $r > 0$.

$L^{p,\lambda}(\mathbb{R}^n)$ is a Fréchet space, and a Banach space if $p \ge 1$. If $\lambda = 0$, then $L^{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\lambda = n$, then $L^{p,\lambda}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ by Lebesgue differentiation theorem (see [6]).

1.3. Hölder inequality for Morrey spaces

For $1/p_1 + 1/p_2 = 1$ and $\lambda_1/p_1 + \lambda_2/p_2 = \lambda$, simple application of classical Hölder inequality of the form (1.1) gives a discrete form of Hölder inequality for Morrey space version:

$$\|fg\|_{L^{p,\lambda}} \le \|f\|_{L^{p_1,\lambda_1}} \|g\|_{L^{p_2,\lambda_2}}. \tag{1.3}$$

1.4. Goal of this paper

We establish a continuous form of Hölder inequality on the settings of Morrey spaces. This will be done in Section 2 and proved in Section 3.

2. A continuous form of Hölder inequality on the settings of Morrey spaces

2.1. Main result

As the main result of this paper, we establish the following theorem which may be regarded as a continuous form of Hölder inequality on Morrey spaces.

Theorem 2.1. *Let $X = (X, \mu)$ be a probability measure space with positive measure μ. Let $0 \le \lambda(x) \le n$ for all $x \in X$. Then for any $p : 0 < p < \infty$ and each measurable function $F : X \times \mathbb{R}^n \to [0, \infty)$,*

$$\left\| \exp\left(\int_X \log F(x,y) d\mu(x)\right)\right\|_{L^{p,\lambda}(\mathbb{R}^n)} \le \exp\left(\int_X \log \|F_x\|_{L^{p,\lambda(x)}(\mathbb{R}^n)} d\mu(x)\right), \tag{2.1}$$

where $F_x(y) = F(x,y),\ y \in \mathbb{R}^n$.

2.2. Discrete form as a Corollary

As a corollary, we have a discrete form of Hölder inequality for Morrey spaces which is a natural generalization of (1.3).

Corollary 2.2. *Let $0 < p < \infty,\ 0 \le \lambda \le n,\ 0 < p_k < \infty,\ 0 \le \lambda_k \le n$, $k = 1,\ 2,\ \ldots, N$, and $\sum_{k=1}^N \frac{1}{p_k} = \frac{1}{p}$, $\sum_{k=1}^N \frac{\lambda_k}{p_k} = \frac{\lambda}{p}$. If $f_k \in L^{p_k,\lambda_k}(\mathbb{R}^n), k = 1, 2, ..., N$, then*

$$\| \prod_{k=1}^N f_k \|_{L^{p,\lambda}(\mathbb{R}^n)} \ \le\ \prod_{k=1}^N \|f_k\|_{L^{p_k,\lambda_k}(\mathbb{R}^n)}. \tag{2.2}$$

3. Proofs

3.1. Proof of Theorem 2.1

Fix $B = B(a, r)$ for a moment and set $d\nu(y) = dy/r^\lambda$. Let $f(x,y) = r^{\lambda-\lambda(x)} F(x,y)^p$ for a $p : 0 < p < \infty$. Then f is a measurable function on $(X, \mu) \times (B, \nu)$. Since X and B are σ-finite, Theorem A guarantees

$$\begin{aligned} &\frac{1}{r^\lambda}\int_B \exp\left(\int_X \log f(x,y)\ d\mu(x)\right) dy \\ &\le\ \exp\left\{\int_X \log\left(\frac{1}{r^\lambda}\int_B f(x,y)\ dy\right) d\mu(x)\right\}, \end{aligned} \tag{3.1}$$

and simple calculation shows that (3.1) becomes

$$\begin{aligned} &\frac{1}{r^\lambda}\int_B \exp\left(\int_X \log F(x,y)^p\ d\mu(x)\right) dy \\ &\le\ \exp\left\{\int_X \log\left(\frac{1}{r^{\lambda(x)}}\int_B F(x,y)^p\ dy\right) d\mu(x)\right\}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\left[\frac{1}{r^\lambda}\int_B \left\{\exp\left(\int_X \log F(x,y)\ d\mu(x)\right)\right\}^p dy\right]^{1/p} \\ &\le\ \exp\left\{\int_X \log\left(\frac{1}{r^{\lambda(x)}}\int_B F(x,y)^p\ dy\right)^{1/p} d\mu(x)\right\}. \end{aligned} \tag{3.2}$$

Note, since F is measurable on $X \times \mathbb{R}^n$, that (3.2) holds for any $B = B(a,r)$. Now, taking the supremum for all $B = B(a,r)$ on both sides of (3.2) we obtain (2.1).

3.2. Proof of Corollary 2.2

We may assume $p = 1$. Let $X = \{1, 2, ..., N\}$ and

$$d\mu(x) = \sum_{k=1}^{N} \frac{1}{p_k} \chi_{\{k\}}(x)\ dm(x),$$

where dm and $\chi_{\{\cdot\}}$ denote the counting measure and the characteristic function of the set$\{\cdot\}$ respectively. Then,

$$\mu(X) = \int_X\ d\mu(x) = \int_{\{1,2,...,N\}}\ \sum_{k=1}^{N} \frac{1}{p_k} \chi_{\{k\}}(x)\ dm(x) = \sum_{k=1}^{N} 1/p_k\ \ =\ 1$$

and

$$\int_X \lambda(x)\ d\mu(y) = \int_{\{1,2,...,N\}} \lambda(x) \sum_{k=1}^{N} \frac{1}{p_k} \chi(x)\ dm(x) = \sum_{k=1}^{N} \frac{\lambda_k}{p_k} = \lambda.$$

Thus, if we set $F(x,y) = |f(x,y)|^{p_x}$, $x \in X$, $y \in \mathbb{R}^n$, then Theorem 2.1 gives (2.1) with f in place of F. Therefore, we get

$$\begin{aligned}
&\| \prod_{k=1}^{N} f_k \|_{L^{1,\lambda}(\mathbb{R}^n)} = \left\| \prod_{k=1}^{N} F_k^{1/p_k} \right\|_{L^{1,\lambda}(\mathbb{R}^n)} \\
&= \left\| \ \exp\left(\sum_{k=1}^{N} \frac{1}{p_k} \log F_k(y) \right) \right\|_{L^{1,\lambda}(\mathbb{R}^n)} \\
&= \left\| \ \exp\left(\int_{\{1,2,...,N\}} \sum_{k=1}^{N} \frac{1}{p_k} \chi_{\{k\}}(x)\ \ \log F_x(y)\ dm(x) \right) \right\|_{L^{1,\lambda}(\mathbb{R}^n)} \\
&= \left\| \ \exp\left(\int_X \log f(x,y)\ d\mu(x) \right) \right\|_{L^{1,\lambda}(\mathbb{R}^n)} \\
&\le \exp\left(\int_X \log \|f_x\|_{L^{1,\lambda(x)}(\mathbb{R}^n)}\ d\mu(x) \right) \\
&= \exp\left(\int_{\{1,2,...,N\}} \sum_{k=1}^{N} \frac{1}{p_k} \chi_{\{k\}}(x)\ \ \log \|F_x(y)\|_{L^{1,\lambda(x)}(\mathbb{R}^n)}\ dm(x) \right) \\
&= \exp\left(\sum_{k=1}^{N} \frac{1}{p_k}\ \ \log \|F_k\|_{L^{1,\lambda_k}(\mathbb{R}^n)} \right)\ =\ \prod_{k=1}^{N} \|F_k\|_{L^{1,\lambda_k}(\mathbb{R}^n)}^{1/p_k} \\
&=\ \prod_{k=1}^{N} \|f_k\|_{L^{p_k,\lambda_k}(\mathbb{R}^n)},
\end{aligned}$$

which gives (2.2).

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