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CERTAIN WEIGHTED MEAN INEQUALITY

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ABSTRACT. In this paper, we report a new sharp inequality of interpolation type in $\mathbb{R}^n$. This inequality is for controlling weighted average of a function via L^n norm of the gradient of a function together with its' certain exponential norm.

1. INTRODUCTION

Inequalities of interpolation type are well-known nowadays and very important in studying PDEs(See for instance [1, 3] and references therein). These inequalities bound roughly L^p norm of a function itself by the derivatives of the function. Concretely, for a function $u \in W^{1,r}(\mathbb{R}^n)$, we have

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^r}^t \|u\|_{L^r}^{1-t}, \quad r \leq p \leq p^* \equiv \frac{nr}{n-r}. \tag{1.1}$$

Here, $t \in [0,1]$ is determined by p, r. But, this type of Sobolev inequality fails when $p < r$. In fact, for a cutoff function ξ and large $R > 0$, if we take

$$u = C\xi(\frac{|x|}{R}), \quad C > 0,$$

then $\|u\|_{L^p} \sim R^{n/p}$, $\|\nabla u\|_{L^r} \sim R^{-1+n/r}$, $\|u\|_{L^r} \sim R^{n/r}$ and (1.1) fails as $R \to \infty$. However, if we consider u which is average zero even under a weighted sense, such inequality remains true in many cases by the Poincare inequality[2]. Thus, it is crucial to bound a weighted average of a function if we want to have inequalities like (1.1). As far as the author knows, an inequality with such spirit appears in [6] firstly and turns out to be helpful to problems in gauge theory. The purpose of this paper is to report such inequality using certain exponential integral of a function. That is,

$$|\int_{\mathbb{R}^n} gv| \leq \frac{1}{n}(\frac{\omega_n}{n})^{(n-1)/n}\|\nabla v\|_{L^n(\mathbb{R}^n)} \left[\ln(X+1) + C\right]^{\frac{n-1}{n}} + C \tag{1.2}$$

for any $v \in W^{1,n}$. Here, $g = (1+|x|^n)^{-2}$, $X = \int_{\mathbb{R}^n}(e^{-|v|} - 1)^2$, and C is an absolute constant depending only on n. In view of the already existing Poincare inequality, The above

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guarantees that L^p_{loc}, $p \le \frac{np}{n-p}$ norm of v can be bounded by the RHS of (1.2). We also show that (1.2) is sharp providing an example which do not satisfy (1.2) if we replace the coefficient $\frac{1}{n}(\frac{\omega_n}{n})^{(n-1)/n}$ with any smaller constant. We finally remark that exponential integral in the righthand side of (1.2) can be replaced with integral of similar bounded function. In fact, if we replace $(e^{-|v|}-1)^2$ with $\phi^2(v)$ satisfying

$$\phi(t) \ge C\min\{1,t\}, \quad \text{for } t>0,$$

(1.2) holds true due to

$$\int (e^{-|v|}-1)^2 \le \sup_{t>0}\frac{(e^{-t}-1)^2}{\phi^2(t)}\int \phi^2(v) \le C\int \phi^2(v).$$

2. WEIGHTED MEAN INEQUALITY

From now on, we omit the domain of integration if it is $\mathbb{R}^n$. We also denote $|x| = r$ and $g = (1+r^n)^{-2}$ as before. We first remind that symmetrization holds true for any nonnegative smooth function with compact support.

Theorem 2.1. *There exists $C>0$ such that*

$$|\int gv| \le \frac{1}{n}(\frac{\omega_n}{n})^{(n-1)/n}\|\nabla v\|_{L^n}\left[\ln(X+1)+C\right]^{(n-1)/n}+C \tag{2.1}$$

for $v \in W^{1,n}$. Here, X is as before and ω_n is the volume of the $n-1$ dimensional unit sphere.

Proof) It is clear that

$$|\int gv| \le |\int g \times -|v||, \quad \|\nabla|v|\|_{L^n} \le \|\nabla v\|_{L^n}.$$

Thus, it is enough to show (1.2) replacing v with $-|v|$. Since $-|v|$ itself belongs to $W^{1,n}$, we can assume $v \le 0$ without loss of generality. Then, it is again enough to show (1.2) for $v \in C_0^\infty$ with $v \le 0$ by the standard density theorem. If v is nonpositive and of compact support, we can apply the symmetrization on $-v$. Let us denote the nonincreasing rearrangement of $-v$ by $(-v)_S$. Clearly, $v^* \equiv -(-v)_S$ is nondecreasing with respect to r. Further, g is decreasing with respect to r and, due to $v \le 0$, $1-e^v$ and $1-e^{v^*}$ are equi-measurable. Therefore,

$$\int gv \ge \int gv^*, \quad \int (e^v-1)^2 = \int (e^{v^*}-1)^2, \quad \|\nabla v^*\|_{L^r} \le \|\nabla v\|_{L^r}.$$

Thus, it is enough to show (1.2) for radially symmetric nonpositive smooth v with compact support.

Now, for $R>0$,

$$v(r) = v(R) + \int_R^r \partial_s v(s)ds.$$

We multiply the above by $g(r)r^{n-1}$ and integrate with respect to r on $(0,\infty)$ to get

$$LHS = \int_0^\infty v(r)g(r)r^{n-1}dr,$$

$$\begin{aligned} RHS =& Cv(R) + \int_0^\infty g(r)r^{n-1} \int_R^r \partial_s v(s) ds dr \\ =& Cv(R) + \left(\int_0^R + \int_R^\infty \right) g(r)r^{n-1} \int_R^r \partial_s v(s) ds dr = Cv(R) + I + II. \end{aligned} \tag{2.2}$$

By the Fubini theorem,

$$\begin{aligned} |I| &\leq \int_0^R |\nabla v(s)| \int_0^s g(r)r^{n-1} dr ds = \int_0^R |\nabla v(s)| \frac{s^n}{n(1+s^n)} ds \\ &\leq \frac{1}{n}\omega_n^{-\frac{1}{n}} \|\nabla v\|_{L^n} \left(\int_0^R (\frac{s^{n-(n-1)/n}}{1+s^n})^{\frac{n}{n-1}} ds \right)^{(n-1)/n}, \\ |II| &\leq \int_R^\infty |\partial_s v(s)| \int_s^\infty g(r)r^{n-1} dr ds = \int_R^\infty |\partial_s v(s)| \frac{1}{n(1+s^n)} ds \\ &\leq \frac{1}{n}\omega_n^{-\frac{1}{n}} \|\nabla v\|_{L^n} \left(\int_R^\infty \frac{1}{s}(1+s^n)^{-\frac{n}{n-1}} ds \right)^{(n-1)/n} \end{aligned}$$

Here, ω_n is the volume of S^{n-1}. By change of variables $s^n = t$, when $R > 1$,

$$\begin{aligned} \int_0^R (\frac{s^{n-(n-1)/n}}{1+s^n})^{\frac{n}{n-1}} ds &= \frac{1}{n} \int_0^{R^n} (\frac{1}{1+t})^{\frac{n}{n-1}} t^{\frac{1}{n-1}} dt \\ &\leq \frac{1}{n} \int_0^{R^n} \frac{1}{1+t} = \frac{1}{n} \ln(1+R^n), \\ \int_R^\infty \frac{1}{s}(1+s^n)^{-\frac{n}{n-1}} ds &\leq \frac{1}{n} \int_1^\infty \frac{1}{t}(1+t)^{-\frac{n}{n-1}} dt \leq C. \end{aligned}$$

Thus, we get

$$|I| + |II| \leq n^{-\frac{2n-1}{n}} \omega_n^{-1/n} \|\nabla v\|_{L^n} (\ln(C+R^n))^{(n-1)/n}.$$

Meanwhile, we claim that for some $C > 0$, there exists $R_1 \leq CX^{1/n}$ such that $v(R_1) > -2$. Indeed, otherwise, we would have $v(r) < -2$ uniformly on the interval $T \equiv [0, CX^{1/n}]$. Then,

$$X = \int (e^v - 1)^2 dx \geq \frac{\omega_n}{4} \int_T r^{n-1} dr = \frac{\omega_n}{4n} C^n X.$$

This gives a contradiction if we choose $C > (4n/\omega_n)^{1/n}$. Thus, the claim is proved. We take $R = R_1$ in (2.2) and arrive at

$$|\int gv| = \omega_n | \int_0^\infty g v r^{n-1} dr| \leq C + \frac{1}{n}(\frac{\omega_n}{n})^{(n-1)/n} \|\nabla v\|_{L^n} ((\ln(C+R_1^n))^{\frac{n-1}{n}} + C).$$

Since $R_1 \leq CX^{1/n}$, we arrive at the desired result redefining C suitably. □

Now, we give an example which shows the sharpness of (1.2). Consider the functions,

$$\psi(x) = \begin{cases} -\ln R & |x| \leq 1 \\ \ln(\frac{r}{R}) & 1 < |x| < R, \\ 0 & |x| \geq R. \end{cases}$$

It is clear that

$$\|\nabla\psi\|_{L^n} = \left(\omega_n \int_1^R \frac{1}{r^n} r^{n-1} dr\right)^{1.n} = (\omega_n \ln R)^{1/n},$$

$$X \sim C + C\int_1^R (1 - \frac{r}{R})^2 r^{n-1} dr \sim C + CR^n,$$

$$\int g\psi = \omega_n \int_0^\infty \frac{-\ln R}{(1+r^n)^2} r^{n-1} dr + C = C - \frac{\omega_n}{n}\ln R.$$

Thus, (1.2) fails for ψ as $R \to \infty$ if we replace $\frac{1}{n}(\frac{\omega_n}{n})^{(n-1)/n}$ in (1.2) with smaller coefficient.

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