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INTERNAL FEEDBACK CONTROL OF THE BENJAMIN-BONA-MAHONY-BURGERS EQUATION

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ABSTRACT. A numerical scheme is proposed to control the BBMB (Benjamin-Bona-Mahony-Burgers) equation, and the scheme consists of three steps. Firstly, BBMB equation is converted to a finite set of nonlinear ordinary differential equations by the quadratic B-spline finite element method in spatial. Secondly, the controller is designed based on the linear quadratic regulator (LQR) theory; Finally, the system of the closed loop compensator obtained on the basis of the previous two steps is solved by the backward Euler method. The controlled numerical solutions are obtained for various values of parameters and different initial conditions. Numerical simulations show that the scheme is efficient and feasible.

1. INTRODUCTION

The mathematical model of propagation of small amplitude long waves in a nonlinear dispersive media is described by the following BBMB equation [1]:

$$\begin{cases} y_t - y_{xxt} - \alpha y_{xx} + \beta y_x + y y_x = f \quad \text{in } [0,L]\times[0,T], \\ y(0,t) = y(L,t) = 0 \quad \text{on } [0,T], \\ y(x,0) = y_0(x) \quad \text{in } [0,L], \end{cases} \tag{1.1}$$

where $\alpha > 0$ and β are constants and f and y_0 given forcing and initial terms. In the physical case, the dispersive effect of (1.1) is the same as the BBM (Benjamin-Bona-Mahony) equation, while the dissipative effect is the same as the Burgers equation, and which is an alternative model for the Korteweg-de Vries-Burgers (KdVB) equation [2]. A Galerkin finite element method of BBM equation has been discussed in references [3]-[6]. The quadratic B-spline finite element method for approximating the solution of Burgers equation can be found in [7, 8]. The stabilization of boundary feedback control for the equations BBMB, KdVB and Burgers equation has been investigated in [9]-[12]. The articles [13, 14] demonstrated that the feedback controller is locally/globally controllable and stabilizable at the case of KdVB equation.

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A conclusion obtained from [15] is that the solution of the generalized regularized long wave-Burgers (GRLWB) equation decays like the solution of the corresponding linear equation, when this solution has a small value. System (1.1) is an important case of the GRLWB equation. Therefore, we can design the feedback control law of the BBMB equation by LQR method. So far, after the discretization in space with the B-spline finite element method, there are lack of deeper researches on designing controller. In this paper, we use the quadratic B-spline finite element method to convert BBMB equation to a finite set of nonlinear ordinary differential equations, and then, design the controller using linear quadratic regulator theory.

In the subsequent section, we describe the B-spline finite element approximations of solutions of the BBMB equations. In Section 3, we design feedback control of the BBMB equations using the LQR method. Numerical results for the linear feedback control is analyzed in the last section.

2. Finite element approximation

Standard Lagrangian finite element basis functions offer only simple C^0-continuity and therefore they cannot be used for the spatial discretization of the higher-order differential equations(e.g., third-order differential equation or forth-order differential equation), but the B-spline basis function can at least achieve C^1-continuous globally, and such basis function is often used to solve the higher order differential equations.

Let us consider the BBMB equation with boundary conditions and the initial condition. We use a variational formulation to approximate (1.1) with a finite element method. A variational formulation of the problem (1.1) is as the following: find $y \in L^2(0,T;H_0^1(0,L))$ such that

$$\begin{cases} \int_0^L y_t v dx + \int_0^L y_{xt} v' dx + \alpha \int_0^L y_x v' dx + \beta \int_0^L y_x v dx \\ \qquad + \int_0^L y y_x v dx = \int_0^L f v dx \quad \text{for all } v \in H_0^1(0,L), \\ y(0,x) = y_0(x) \quad \text{in } [0,L], \end{cases} \tag{2.1}$$

where $H_0^1 = \{y \in H^1(0,L) : y|_{x=0} = y|_{x=1} = 0\}$ and $H^1(0,L) = \{v \in L^2(0,L) : \frac{\partial v}{\partial x} \in L^2(0,L)\}$.

A typical finite element approximation of (2.1) is defined as follows: we first choose conforming finite element subspaces $V^h \subset H^1(0,L)$ and then define $V_0^h = V^h \cap H_0^1(0,L)$. One then seeks $y^h(t,\cdot) \in V_0^h$ such that

$$\begin{cases} \int_0^L y_t^h v^h dx + \int_0^L y_{xt}^h (v^h)' dx + \alpha \int_0^L y_x^h (v^h)' dx + \beta \int_0^L y_x^h v^h dx \\ \qquad + \int_0^L y^h y_x^h v^h dx = \int_0^L f v^h dx \quad \text{for all } v^h \in V_0^h(0,L), \\ y^h(0,x) = y_0^h(x) \quad \text{in } [0,L], \end{cases} \tag{2.2}$$

where $y_0^h(x) \in V_0^h$ is an approximation, e.g., a projection, of $y_0(x)$.

The interval $[0, L]$ is divided into n finite elements of equal length h by the knots x_i such that $0 = x_0 < x_1 < \cdots < x_n = L$. The set of splines $\{\eta_{-1}, \eta_0, \cdots, \eta_n\}$ form a basis for functions defined on $[0, L]$. Quadratic B-splines $\eta_i(x)$ with the required properties are defined by [16]

$$\eta_i(x) = \frac{1}{h^2}\begin{cases} (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2 + 3(x_i - x)^2), & [x_{i-1}, x_i], \\ (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2, & [x_i, x_{i+1}], \\ (x_{i+2}-x)^2, & [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise}, \end{cases}$$

where $h = x_{i+1} - x_i, i = -1, 0, \cdots, n$.

The quadratic spline and its first derivative vanish outside the interval $[x_{i-1}, x_{i+2}]$. The spline function values and its first derivative at the knots are given by

$$\begin{cases} \eta_i(x_{i-1}) = \eta_i(x_{i+2}) = 0, \ \eta_i(x_i) = \eta_i(x_{i+1}) = 1; \\ \eta_i^{'}(x_{i-1}) = \eta_i^{'}(x_{i+2}) = 0, \ \eta_i^{'}(x_i) = \eta_i^{'}(x_{i+1}) = 1. \end{cases} \tag{2.3}$$

Thus an approximate solution can be written in terms of the quadratic spline functions as

$$y^h(x,t) = \sum_{i=-1}^{n} a_i(t)\eta_i(x), \tag{2.4}$$

where $a_i(t)$ are yet undetermined coefficients.

Each spline covers three intervals so that three splines $\eta_{i-1}(x), \eta_i(x), \eta_{i+1}(x)$ cover each finite element $[x_i, x_{i+1}]$. All other splines are zero in this region. Using Eq.(2.4) and spline function properties (2.3), the nodal values of function $y^h(x,t)$ and its derivative at the knot x_i and fixed time $\tilde{t}$ can be expressed in terms of the coefficients $a_i(\tilde{t})$ as

$$y^h(x_i, \tilde{t}) = a_{i-1}(\tilde{t}) + a_i(\tilde{t}), \quad \frac{\partial y^h(x,\tilde{t})}{\partial x}\Big|_{x=x_i} = \frac{2}{h}(a_i(\tilde{t}) - a_{i-1}(\tilde{t})). \tag{2.5}$$

From (2.5) and homogeneous boundary conditions we get $a_{-1}(t) = -a_0(t)$ and $a_n(t) = -a_{n-1}(t)$. Hence we have

$$y^h(x,t) = \sum_{i=0}^{n-1} a_i(t)\xi_i(x), \tag{2.6}$$

where $\xi_0(x) = (\eta_0(x) - \eta_{-1}(x)), \xi_i(x) = \eta_i(x)(i = 1, 2, \cdots, n-2), \xi_{n-1}(x) = \eta_{n-1}(x) - \eta_n(x)$. Hence n unknowns $a_i(t)(i = 0, 1, \cdots, n-1)$ for every moment of t can be determined.

According to Galerkin method the weighted function $v^h(x)$ in (2.2) is chosen as $v_i^h(x) = \xi_i(x)(i = 0, 1, \cdots, n-1)$. Substituting (2.6) into (2.2) we obtain

$$\begin{cases} \sum_{i=0}^{n-1}\left(\int_0^L \xi_i\xi_j dx\right)\frac{da_i(t)}{dt} + \sum_{i=0}^{n-1}\left(\int_0^L \xi_i^{'}\xi_j^{'} dx\right)\frac{da_i(t)}{dt} \\ \qquad + \alpha\sum_{i=0}^{n-1}\left(\int_0^L \xi_i^{'}\xi_j^{'} dx\right)a_i(t) + \beta\sum_{i=0}^{n-1}\left(\int_0^L \xi_i^{'}\xi_j dx\right)a_i(t) \\ \qquad + \sum_{i=0}^{n-1}\sum_{k=0}^{n-1}\left(\int_0^L \xi_i\xi_k^{'}\xi_j dx\right)a_i(t)a_k(t) = \int_0^L f\xi_j dx, \\ \sum_{i=0}^{n-1}\left(\int_0^L \xi_i\xi_j dx\right)a_i(0) = \int_0^L y_0(x)\xi_j dx, \quad j = 0, 1, \cdots, n-1. \end{cases} \tag{2.7}$$

Assumimg $m_{ij} = (\xi_i, \xi_j)$, $s_{ij} = (\xi_i^{'}, \xi_j^{'})$, $d_{ij} = (\xi_i^{'}, \xi_j)$, $n_{ijk} = (\xi_i\xi_k^{'}, \xi_j)$, $f_j = (f, \xi_j)$, $y_0^j = (y_0, \xi_j)$, and $M = (m_{ij})$, $S = (s_{ij})$, $D = (d_{ij})$, $N = (n_{ijk})$, $\vec{f} = (f_0, f_1, \cdots, f_{n-1})^T$, $\vec{y}_0 = (y_0^0, y_0^1, \cdots, y_0^{n-1})$, $\vec{a}_0 = (a_0(0), a_1(0), \cdots, a_{n-1})^T$, $\vec{a}(t) = (a_0(t), a_1(t), \cdots, a_{n-1}(t))^T$, the system (2.7) can be written in the matrix form

$$\begin{cases} (M+S)\frac{d\vec{a}}{dt} + (\alpha S + \beta D)\vec{a} + (\vec{a})^T N\vec{a} = \vec{f}, \\ M\vec{a}_0 = \vec{y}_0. \end{cases} \tag{2.8}$$

system (2.8) is a nonlinear ordinary differential equations which consists of n equations and n unknowns. Because M and $(M+S)$ are invertible matrices, the system (2.8) can be written as a standard first order nonlinear ordinary differential equations with initial condition, namely,

$$\frac{d\vec{a}}{dt} = (M+S)^{-1}(\vec{f} - ((\alpha S + \beta D)\vec{a} + (\vec{a})^T N\vec{a})), \quad \vec{a}_0 = \vec{y}_0, \tag{2.9}$$

where, $\vec{y}_0 = M^{-1}\vec{y}_0$. Because the right terms in (2.9) are continuously differentiable, and the system (2.9) has one and only one solution and has a zero equilibrium solution when forcing term $f(x,t)$ tends to zero while time approach infinity. Therefore, letting the equilibrium solution as the starting point, we obtain the numerical solution of the system (1.1) by using Newton method. The system (2.9) will be applied to the feedback control design in the next section and the approximate solution we obtained here will be used to compare with the controlled solution in the final section.

3. Feedback control design

Now we describe our control problem: Find an optimal control $u^*(t)$ which minimizes the cost functional

$$J(u) = \int_0^\infty ||y(t,\cdot)||^2_{L^2(\Omega)} + |u(t)|^2 dt$$

subject to the constraint equations

$$\begin{cases} y_t - y_{xxt} = \alpha y_{xx} - \beta y_x - y y_x + f(x,t) \quad \text{in } [0,L]\times[0,T], \\ y(0,t) = y(L,t) = 0 \quad \text{on } [0,T], \\ y(x,0) = y_0(x) \quad \text{in } [0,L]. \end{cases} \tag{3.1}$$

We will replace the forcing term by the special form $b(x)u(t)$ in the system (3.1), then $u(t)$ is the control input and $b(x)$ is a given function used to distribute the control over the domain.

3.1. **Linear Quadratic Regulator Design.** Assuming that the nonlinear term in the BBMB equation is small, a suboptimal feedback control u^* can be obtained by using the well-known linear quadratic regulator theory [17]-[19]. A full state feedback control is to find an optimal control $u^* \in L^2([0,T), L^2(\Omega))$ by minimizing the cost functional

$$J(u) = \int_0^\infty (Qy(t,\cdot), y(t,\cdot))_{L^2(\Omega)} + (Ru(t), u(t)))dt$$

subject to the constraint equations

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0, \quad \text{for} \quad t > 0$$

where $Q : L^2(\Omega) \to L^2(\Omega)$ is a nonnegative definite self-adjoint weighting operator for state and $R : L^2(\Omega) \to L^2(\Omega)$ is a positive definite weighting operator for the control. The optimal control $u^*(t)$ can be found as

$$u^*(t) = -\frac{1}{2}R^{-1}B^T\Pi y(t) = -Ky(t),$$

where K is called the feedback operator and Π is symmetric positive definite solution of the algebraic Riccati equation

$$\Pi A + A^T\Pi - \Pi BR^{-1}B^T\Pi + Q = 0. \tag{3.2}$$

Remark 3.1. In the actual calculation, we take $A = (M+S)^{-1}(\alpha S+\beta D)$, and the operator B is constructed by $b(x)$ and test function $\xi(x)$. Here, the method of design controller is the same as the ordinary LQR scheme, but the properties (2.5) are considered in the whole process of the controller design (2.5).

3.2. **Linear feedback controllers with state estimate feedback.** A simple, classical feedback control design, linear quadratic regulator (LQR), assumes the full state is "feed back" into the system by the control. However, knowledge of the full state is not possible for many complicated physical systems. As a realistic alternative, a compensator design provides a state estimate based on state measurements to be used in the feedback control law.

We do not assume that we have knowledge of the full state. Instead, we assume a state measurement of the form

$$z(t) = Cy(t), \tag{3.3}$$

where $C \in \mathcal{L}(L^2(\Omega), \mathbb{R}^m)$. We can apply the theory and results to show that a stabilizing compensator based controller can be applied to the system [20].

The observer design is mainly needed in order to provide the feedback control law with estimated state variables. Therefore, the control law and observer are combined together into a complete system. The combined system is called compensator. This technique needs the availability of a limited measurement of the state as a condition. we assume that we have a system in the abstract form

$$\dot{y}(t) = Ay(t) + G(y(t)) + Bu(t), \qquad y(0) = y_0, \tag{3.4}$$

where $y(t)$ is in a state space $L^2(\Omega)$ and $u(t)$ is in a control space U.

According to the given state measurement (3.3), a state estimate $\tilde{y}(t)$ is computed by solving the observer equation

$$\dot{\tilde{y}}(t) = A\tilde{y}(t) + G(\tilde{y}(t)) + Bu(t) + L[z(t) - C\tilde{y}(t)], \qquad \tilde{y}(0) = \tilde{y}_0. \tag{3.5}$$

The feedback control law is given by

$$u(t) = -K\tilde{y}(t), \tag{3.6}$$

where K is called the feedback operator. Functional gain operator K and estimator gain operator L are determined by linear quadratic regulator (LQR) and Kalman estimator (LQE), respectively, in usual manner. According to the result of the above, we already know

$$K = R^{-1}B^T\Pi. \tag{3.7}$$

Next, P is found as the non-negative definite solution of

$$AP + PA^T - PC^TCP + \bar{Q} = 0,$$

where $\bar{Q}$ is a non-negative definite weighting operator. If the solution P exists, we can define

$$L = PC^T. \tag{3.8}$$

Form (3.3)-(3.8), we obtain the closed loop compensator as

$$\begin{cases} \dot{y}(t) = Ay(t) - BK\tilde{y}(t) + G(y(t)), \\ \dot{\tilde{y}}(t) = LCy(t) + (A - LC - BK)\tilde{y}(t) + G(\tilde{y}(t)), \\ \quad y(0) = y_0, \quad \tilde{y}(0) = \tilde{y}_0, \end{cases} \tag{3.9}$$

Remark 3.2. In this subsection, the matrix A and B are the same as the above in remark 3.1, and nonlinear term is $G(y) = (M + S)^{-1}y^TNy$. Backward Euler method is applied to solve numerical solution of the system (3.9).

4. Computational Experiments

For numerical computations, two cases are considered as follows.

Case 1: parameters $\alpha = 0.5$, $\beta = 1$, $T = 5$ and initial condition

$$y_0(x) = \exp(-x)\sin(\pi x).$$

Case 2: parameters $\alpha = 0.0001$, $\beta = 10$, $T = 10$, and initial condition

$$y_0(x) = 4x(1 - x).$$

The following quantities can be used for the numerical computation in this paper. The spatial interval is taken to be $[0, 1]$. The spatial and time step sizes are $h = 1/64$ and $1/100$, respectively. The control input operator is $B = \int_0^L b(x)\xi_i(x)dx$, where $b(x) = x$ and $\xi_i(x)$ is a test function($i = 0, 1, \cdots, n-1$). The state weighting operator (used in Riccatheti equation calculations) Q is taken to be $(M + S)$. We set control weighting operator $R(1, 1) = 10^{-6}$. And the weighting operator $\bar{Q}$ is also chosen as $(M + S)$. Finally, we create the measurement matrix C with $Cy(t, x) = 8\int_{3/4}^{5/6} y(t, x)dx$ for the state estimate feedback controller.

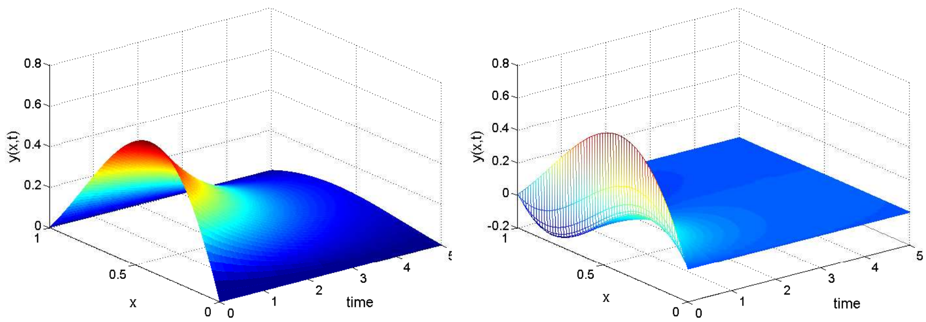

FIGURE 1. Uncontrolled solution (left) and state estimate feedback controlled solution for Case 1

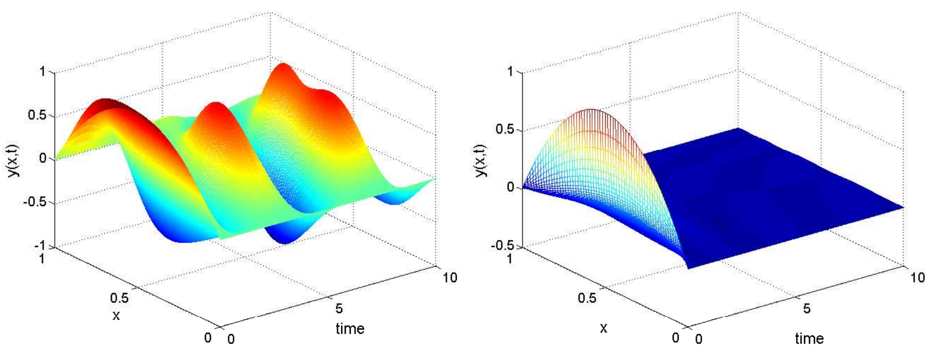

FIGURE 2. Uncontrolled solution (left) and state estimate feedback controlled solution for Case 2

Fig.1 and Fig.2 present uncontrolled solution (left) and feedback controlled solution for the Case 1 and the Case 2, respectively. Fig. 3 shows that the L^2-norms of $y(t)$ for the Case 1 and the Case 2. The finite element approximate solution of the BBMB equation is smooth and slowly tends to zero when the time goes infinite, but the norm of the controlled solution is

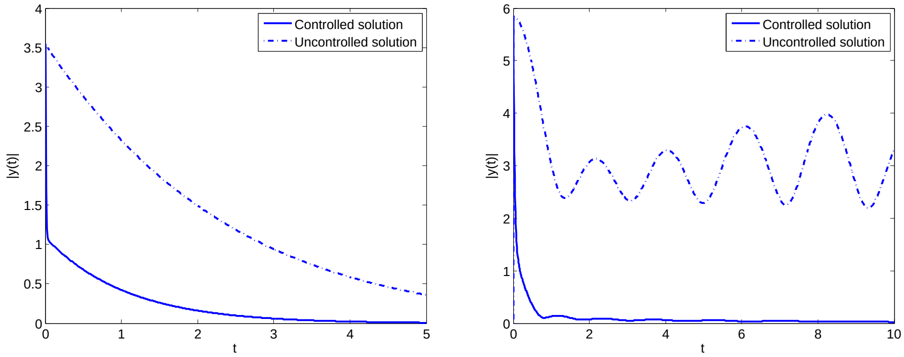

FIGURE 3. L^2-norms for the solutions of the uncontrolled and controlled problem vs. time for Case 1 (left) and Case 2.

rapidly becoming small just at the begging, and then, slowly close to zero for the Case 1. In the Case 2, for small coefficient α and the appropriate constant β, the finite element approximate solution of the BBMB equation fluctuates in a certain range, but the control solution is rapidly becoming smaller just at the begging, and then, close to zero with very small fluctuations.

The BBMB equation have appeared frequently as models of physical phenomena[21]-[23]. In the BBMB equation, the appropriate parameter β can control the dissipative phenomena, but it is difficult to achieve our desired situation. In this paper, our control scheme got a desired solution successfully. And the numerical solution results also shows that the scheme is efficient and feasible.

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