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DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR PARABOLIC PROBLEMS WITH MIXED BOUNDARY CONDITION†

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Abstract. In this paper we consider the nonlinear parabolic problems with mixed boundary condition. Under comparatively mild conditions of the coefficients related to the problem, we construct the discontinuous Galerkin approximation of the solution to the nonlinear parabolic problem. We discretize spatial variables and construct the finite element spaces consisting of discontinuous piecewise polynomials of which the semidiscrete approximations are composed. We present the proof of the convergence of the semidiscrete approximations in $L^\infty(H^1)$ and $L^\infty(L^2)$ normed spaces.

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1. Introduction

Discontinuous Galerkin (DG) finite element methods employ discontinuous piecewise polynomials to approximate the solutions of differential equations and impose interelement continuity weakly. Even though DG methods often have been involved with large number of degrees of freedom than the classical Galerkin method, DG methods are adopted widely in a variety of differential equations. DG methods were introduced for the numerical solutions of first-order hyperbolic system, but independently they are proposed as nonconforming schemes for the numerical solutions of 2nd order elliptic problems by Nitsche [10]. Recently there has been renewed interest in DG methods due to their efficient properties which include a high degree of locality, the flexibility of locally varying the degree of polynomial in adaptive hp version approximations since no continuity requirement is imposed.
Much attention have been devoted to the analysis of DG methods applied to

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elliptic problem [6, 7, 8] as well as to several other types of nonlinear equations including time-dependent convection-diffusion equations [3], non-Fickian diffusion equation [14], Camassa-Holm equation [18], solid viscoelasticity problems [15], Maxwell equations [4], Navier Stokes equations [16], Keller-Segel chemotaxis model [5] and reactive transport problem [17].
In this paper we consider the DG methods applied to parabolic problems. In [13] Rieviere and Wheeler initiated to adopt DG method and develop DG approximations to parabolic problems. They constructed discontinuous and time discretized approximations and obtained the optimal convergence order of spatial error in H^1 and time truncation error in L^2 normed space. In [11] the authors applied DG method to parabolic problem with homogeneous Neumann boundary condition and constructed DG spatial discretized approximations using a penalty term and obtained an optimal $L^\infty(L^2)$ error estimate. In addition the authors [12] applied DG method to construct the fully discrete approximations for the parabolic problems with homogeneous Neumann boundary condition and obtained the optimal order of convergence in $\ell^\infty(L^2)$ normed space.
And also Lasis and Süli [9] considered the hp-version DG method with interior penalty for semilinear parabolic equations to construct spatial discretized approximations and obtained an optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ error estimates.
In this paper we consider the semidiscrete DG approximations of the nonlinear parabolic equations. Compared to the previous works in this paper we require very weak conditions on the terms characterizing the nonlinearity of the parabolic problem. In this paper we weaken the conditions of the tensor coefficient and the forcing term so that they are assumed to be locally Lipschitz continuous only. In addition, the parabolic problem considered in this paper is related with mixed nonhomogeneous Dirichlet-nonhomogeneous Neumann boundary conditon so that we manage the most generalized boundary condition. The rest of this paper is as follows. In Section 2 we introduce our parabolic problem to be considered and some notations and we construct finite element space. In Section 3, we develop some auxiliary projection onto finite element space and we prove its convergence of optimal order. In Section 4 we construct the semidiscrete approximation and prove its existence and finally we provide the error analysis of the semidiscrete approximations.

2. The problem and notations

Consider the following nonlinear parabolic differential equation:

$$\begin{cases} u_t - \nabla\cdot(a(x,u)\nabla u) = f(x,t,u) & (x,t)\in\Omega\times(0,T] \\ a(x,u)\nabla u\cdot\boldsymbol{n} = g_N(x,t) & (x,t)\in\partial\Omega_N\times(0,T] \\ u(x,t) = g_D(x,t) & (x,t)\in\partial\Omega_D\times(0,T] \\ u(x,0) = u_0(x) & x\in\Omega. \end{cases} \tag{2.1}$$

where Ω is a bounded open convex domain in $\mathbb{R}^d$, $1\le d\le 3$, $\partial\Omega$ is the boundary of Ω, $\partial\Omega_N\cup\partial\Omega_D=\partial\Omega$, $\partial\Omega_N\cap\partial\Omega_D=\phi$ and $\boldsymbol{n}$ is a unit outward normal vector to $\partial\Omega$.

Assume that

(A1). $a(x, u(x,t))$ is continuous at $(x,t) \in \overline{\Omega} \times [0,T]$

(A2). There exists a positive constant a_* such that $a(x, u(x,t)) \geq a_*$, $\forall (x,t) \in \overline{\Omega} \times [0,T]$.

Let $\Omega_h = \{E_1, E_2, \cdots, E_{N_h}\}$ be a subdivision of Ω, where E_i is an interval if $d = 1$, and in case of $d = 2(d = 3)$ E_i is a triangle or a quadrilateral (a symplex or parallelogram) which may have one curved edge (face). Let h_i be the diameter of E_i and $h = \max\{h_i : 1 \leq i \leq N_h\}$. We assume that there exists a constants δ such that $\delta^{-1} h \leq h_i \leq \delta h,\ 1 \leq i \leq N_h$.

Let $\mathcal{E}_h$ be the set of the edge of E_i, $1 \leq i \leq N_h$ and we let

$$\begin{aligned}
&\mathcal{E}_I = \{e \in \mathcal{E}_h \mid \tilde{m}(e \cap \Omega) \neq 0\},\ \mathcal{E}_D = \{e \in \mathcal{E}_h \mid \tilde{m}(e \cap \partial\Omega_D) \neq 0\}, \\
&\mathcal{E}_N = \{e \in \mathcal{E}_h \mid \tilde{m}(e \cap \partial\Omega_N) \neq 0\},\ \mathcal{E}_{DN} = \mathcal{E}_D \cup \mathcal{E}_N,\ \mathcal{E}_{ID} = \mathcal{E}_I \cup \mathcal{E}_D,
\end{aligned}$$

where $\tilde{m}$ is $(d-1)$ dimensional measure defined in $\mathbb{R}^{d-1}$. If $e = \partial E_i \cap \partial E_j$ with $i < j$, the unit outward normal vector $\boldsymbol{n}_i$ to E_i is taken as the unit vector $\boldsymbol{n}$ associated with e

The L^2 inner product is denoted by $(\cdot,\cdot)$ and we denote usual L^2 norm defined on E by $\|\cdot\|_E$, and usual L^∞ norm by $\|\cdot\|_{\infty,E}$. In both cases we may skip E if $E = \Omega$. Let $H^s(E)$ be the Sobolev space equipped with the usual Sobolev norm $\|v\|_{s,E} = \sum\limits_{|\delta| \leq s} \int_E |D^l v|^2 dx$ where $D^l v = \frac{\partial^{|l|} v}{\partial^{l_1} x_1 \cdots \partial^{l_d} x_d}, l = (l_1, \cdots, l_d)$. If $E = \Omega$, we simply denote it by $\|\cdot\|_s$ and if $s = 0$ denote it by $\|\cdot\|_E$. We denote the usual seminorm defined on E by $|\cdot|_{s,E}$. And also we denote $W^{s,\infty}(E) = \{v \mid D^l v \in L^\infty,\ \ \forall\ |l| \leq s\}$ equipped with the norm $\|v\|_{W^{s,\infty}(E)} = \max\limits_{|l| \leq s} \operatorname{ess\,sup} |D^l v|$. If $E = \Omega$ then for our convevience we skip E in the notation of $W^{s,\infty}(E)$. Now we let $H^s(\Omega_h) = \{v \mid v|_{E_i} \in H^s(E_i)\ 1 \leq i \leq N_h\}$. If $v \in H^s(\Omega_h)$ with $s > \frac{1}{2}$ we define the average $\{v\}$ and the jump $\langle v \rangle$ functions as follows: For $e \in \partial E_i \cap \partial E_j$ with $i < j$ then

$$\langle v(x) \rangle = \frac{1}{2}(v|_{\overline{E}_i \cap e} + v|_{\overline{E}_j \cap e}),\ [v(x)] = v(x)|_{\overline{E}_j \cap e} - v(x)|_{\overline{E}_i \cap e},\ \forall x \in e.$$

For $e \in \partial\Omega_D$,

$$\langle v(x) \rangle = [v(x)] = v(x),\ \forall x \in e.$$

Now we define the following broken norm on $H^2(\Omega_h)$:

$$|||v|||_1^2 = \sum_{j=1}^{N_h} \|v\|_{1,E_j}^2 + \sum_{e \in \mathcal{E}_{ID}} h \|\langle \nabla v \cdot \boldsymbol{n} \rangle\|_e^2 + \sum_{e \in \mathcal{E}_{ID}} h^{-1} \|[v]\|_e^2.$$

To continue our analysis we may assume that E_i is a triangle. For the case that E_i is a rectangle we may develop the analogous theories. We let V_h^r be the space of piecewise polynomials defined as

$$V_h^r = \{v | \; v|_{E_i} \in P_r(E_i) \;\; 1 \le i \le N_h\},$$

where $P_r(E_j)$ is the set of polynomials of total degree $\le r$.

3. Approximation properties and elliptic projection

Hereafter C denotes a generic positive constant independent of h and any two Cs in different positions don't need to be the same. The following approximation properties are proved in [1, 2].

Lemma 3.1. *Let $E \in \Omega_h$, e be an edge of E and $v \in H^s(E)$. Then there exist a positive constant independent of v, r and h and a sequence $\hat{v}_r^h \in P_r(E)$, $r = 1, 2, \cdots$, such that for any $0 \le q \le s$,*

$$\begin{aligned}
&\|v - \hat{v}_r^h\|_{q,E} \le Ch^{\mu-q}\|v\|_{s,E}, \qquad s \ge 0,\\
&\|v - \hat{v}_r^h\|_{e} \le Ch^{\mu-\frac{1}{2}}\|v\|_{s,E}, \qquad s > \frac{1}{2},\\
&\|v - \hat{v}_r^h\|_{1,e} \le Ch^{\mu-\frac{3}{2}}\|v\|_{s,E}, \qquad s > \frac{3}{2},\\
&\|v - \hat{v}_r^h\|_{\infty,E} \le Ch^{\mu}\|v\|_{W^{s,\infty}(E)},
\end{aligned}$$

hold where $\mu = \min(r+1, s)$. Moreover if $e = \partial E_i \cap \partial E_j$ then $\|\nabla \hat{v}_r^h\|_{\infty,e} \le C\|\nabla v\|_{\infty, E_i \cup E_j}$ holds.

Lemma 3.2. *Let $E \in \Omega_h$, e be an edge of E and $\boldsymbol{n}$ be a normal vector associated with e. Then there exists a positive constant C such that $\forall v \in H^1(E)$*

$$\begin{aligned}
&\|v\|_e^2 \le C(h^{-1}\|v\|_E^2 + h\|\nabla v\|_E^2),\\
&\|\nabla v \cdot \boldsymbol{n}\|_e^2 \le C(h^{-1}\|\nabla \phi\|_E^2 + h|\phi|_{2,E}^2),\\
&\|v\|_e^2 \le Ch^{-1}\|v\|_E^2.
\end{aligned}$$

Now we let $\hat{u}$ be the interpolation of u satisfying the approximation properties of Lemma 3.1. By applying Lemma 3.1 we obviously have the following Lemma.

Lemma 3.3. *If $u \in H^s(\Omega)$, then $\hat{u}$ satisfies the following approximation property*

$$|||u - \hat{u}|||_1 \le Ch^{\mu-1}\|u\|_s$$

where $\mu = \min(r+1, s)$.

Proof. By applying Lemma 3.1 and 3.2, we get

$$\begin{aligned}
|||u - \hat{u}|||_1^2 =& \sum_{i=1}^{N_h} \|\nabla(u - \hat{u})\|_{E_i}^2 + \sum_{e \in \mathcal{E}_{ID}} h\|\langle \nabla(u - \hat{u}) \cdot \boldsymbol{n}\rangle\|_e^2\\
&+ \sum_{e \in \mathcal{E}_{ID}} h^{-1}\|[u - \hat{u}]\|_e^2 + \alpha\|u - \hat{u}\|^2\\
\le& C\sum_{i=1}^{N_h} h^{2(\mu-1)}\|u\|_{s,E_i}^2 \le Ch^{2(\mu-1)}\|u\|_s^2
\end{aligned}$$

□

We define the following bilinear form $A^\beta(a,u:v,w)$ on $V_h^r \times V_h^r$:

$$\begin{aligned}A^\beta(a,u:v,w) =& \sum_{i=1}^{N_h} \big(a(x,u)\nabla v, \nabla w\big)_{E_i} \\ &- \sum_{e\in\mathcal{E}_{ID}} \big(\langle a(x,u)\nabla v\cdot \boldsymbol{n}\rangle, [w]\big)_e - \sum_{e\in\mathcal{E}_I} \big([v], \langle a(x,u)\nabla w\cdot \boldsymbol{n}\rangle\big)_e \\ &+ \sum_{e\in\mathcal{E}_{ID}} \big(\beta h^{-1}[v],[w]\big)_e,\ \forall\,(v,w)\in V_h^r\times V_h^r.\end{aligned}$$

For a $\alpha > 0$ we let $A_\alpha^\beta(a,u:v,w) = A^\beta(a,u:v,w) + \alpha(v,w)$.

Lemma 3.4. *For any $v, w \in H^s(\Omega_h)$ with $s \geq 2$*

$$|A_\alpha^\beta(a,u:v,w)| \leq C|\!|\!|v|\!|\!|_1 |\!|\!|w|\!|\!|_1$$

holds.

Proof. For any $v, w \in H^s(\Omega_h)$ with $s \geq 2$,

$$\begin{aligned}|A_\alpha^\beta(a,u:v,w)| \leq& \sum_{i=1}^{N_h} |(a(x,u)\nabla v,\nabla w)_{E_i}| + \sum_{e\in\mathcal{E}_{ID}} |\big(\langle a(x,u)\nabla v\cdot\boldsymbol{n}\rangle,[w]\big)_e| \\ &+ \sum_{e\in\mathcal{E}_I} |\big([v],\langle a(x,u)\nabla w\cdot\boldsymbol{n}\rangle\big)_e| + \sum_{e\in\mathcal{E}_{ID}} |\big(\beta h^{-1}[v],[w]\big)_e| \\ &+ \alpha(v,w) \\ \leq& a^* \sum_{i=1}^{N_h} \|\nabla v\|_{E_i}\|\nabla w\|_{E_i} \\ &+ a^*\big(\sum_{e\in\mathcal{E}_{ID}} \frac{\sigma}{h}\|[w]\|_e^2\big)^{\frac12}\big(\sum_{e\in\mathcal{E}_{ID}} \frac{h}{\sigma}\|\langle\nabla v\cdot\boldsymbol{n}\rangle\|_e^2\big)^{\frac12} \\ &+ a^*\big(\sum_{e\in\mathcal{E}_I} \frac{h}{\sigma}\|\langle\nabla w\cdot\boldsymbol{n}\rangle\|_e^2\big)^{\frac12}\big(\sum_{e\in\mathcal{E}_I} \frac{\sigma}{h}\|[v]\|_e^2\big)^{\frac12} \\ &+ \big(\sum_{e\in\mathcal{E}_{ID}} \beta h^{-1}\|[v]\|_e^2\big)^{\frac12}\big(\sum_{e\in\mathcal{E}_{ID}} \beta h^{-1}\|[w]\|_e^2\big)^{\frac12} + \alpha(v,w) \\ \leq& C|\!|\!|v|\!|\!|_1|\!|\!|w|\!|\!|_1.\end{aligned}$$

□

Lemma 3.5. *If β is sufficiently large, then there is a constant $\underset{\sim}{c} > 0$ such that*

$$A_\alpha^\beta(a,u:v,v) \geq \underset{\sim}{c}|\!|\!|v|\!|\!|_1^2, \qquad \forall v\in V_h^r.$$

Proof. For $\forall v \in V_h^r$ and $\delta > 0$

$$
\begin{aligned}
A_\alpha^\beta(a,u:v,v) = &\sum_{i=1}^{N_h} \big(a(x,u)\nabla v, \nabla v\big)_{E_i} - \sum_{e\in\mathcal{E}_{ID}} \big(\langle a(x,u)\nabla v\cdot \boldsymbol{n}\rangle, [v]\big)_e \\
&- \sum_{e\in\mathcal{E}_I} (\langle a(x,u)\nabla v\cdot \boldsymbol{n}\rangle, [v])_e + \sum_{e\in\mathcal{E}_{ID}} \beta h^{-1}\|[v]\|_e^2 + \alpha(v,v) \\
\geq &a_* \sum_{i=1}^{N_h} \|\nabla v\|_{E_i}^2 - \big\{ \sum_{e\in\mathcal{E}_{ID}} \big(\frac{a^*}{\delta} h\|\langle \nabla v\cdot \boldsymbol{n}\rangle\|_e^2 + \delta h^{-1}\|[v]\|_e^2\big)\big\} \\
&- \big\{ \sum_{e\in\mathcal{E}_I} \big(\frac{a^*}{\delta} h\|\langle \nabla v\cdot \boldsymbol{n}\rangle\|_e^2 + \delta h^{-1}\|[v]\|_e^2\big)\big\} + \sum_{e\in\mathcal{E}_{ID}} \beta h^{-1}\|[v]\|_e^2 + \alpha\|v\|^2,
\end{aligned}
$$

holds. By applying Lemma 3.2 we get for sufficiently large β and δ,

$$
\begin{aligned}
A_\alpha^\beta(a,u:v,v) &\geq (a_* - \frac{a^*C}{\delta}) \sum_{i=1}^{N_h} \|\nabla v\|_{E_i}^2 + \sum_{e\in\mathcal{E}_{ID}} (\beta - 2\delta) h^{-1}\|[v]\|_e^2 + \alpha\|v\|^2 \\
&\geq C(\sum_{i=1}^{N_h} \|\nabla v\|_{E_i}^2 + \sum_{e\in\mathcal{E}_{ID}} h^{-1}\|[v]\|_e^2) + \alpha\|v\|^2 \\
&\geq C((\sum_{i=1}^{N_h} \|\nabla v\|_{E_i}^2 + \sum_{e\in\mathcal{E}_{ID}} h\|\langle \nabla v\cdot \boldsymbol{n}\rangle\|_e^2 + \sum_{e\in\mathcal{E}_{ID}} h^{-1}\|[v]\|_e^2) + \alpha\|v\|^2 \\
&\geq \underset{\sim}{c} \|\!|v|\!\|_1^2,
\end{aligned}
$$

since

$$
\sum_{e\in\mathcal{E}_{ID}} h\|\langle \nabla v\cdot \boldsymbol{n}\rangle\|_e^2 \leq C\sum_{i=1}^{N_h} (\|\nabla v\|_{E_i}^2 + h^2\|\nabla^2 v\|_{E_i}^2) \leq C\sum_{i=1}^{N_h} \|\nabla v\|_{E_i}^2.
$$

□

By applying Lemmas 3.4 and 3.5 there exists $\tilde{u}(t) \in V_h^r$ satisfying

$$
\begin{cases} A_\alpha^\beta(a,u:u-\tilde{u},v) = 0 \quad \forall v\in V_h^r, \quad \forall t>0 \\ \tilde{u}(0) = P_h(u_0(x)). \end{cases} \tag{3.1}
$$

By Lax-Milgram Lemma, $\tilde{u}$ satisfies

$$
\|\!|u-\tilde{u}|\!\|_1 \leq Ch^{\mu-1}\|u\|_s. \tag{3.2}
$$

Now we let

$$
\eta = u - \tilde{u} \text{ and } \theta = \hat{u} - \tilde{u}.
$$

Lemma 3.6. *Let G be a linear mappping defined on $H^2(\Omega_h)$ and suppose that there exists $w \in H^2(\Omega_h)$ satisfying*

$$
A_\alpha^\beta(a,u:w,v) = G(v), \quad \forall v\in V_h^r.
$$

Suppose that there exist positive constants K_1 and K_2 such that

$$|G(v)| \le K_1 |\!|\!| v |\!|\!|_1, \qquad \forall v \in H^2(\Omega_h),$$
$$|G(\phi)| \le K_2 \|v\|_2, \qquad \forall \phi \in H^2(\Omega) \cap \tilde{H}(\Omega)$$

where $\tilde{H}(\Omega) = \{v \in H^1(\Omega) \mid g = 0 \text{ on } \partial\Omega_D\}$. Then we have the following estimation

$$\|w\| \le C\ (h|\!|\!| w |\!|\!|_1 + K_1 h + K_2).$$

Proof. Let ϕ be the solution of

$$\begin{cases} -\nabla \cdot (a(x,u)\nabla\phi) + \alpha\phi = w, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega_D. \end{cases} \tag{3.3}$$

Then by the regularity property of elliptic problem we get $\|\phi\|_2 \le C\|w\|$. By Lemma 3.3 there exists $\hat{\phi}$, an interpolation ϕ satisfying $|\!|\!| \phi - \hat{\phi} |\!|\!|_1 \le Ch\|\phi\|_2$. By (3.3) we have

$$\begin{aligned} \|w\|^2 &= \big(w, -\nabla \cdot (a(x,u)\nabla\phi)\big) + \alpha(w,\phi) = A_\alpha^\beta(a, u : w, \phi) \\ &= A_\alpha^\beta(a, u : w, \phi - \hat{\phi}) + A_\alpha^\beta(a, u : w, \hat{\phi}) \\ &\le C|\!|\!| w |\!|\!|_1 |\!|\!| \phi - \hat{\phi} |\!|\!|_1 + G(\phi) - G(\phi - \hat{\phi}) \\ &\le C\big(h|\!|\!| w |\!|\!|_1 \|\phi\|_2 + K_2\|\phi\|_2 + K_1 h\|\phi\|_2\big) \\ &\le C\big(h|\!|\!| w |\!|\!|_1 + K_2 + K_1 h\big)\|w\| \end{aligned}$$

which implies that

$$\|w\| \le C\big(h|\!|\!| w |\!|\!|_1 + K_1 h + K_2\big).$$

□

Theorem 3.1. *Suppose that $u(\cdot,t) \in H^s,\ u_t(\cdot,t) \in H^s$ then the following error estimations hold:*

$$\|\eta\| \le Ch^\mu\|u\|_s,\ \|\eta_t\| \le Ch^\mu(\|u\|_s + \|u_t\|_s),\ |\!|\!| \eta_t |\!|\!|_1 \le Ch^{\mu-1}(\|u\|_s + \|u_t\|_s),$$

where $\mu = \min(r+1, s)$. And also if $\mu \ge \frac{d}{2}+1$, then $\|\nabla\tilde{u}\|_{L^\infty(E)}$ and $\|\nabla\tilde{u}\|_{L^\infty(e)}$ are bounded for $e \in \mathcal{E}_h$.

Proof. By Lemma 3.4 and 3.5, we have

$$|\!|\!| \theta |\!|\!|_1^2 \le CA_\alpha^\beta(a, u : \theta, \theta) = CA_\alpha^\beta(a, u : \hat{u} - u, \theta) \le C|\!|\!| u - \hat{u} |\!|\!|_1 |\!|\!| \theta |\!|\!|_1$$

so that

$$|\!|\!| \theta |\!|\!|_1 \le C|\!|\!| u - \hat{u} |\!|\!|_1 \le Ch^{\mu-1}\|u\|_s. \tag{3.4}$$

By (3.1), (3.2) and Lemma 3.6 with $G(v) = 0$, we get

$$\|\eta\| \le Ch|\!|\!| \eta |\!|\!|_1 \le Ch^\mu\|u\|_s. \tag{3.5}$$

Now we differentiate $A_\alpha^\beta(a, u : \eta, v) = 0$ with respect to t to obtain

$$A_\alpha^\beta(a, u : \eta_t, v) = \tilde{G}(v)$$

where

$$\begin{aligned}\tilde{G}(v) = &-\sum_{i=1}^{N_h}\big(\frac{\partial}{\partial t}(a(x,u))\nabla\eta, \nabla v\big)_{E_i} + \sum_{e\in\mathcal{E}_{ID}}(\langle\frac{\partial}{\partial t}(a(x,u))\nabla\eta\cdot\boldsymbol{n}\rangle, [v])_e \\ &+ \sum_{e\in\mathcal{E}_I}([\eta], \langle\frac{\partial}{\partial t}(a(x,u))\nabla v\cdot\boldsymbol{n}\rangle)_e.\end{aligned}$$

By the similar process as the proof of Lemma 3.4 we have with $v \in H^2(\Omega_h)$,

$$|\tilde{G}(v)| \le C|||\eta|||_1 |||v|||_1, \tag{3.6}$$

and with $v \in H^2(\Omega) \cap \tilde{H}(\Omega)$,

$$|\tilde{G}(v)| \le C\|\eta\|\|v\|_2.$$

Therefore, by Lemma 3.6

$$\|\eta_t\| \le C(h|||\eta_t|||_1 + h|||\eta|||_1 + \|\eta\|). \tag{3.7}$$

By Lemma 3.5, Lemma 3.4 and (3.6) we have

$$\begin{aligned}|||\theta_t|||_1^2 &\le CA_\alpha^\beta(a,u:\theta_t,\theta_t) \\ &= C(A_\alpha^\beta(a,u:\eta_t,\theta_t) - A_\alpha^\beta(a,u:u_t-\hat{u}_t,\theta_t)) \\ &\le C(|\tilde{G}(\theta_t)| + |||u_t-\hat{u}_t|||_1|||\theta_t|||_1) \le C(|||\eta|||_1 + |||u_t-\hat{u}_t|||_1)|||\theta_t|||_1,\end{aligned}$$

which implies that by Lemma 3.3,

$$|||\theta_t|||_1 \le C(|||\eta|||_1 + |||u_t-\hat{u}_t|||_1) \le Ch^{\mu-1}(\|u\|_s + \|u_t\|_s).$$

Hence we get

$$|||\eta_t|||_1 \le |||\theta_t|||_1 + |||u_t-\hat{u}_t|||_1 \le Ch^{\mu-1}(\|u\|_s + \|u_t\|_s). \tag{3.8}$$

Now we substitute (3.8), (3.2) and (3.5) into (3.7) to get

$$\|\eta_t\| \le Ch^\mu(\|u\|_s + \|u_t\|_s).$$

If $\mu \ge \frac{d}{2}+1$ then

$$\begin{aligned}\|\nabla\tilde{u}\|_{L^\infty(E)} &\le \|\nabla u\|_{L^\infty(E)} + \|\nabla\theta\|_{L^\infty(E)} \\ &\le \|\nabla u\|_{L^\infty(E)} + ch^{\mu-1}\|u\|_s \le C\end{aligned}$$

holds. Now we let $e = E_i \cap E_j$. By Lemma 3.1

$$\begin{aligned}\|\nabla\tilde{u}\|_{L^\infty(e)} &\le \|\nabla\hat{u}\|_{L^\infty(e)} + \|\nabla\theta\|_{L^\infty(e)} \\ &\le C\big(\|\nabla u\|_{L^\infty(E_i\cup E_j)}\big) + h^{-\frac{(d-1)}{2}}\|\nabla\theta\|_{L^2(e)}\big) \\ &\le C\big(\|\nabla u\|_{L^\infty(E_i\cup E_j)}\big) + h^{-\frac{(d-1)}{2}}(h^{-\frac{1}{2}}\|\nabla\theta\|_{L^2(E_i\cup E_j)})\big) \\ &\le C(\|\nabla u\|_{L^\infty(E_i\cup E_j)} + h^{-\frac{d}{2}+\mu-1}) \le C\end{aligned}$$

holds. □

4. Spatial discretized approximation and error analysis.

The discontinuous Galerkin method of the problem (1.1) reads as follows: find $u_h(\cdot,t) \in V_h^r$ such that

$$\begin{cases} (u_{ht}, v) + A^\beta(a, u_h : u_h, v) = (f(x,t,u_h), v) + l_\beta(v), \quad \forall v \in V_h^r,\ \forall t > 0, \\ u_h(0) = P_h(u_0(x)) \end{cases} \tag{4.1}$$

where $l_\beta(v) = \sum_{e \in \mathcal{E}_N} (g_N, v)_e + \sum_{e \in \mathcal{E}_D} (g_D, \beta h^{-1} v)_e$ and $P_h(u_0(x))$ denotes the approximation of $u_0(x)$ generated by Lemma 3.1. From (2.1) $u(x,t)$ satisfies

$$(u_t, v) + A^\beta(a, u : u, v) = (f(x,t,u), v) + l_\beta(v), \quad \forall v \in V_h^r,\ \forall t > 0. \tag{4.2}$$

Theorem 4.1. *There exists $u_h(x,t)$ satisfying (4.1). If $f(x,t,u)$ and $a(x,u)$ are locally Lipschitz continuous in t and u then there exists a unique $u_h(x,t)$ locally. And also if $f(x,t,u)$ and $a(x,u)$ are globally Lipschitz continuous in t and u then the unique existence holds globally.*

Proof. Let $\{v_i(x)\}_{i=1}^m$ be a basis of V_h^r and $u_h(x,t) = \sum_{i=1}^m \alpha_i(t) v_i(x)$ and suppose that $P_h(u_0(x)) = \sum_{i=1}^m \alpha_{0i} v_i(x)$. From (4.1) we have

$$\begin{cases} \big(\sum_{i=1}^m \alpha_i'(t) v_i(x), v_j(x)\big) + A^\beta\big(a, \sum_{i=1}^m \alpha_i(t) v_i(x) : \sum_{m=1}^m \alpha_i(t) v_i(x), v_j(x)\big) \\ = \big(f(x,t, \sum_{i=1}^m \alpha_i(t) v_i(x), v_j(x)\big) + l_\beta\big(v_j(x)\big), \quad \forall t > 0,\ 1 \le j \le m \\ \alpha(0) = \alpha_0 = (\alpha_{01}, \alpha_{02}, \cdots, \alpha_{0m})^T. \end{cases} \tag{4.3}$$

Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \cdots, \alpha_m(t))^T$. (4.3) can be represented as the following system

$$M\alpha'(t) = -N(\alpha(t)) + F(\alpha(t)) + L(t)$$

where $M = (M_{ij})_{1 \le i,j \le m}$, $N(\alpha(t)) = (N_{ij}(\alpha))_{1 \le i,j \le m}$ are symmetric matrices and $F(\alpha(t)) = F_j(\alpha(t))_{1 \le j \le m}$ and $L(t) = (L_j(t))_{1 \le j \le m}$ are vectors. $M, N(\alpha(t))$ and $L(t)$ are defined by $M_{ij} = (v_i(x), v_j(x)), N_{ij}(\alpha) = A^\beta(a, u_h : v_i(x), v_j(x))$ and $F_j(\alpha(t)) = (f(x,t,u_h), v_j(x)), L_j(t) = l_\beta(v_j(x))$.

For $y = (y_1, y_2, \cdots, y_m)^T \in \mathbb{R}^m$ we let $v(x) = \sum_{i=1}^m y_i v_i(x)$ then

$$y^T M y = \sum_{i=1}^m y_i \big(\sum_{j=1}^m M_{ij} y_j\big) = \|v\|^2.$$

Therefore M is a positive definite matrix. By applying the theory on the existence of the solution of the system of the ordinary differential equations, we acquire the existence of the solution of the system (4.3).

Since $f(x,t,u)$ and $a(x,u)$ are locally Lipschitz continuous in u, $-N(\alpha(t))\alpha(t) + F(\alpha(t)) + L(\alpha(t))$ is also locally Lipschitz in $\alpha(t)$. Thus from the theory on the uniqueness property of the system of the ordinary differential equations the unique existence can be quaranteed locally at $(0, \alpha(0))^T$.

By the similar analysis we may prove that the uniqueness property of $u_h(x,t)$ holds globally if $a(x,t)$ and $f(x,t,u)$ are globally Lipschitz in t and u. $\square$

Remark 1. From theorem 4.1 we obviously deduce that $\|u^h(t)\|_{L^\infty}$ is continuous with respect to t and

$$\|u(x,0) - u_h(x,0)\|_{L^\infty} = \|u_0(x) - P_h u_0(x)\|_{L^\infty} \le Ch^\mu \|u_0(x)\|_{W^{s,\infty}(\Omega)} \le K^*. \quad (4.4)$$

holds for some positive constant K^* and sufficiently small h provided that $\mu = \min(r+1, s) \ge 1$. We define K^* satisfying (4.11) which appears in the end this paper as well as (4.4).
Now we let $\chi = \tilde{u} - u_h$, then $u - u_h = \eta + \chi$.

Theorem 4.2. *We assume that the hypotheses of Theorem 3.1 hold. Suppose that $f(x,t,u)$ and $a(x,u)$ satisfy that*

$$\begin{gathered} \text{if } |u(x,t) - p| \le 2K^*, \forall (x,t) \in \Omega \times [0,T], \text{ then} \\ |f(x,t,u(x,t)) - f(x,t,p)| \le C(u,K^*)|u(x,t) - p| \text{ and} \\ |a(x,u(x,t)) - a(x,p)| \le C(u,K^*)|u(x,t) - p|, \ \forall (x,t) \in \Omega \times [0,T], \end{gathered} \quad (4.5)$$

hold. If $\mu > \frac{d}{2} + 1$ then there is a generic positive constant C such that

$$\|u - u_h\|_{L^\infty(L^2)} \le Ch^\mu (\|u\|_s + \|u_t\|_s),$$

where $\mu = \min(r+1, s)$.

Proof. To get the error bound of $u - u_h$ we temporarily assume that

$$\|u(t) - u_h(t)\|_{L^\infty} < 2K^*, \quad \forall t,\ 0 \le t \le T, \quad \forall h < h^* \quad (4.6)$$

holds for sufficiently small h^*. We subtract (4.1) from (4.2) and obtain the following error equation:

$$\begin{aligned} &(u_t - u_{ht}, v) + A_\alpha^\beta(a, u : u, v) - A_\alpha^\beta(a, u_h : u_h, v) \\ &\qquad = (f(x,t,u) - f(x,t,u_h), v) + \alpha(u - u_h, v), \quad \forall v \in V_h^r, \end{aligned}$$

from which we have

$$\begin{aligned} &(\chi_t, v) + A_\alpha^\beta(a, u_h : \chi, v) \\ = &-(\eta_t, v) - A_\alpha^\beta(a, u : u, v) + A_\alpha^\beta(a, u_h : \tilde{u}, v) \\ &+ (f(x,t,u) - f(x,t,u_h), v) + \alpha(u - u_h, v). \end{aligned} \quad (4.7)$$

By (3.1) we have

$$\begin{aligned}
&A_\alpha^\beta(a,u_h:\tilde{u},\chi)-A_\alpha^\beta(a,u:u,\chi)\\
=&A_\alpha^\beta(a,u_h:\tilde{u},\chi)-A_\alpha^\beta(a,u:\tilde{u},\chi)\\
=&\sum_{i=1}^{N_h}\big((a(x,u_h)-a(x,u))\nabla\tilde{u},\nabla\chi\big)_{E_i}\\
&-\sum_{e\in\mathcal{E}_{ID}}\big(\langle(a(x,u_h)-a(x,u))\nabla\tilde{u}\cdot\boldsymbol{n}\rangle,[\chi]\big)_e\\
&-\sum_{e\in\mathcal{E}_I}\big([\tilde{u}],\langle(a(x,u_h)-a(x,u))\nabla\chi\cdot\boldsymbol{n}\rangle\big)_e=L_1+L_2+L_3.
\end{aligned}$$

By (4.5), (4.6), Lemma 3.1 and Theorem 3.1 we have for $\epsilon>0$,

$$\begin{aligned}
|L_1|&\le C(K^*)\sum_{i=1}^{N_h}\big(\|u_h-u\|_{E_i}\|\nabla\tilde{u}\|_{L^\infty(E_i)}\|\nabla\chi\|_{E_i}\big)\\
&\le C(K^*)\sum_{i=1}^{N_h}(\|\eta\|_{E_i}+\|\chi\|_{E_i})\|\nabla\chi\|_{E_i}\\
&\le C(\|\eta\|^2+\|\chi\|^2)+\epsilon\|\nabla\chi\|^2\le Ch^{2\mu}\|u\|_s^2+C\|\chi\|^2+\epsilon|\!|\!|\chi|\!|\!|_1^2.
\end{aligned}$$

By Theorem 3.1 we get the following estimations:

$$\begin{aligned}
|L_2|&\le C\sum_{e\in\mathcal{E}_{ID}}\big(\|\nabla\tilde{u}\|_{L^\infty(e)}\|u-u_h\|_e\|[\chi]\|_e\big)\\
&\le\epsilon\sum_{e\in\mathcal{E}_{ID}}h^{-1}\|[\chi]\|_e^2+Ch\sum_{i=1}^{N_h}\big(h^{-1}\|\eta\|_{E_i}^2+h\|\nabla\eta\|_{E_i}^2+h^{-1}\|\chi\|_{E_i}^2\big)\\
&\le\epsilon\sum_{e\in\mathcal{E}_{ID}}h^{-1}\|[\chi]\|_e^2+C\big(\|\eta\|^2+h^2\|\nabla\eta\|^2+\|\chi\|^2\big)\\
&\le Ch^{2\mu}\|u\|_s^2+C\|\chi\|^2+\epsilon|\!|\!|\chi|\!|\!|_1^2,
\end{aligned}$$

$$\begin{aligned}
|L_3|&=|\sum_{e\in\mathcal{E}_I}\big(\langle(a(x,u_h)-a(x,u))\nabla\chi\cdot\boldsymbol{n}\rangle,[\tilde{u}]\big)_e|\\
&=|\sum_{e\in\mathcal{E}_I}\big(\langle(a(x,u_h)-a(x,u))\nabla\chi\cdot\boldsymbol{n}\rangle,[u-\tilde{u}]\big)_e|\\
&\le C\sum_{e\in\mathcal{E}_I}\|\nabla\chi\|_{L^2(e)}h^{-\frac{1}{2}(d-1)}(\|\langle\eta\rangle\|_e+\|\langle\chi\rangle\|_e)\|[\eta]\|_e\\
&\le\sum_{i=1}^{N_h}Ch^{-\frac{(d-1)}{2}}h^{-\frac{3}{2}}(h\|\nabla\chi\|_{E_i})(\|\eta\|_{E_i}+h\|\nabla\eta\|_{E_i}+h\|\nabla\chi\|_{E_i})(\|\eta\|_{E_i}+h\|\nabla\eta\|_{E_i}).
\end{aligned}$$

Therefore

$$L_3\le Ch^{-\frac{d}{2}}\|\nabla\chi\|h^\mu h^{\frac{d}{2}}\|u\|_{\frac{d}{2}}\le\epsilon|\!|\!|\chi|\!|\!|_1^2+Ch^{2\mu},$$

where C depends on $\|u\|_{\frac{d}{2}}$ and $\|u\|_s$. By the assumption (4.5) and (4.6) we get

$$\begin{aligned}
&|(f(x,t,u)-f(x,t,u_h),\chi)| \\
&\le C(K^*)\big(|u-u_h|,|\chi|\big) = C(K^*)\big(\|\eta\|+\|\chi\|\big)\|\chi\| \\
&\le C(K^*)\big(\|\eta\|^2+\|\chi\|^2\big) \le C(K^*)(h^{2\mu}+\|\chi\|^2).
\end{aligned} \tag{4.8}$$

Now we substitute the estimations of $L_1 \sim L_3$ and (4.8) into (4.7) with $v=\chi$ we get for some $\underset{\sim}{c} > 0$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\chi\|^2 + \underset{\sim}{c}|\!|\!|\chi|\!|\!|_1^2 \\
&\le C\|\eta_t\|^2 + \alpha\|\eta\|^2 + C\|\chi\|^2 + Ch^{2\mu}\|u\|_s^2 + 3\epsilon|\!|\!|\chi|\!|\!|_1^2 \\
&\le C(h^{2\mu}+\|\chi\|^2) + 3\epsilon|\!|\!|\chi|\!|\!|_1^2.
\end{aligned}$$

Now we choose sufficiently small $\epsilon > 0$ and apply the Gronwall inequality to get

$$\|\chi(t)\|^2 \le Ch^{2\mu}. \tag{4.9}$$

Therefore $\|u-u_h\| \le \|u-\tilde{u}\| + \|\tilde{u}-\hat{u}\| \le Ch^\mu$. Now we will verify that we may without loss of generality assume that (4.6) holds.
By (4.4), obviously (4.6) holds for $t=0$. Suppose that there exist t^* such that $\|u(t)-u_h(t)\|_{L^\infty} < 2K^*, \quad \forall t < t^*$ but

$$\|u(t^*)-u_h(t^*)\|_{L^\infty} \ge 2K^*. \tag{4.10}$$

Now we choose a sequence of $\{t_n\}_{n=1}^\infty \subset (0,t^*)$ converging to t^*. By following the preceding process below (4.6) we obtain the result $\|\chi(t_n)\|^2 \le Ch^{2\mu}$. By applying Lemma 3.1, (3.4) and (4.9) we get

$$\begin{aligned}
&\|(u-u_h)(t_n)\|_{L^\infty} \\
&\le \|u(t_n)-\hat{u}(t_n)\|_{L^\infty} + \|\hat{u}(t_n)-\tilde{u}(t_n)\|_{L^\infty} + \|\tilde{u}(t_n)-u_h(t_n)\|_{L^\infty} \\
&\le Ch^\mu\|u\|_{W^{s,\infty}} + Ch^{-\frac{d}{2}}(\|\theta(t_n)\| + \|\chi(t_n)\|) \\
&\le Ch^\mu\|u\|_{W^{s,\infty}} + Ch^{-\frac{d}{2}}(h^{\mu-1}\|u\|_s + h^\mu\|u_t\|_s) \le \frac{3}{2}K^*
\end{aligned} \tag{4.11}$$

provided that $\mu > \frac{d}{2}+1$.
By Theorem 4.1 we notice that $u_h(t)$ is continuous with respect to t, therefore this implies that $\|(u-u_h)(t)\|_\infty$ is continuous with respect to t. Hence we get

$$\|(u-u_h)(t^*)\|_{L^\infty} = \lim_{n\to\infty}\|(u-u_h)(t_n)\|_{L^\infty} < 2K^*$$

which contradicts to (4.10). Therefore we may assume that (4.6) holds for any $h < h^*$ with sufficiently small h^*. $\square$

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