J. Appl. Math. & Informatics Vol. **32**(2014), No. 5 - 6, pp. 567 - 584
http://dx.doi.org/10.14317/jami.2014.567

RICHARDSON EXTRAPOLATION OF ITERATED DISCRETE COLLOCATION METHOD FOR EIGENVALUE PROBLEM OF A TWO DIMENSIONAL COMPACT INTEGRAL OPERATOR

BIJAYA LAXMI PANIGRAHI*, GNANESHWAR NELAKANTI

Abstract. In this paper, we consider approximation of eigenelements of a two dimensional compact integral operator with a smooth kernel by discrete collocation and iterated discrete collocation methods. By choosing numerical quadrature appropriately, we obtain convergence rates for gap between the spectral subspaces, and also we obtain superconvergence rates for eigenvalues and iterated eigenvectors. We then apply Richardson extrapolation to obtain further improved error bounds for the eigenvalues. Numerical examples are presented to illustrate theoretical estimates.

AMS Mathematics Subject Classification : 45C05, 45B05.
Key words and phrases : Eigenvalue problem, compact integral operator, Discrete collocation methods, Richardson extrapolation.

1. Introduction

Consider the following integral operator $\mathcal{K}$ defined on $\mathbb{X} = L^\infty(D)$ by

$$\mathcal{K}u(s,t) = \int_a^b \int_c^d K(s,t,x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad (s,t) \in D, \tag{1}$$

where kernel $K(.,.,.,.) \in \mathcal{C}(D) \times \mathcal{C}(D),\ D = [a,b] \times [c,d] \subset \mathbb{R}^2$ is a given rectangular domain. Then $\mathcal{K}$ is a compact linear operator on $\mathbb{X}$.

We are interested in the following eigenvalue problem: find $u \in \mathbb{X}$ and $\lambda \in \mathbb{C} - \{0\}$ such that

$$\mathcal{K}u = \lambda u, \ \|u\| = 1. \tag{2}$$

Many practical problems in science and engineering are formulated as eigenvalue problems (2) of compact linear integral operators $\mathcal{K}$ (cf.,[3]). For many years, numerical solution of eigenvalue problems have attracted much attention.

Received February 14, 2012. Revised March 20, 2014. Accepted April 16, 2014. *Corresponding author.

During the last some years, significant work has been done in the numerical analysis of the one-dimensional eigenvalue problem for the compact integral operator $\mathcal{K}$. The Galerkin, petrove-Galerkin, collocation, Nyström and degenerate kernel methods are the commonly used methods for the approximation of eigenelements of the compact integral operator $\mathcal{K}$. The analysis for the convergence of Galerkin, petrove-Galerkin, collocation, Nyström and degenerate kernel methods for the one dimensional eigenvalue problems are well documented in ([1], [3], [12], [13], [14]). By replacing the various integrals appearing in these methods by numerical quadrature leads to discrete methods. In ([9]) discrete and iterated discrete Galerkin methods and in ([4]) discrete and iterated discrete collocation methods were discussed for the one dimensional eigenvalue problem (2) with a smooth kernel.

In [15], we were interested to solve the eigenvalue problem of a two dimensional compact integral operator with smooth kernel taking the help of discrete Galerkin and iterated discrete Galerkin methods and obtained the error bounds for approximated eigenelements. Further, to improve the convergence rates for the eigenvalues, we derived an asymptotic expansion for the iterated discrete Galerkin operator and then using Richardson extrapolation we improved the convergence rates for the eigenvalues. Meanwhile, to do so, we replace the various integrals arise in L^2 inner product, when the projection is an orthogonal projection and the two-dimensional integral operator $\mathcal{K}$ by using Gauss quadrature rule, which improves the computational cost to generate the matrix eigenvalue problem. To avoid such, discrete collocation method receive favorable attention due to lower computational cost in generating matrix eigenvalue problem. In fact, in comparison to discrete Galerkin and discrete petrove Galerkin methods, the discrete collocation method requires much less computational effort in evaluation of its entries defined by integrals. This motivates us to do this work.

In section-2, we develop discrete collocation, iterated discrete collocation methods and theoretical frame work for the eigenvalue problem using interpolatory projections. In section-3, we discuss the convergence rates for the approximated eigenfunctions to the exact eigenfunctions. In section-4, we discuss Richardson extrapolation for eigenvalue problem to improve convergence rates. In section-5, we present numerical results, which agree with the theoretical results. Throughout the paper, we assume c as the generic constant.

2. Discrete and Iterated discrete collocation methods

Consider the following compact integral operator $\mathcal{K}$ defined on $\mathbb{X} = L^\infty(D)$ by

$$\mathcal{K}u(s,t) = \int_a^b \int_c^d K(s,t,x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad (s,t) \in D, \tag{3}$$

where the kernel $K(.,.,.,.) \in \mathcal{C}(D) \times \mathcal{C}(D),\ D = [a,b] \times [c,d] \subset \mathbb{R}^2$.

We are interested in the eigenvalue problem (2). Assume λ be the eigenvalue of $\mathcal{K}$ with algebraic multiplicity m and ascent ℓ. Let $\Gamma \subset \rho(\mathcal{K})$ be a simple closed rectifiable curve such that $\sigma(\mathcal{K}) \cap int\Gamma = \{\lambda\}$, $0 \neq int\Gamma$, where $int\Gamma$ denotes the interior of Γ. Now we describe the collocation method for the eigenvalue problem (2).

Let $\Delta_M^{(1)}$ and $\triangle_N^{(2)}$ be the uniform partitions of finite intervals $[a,b]$ and $[c,d]$, respectively, defined by $\Delta_M^{(1)} : a = x_0 < x_1 < \cdots < x_M = b$ and $\Delta_N^{(2)} : c = y_0 < y_1 < \cdots < y_N = d$ with $h = x_{m+1} - x_m = \frac{b-a}{M}$ and $k = y_{n+1} - y_n = \frac{d-c}{N}$, for $m = 0,1,2,\ldots,M-1$ and $n = 0,1,2,\ldots,N-1$. These partitions define a grid for D, $\Delta_{MN} = \Delta_M^{(1)} \times \Delta_N^{(2)} = \{(x_m, y_n) : 0 \le m \le M-1,\ 0 \le n \le N-1\}$. Set $I_0^{(1)} = [x_0, x_1]$, $I_m^{(1)} = (x_m, x_{m+1}]$, $I_0^{(2)} = [y_0, y_1]$, $I_n^{(2)} = (y_n, y_{n+1}]$ and $I_{mn} = I_m^{(1)} \times I_n^{(2)}$, $m = 0,1,\ldots,M-1$ and $n = 0,1,\ldots,N-1$. For any given positive integer p and q, let $\mathcal{P}_{p-1,q-1}$ denotes the space of polynomials of degree $p-1$ in x and $q-1$ in y, then for $0 \le m \le M-1$, $0 \le n \le N-1$,

$$\mathbb{X}_{MN} = S_{p-1,q-1}^{(-1)}(\Delta_{MN}) = \{u : u\,|_{I_{mn}} = u_{m,n} \in \mathcal{P}_{p-1,q-1}\},$$

is the finite element space of dimension $MNpq$, which is the tensor product space of univariate spline spaces $S_{p-1}^{(-1)}(\Delta_M^{(1)})$ on $[a,b]$ and $S_{q-1}^{(-1)}(\Delta_N^{(2)})$ on $[c,d]$. The use of superscript (-1) in the notation for the above finite element space is to emphasize that it is not a subspace of $\mathcal{C}(D)$.

Let $s_{m,i} = x_m + \tau_i h$ and $t_{n,j} = y_n + \theta_j k$, be the collocation points on $[x_m, x_{m+1}]$, $m = 0,1,\ldots,M-1$, and $[y_n, y_{n+1}], n = 0,1,\ldots,N-1$, respectively, where $\tau_0, \tau_1, \ldots, \tau_{p-1}$ and $\theta_0, \theta_1, \ldots, \theta_{q-1}$ are zeros of Legendre polynomials of degree p and q, respectively on $[0,1]$.

Let $\mathcal{P}_h : L^\infty([a,b]) \to S_{p-1}^{(-1)}(\triangle_M^{(1)})$ and $\mathcal{P}_k : L^\infty([c,d]) \to S_{q-1}^{(-1)}(\triangle_N^{(2)})$ be the interpolatory projection with respect to the nodes $\{s_{m,i}\}$ and $\{t_{n,j}\}$, respectively, that is, for $u \in L^\infty([a,b])$,

$$\mathcal{P}_h u \in S_{p-1}^{(-1)}(\triangle_M^{(1)}) \text{ and } \mathcal{P}_h u(s_{m,i}) = u(s_{m,i}) \tag{4}$$

$$\mathcal{P}_k u \in S_{q-1}^{(-1)}(\triangle_N^{(2)}) \text{ and } \mathcal{P}_k u(t_{n,j}) = u(t_{n,j}), \tag{5}$$

then there holds([3]), $\|\mathcal{P}_h\|_\infty \le c_1 < \infty$, $\|\mathcal{P}_k\|_\infty \le c_2 < \infty$ and for any $u \in \mathcal{C}^p[a,b]$ and $u \in \mathcal{C}^q[c,d]$, there holds,

$$\|(\mathcal{I} - \mathcal{P}_h)u\| \le c\, h^p \|u^{(p)}\|_\infty, \text{ and } \|(\mathcal{I} - \mathcal{P}_k)u\| \le c\, k^q \|u^{(q)}\|_\infty. \tag{6}$$

Note that

$$\mathcal{I} - \mathcal{P}_h\mathcal{P}_k = (\mathcal{I} - \mathcal{P}_h) + (\mathcal{I} - \mathcal{P}_k) - (\mathcal{I} - \mathcal{P}_h)(\mathcal{I} - \mathcal{P}_k). \tag{7}$$

As a consequence, from (6), it follows that

$$\begin{aligned} \|(\mathcal{I} - \mathcal{P}_h\mathcal{P}_k)u\|_\infty &\le \|(\mathcal{I} - \mathcal{P}_h)u\|_\infty + \|(\mathcal{I} - \mathcal{P}_k)u\|_\infty + (1 + c_1)\|(\mathcal{I} - \mathcal{P}_k)u\|_\infty \\ &\to 0, \text{ as } h, k \to 0. \end{aligned} \tag{8}$$

Let ϕ_{im}, ψ_{jn} denote the Lagrange polynomials of degree $p-1$ and $q-1$ on the subintervals $[x_m, x_{m+1}]$, $m = 0,1,\ldots,M-1$ and $[y_n, y_{n+1}]$, $n = 0,1,\ldots,N-1$ respectively, where, $j = 0,1,\ldots q-1$, $i = 0,1,\ldots,p-1$. Then it follows that $S_{p-1}^{(-1)}(\triangle_M^{(1)}) = \text{span}\{\phi_{im}\}$ and $S_{q-1}^{(-1)}(\triangle_N^{(2)}) = \text{span}\{\psi_{jn}\}$. Then we have $\mathbb{X}_{MN} = \text{span}\{\phi_{im}\psi_{jn}\}$.

Now the collocation method for solving the eigenvalue problem (2) is defined as follows: find $u_{hk} \in \mathbb{X}_{MN}$, $\|u_{hk}\| = 1$ and $\lambda_{hk} \in \mathbb{C} - \{0\}$ such that

$$\mathcal{K}u_{hk}(s_{m',i'}, t_{n',j'}) = \lambda_{hk} u_{hk}(s_{m',i'}, t_{n',j'}), \tag{9}$$

for $i' = 0, 1, \dots p-1$, $m' = 0, 1, \dots M-1$, $j' = 0, 1, \dots q-1$, $n' = 0, 1, \dots N-1$. Using $u_{hk} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \alpha_{ij} \phi_{im} \psi_{jn} \in \mathbb{X}_{MN}$, the equation (9) can be converted to the matrix eigenvalue problem. The iterated eigenvector is defined by $u'_{hk} = \frac{1}{\lambda_{hk}} \mathcal{K}u_{hk}$.

To solve the matrix eigenvalue problem and the iterated eigenvector, we need to evaluate various integrals arising from the integral operator $\mathcal{K}$. In practice, numerical quadrature has to be used to compute these integrals. This leads to discrete methods. To do this, let for $f, g \in \mathcal{C}[0, 1]$,

$$R(f) = \sum_{i=0}^{k'-1} w_i f(c_i) \approx \int_0^1 f(s)\, \mathrm{d}s, \tag{10}$$

$$S(g) = \sum_{j=0}^{l'-1} \tilde{w}_j g(d_j) \approx \int_0^1 g(t)\, \mathrm{d}t \tag{11}$$

be the numerical quadrature with weights $w_i > 0$, $\tilde{w}_j > 0$ and quadrature points $c_i, i = 0, 1, \dots, k'-1$, $d_j, j = 0, 1, \dots, l'-1$, chosen as Gauss points in $[0, 1]$ which satisfy $0 < c_0 < c_1 < \cdots < c_{k'-1} < 1$ and $0 < d_0 < d_1, \dots, < d_{l'-1} < 1$ having degree of precision $2k'-1$, $2l'-1$, respectively. Then the composite Gauss quadrature rule for any $f \in \mathcal{C}([a, b]), g \in \mathcal{C}([c, d])$ is given by

$$R_h(f) = h \sum_{m=0}^{M-1} \sum_{i=0}^{k'-1} w_i f(x_{m,i}) \approx \int_a^b f(s)\, \mathrm{d}s, \tag{12}$$

$$S_k(g) = k \sum_{n=0}^{N-1} \sum_{j=0}^{l'-1} \tilde{w}_j g(y_{n,j}) \approx \int_c^d g(t)\, \mathrm{d}t, \tag{13}$$

where $x_{m,i} = x_m + c_i h$, $i = 0, 1, \dots, k'-1$, and $y_{n,j} = y_n + d_j k$, $j = 0, 1 \dots, l'-1$, be the quadrature points on the subintervals $[x_m, x_{m+1}]$, $m = 0, 1, \dots, M-1$ of $[a, b]$ and $[y_n, y_{n+1}]$, $n = 0, 1, \dots, N-1$ of $[c, d]$, respectively. Now using (12) and (13), we define the composite quadrature rule for $g \in \mathcal{C}(D)$ by

$$R_h S_k(g) = hk \sum_{i=0}^{k'-1} \sum_{j=0}^{l'-1} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} w_i \tilde{w}_j g(x_{m,i}, y_{n,j}) \approx \int_a^b \int_c^d g(s, t)\, \mathrm{d}s\, \mathrm{d}t.$$

Let $\mathcal{K}_{hk} : \mathbb{X} \to \mathbb{X}$ be the Nyström operator defined for $u \in \mathbb{X}$ is

$$\mathcal{K}_{hk}u(s,t) = hk \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{i=0}^{k'-1} \sum_{j=0}^{l'-1} w_i \tilde{w}_j K(s,t,x_{m,i},y_{n,j}) u(x_{m,i},y_{n,j}). \tag{14}$$

Now replacing the integral operator $\mathcal{K}$ by the Nyström operator (14), the matrix eigenvalue problem leads to the discrete collocation method,

$$\begin{aligned}\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \beta_{ij} \mathcal{K}_{hk} \phi_{im}(s_{m',i'}) \psi_{jn}(t_{n',j'}) \\ = \tilde{\lambda}_{hk} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \beta_{ij} \phi_{im}(s_{m',i'}) \psi_{jn}(t_{n',j'}), \\ i' = 0,1,\ldots p-1,\ m' = 0,1,\ldots M-1,\ j' = 0,1,\ldots q-1,\ n' = 0,1,\ldots N-1.\end{aligned} \tag{15}$$

By solving this discrete matrix eigenvalue problem (15), we find the eigenvalue $\tilde{\lambda}_{hk} \in \mathbb{C} - \{0\}$ and $\beta = [\beta_{ij}, i = 0,1\ldots,p-1,\ j = 0,1,\ldots,q-1]$. Then the discrete collocation eigenvector is defined by, $\tilde{u}_{hk} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \beta_{ij} \phi_{im} \psi_{jn} \in \mathbb{X}_{MN}$. The discrete matrix eigenvalue problem (15) can be written in operator form as

$$\mathcal{P}_h \mathcal{P}_k \mathcal{K}_{hk} \tilde{u}_{hk} = \tilde{\lambda}_{hk} \tilde{u}_{hk}. \tag{16}$$

Next we define the iterated discrete collocation eigenvector by $\tilde{u}'_{hk} = \frac{1}{\tilde{\lambda}_{hk}} \mathcal{K}_{hk} \tilde{u}_{hk}$ Clearly we see that $\mathcal{P}_h \mathcal{P}_k \tilde{u}'_{hk} = \tilde{u}_{hk}$ and

$$\mathcal{K}_{hk} \mathcal{P}_h \mathcal{P}_k \tilde{u}'_{hk} = \tilde{\lambda}_{hk} \tilde{u}'_{hk}. \tag{17}$$

This is the iterated discrete collocation method.

Next we discuss the convergence of approximated eigenvalues and eigenvectors to the exact eigenvalues and eigenvectors of the integral operator $\mathcal{K}$. To do this, first we set the following notations: Set $K(s,t,x,y) = K_{s,t}(x,y)$. For $K(.,.,.,.) \in \mathcal{C}^{(i,j)}(D) \times \mathcal{C}^{(i',j')}(D)$, denote

$$D^{(i,j)}\mathcal{K}u(s,t) = \int_a^b \int_c^d \frac{\partial^{i+j}}{\partial s^i \partial t^j} K(s,t,\xi,\eta) u(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta,$$

$$D^{(i,j,i',j')} K(s,t,\xi,\eta) = \frac{\partial^{i+j+i'+j'}}{\partial s^i \partial t^j \partial \xi^{i'} \partial \eta^{j'}} K(s,t,\xi,\eta).$$

For any $\alpha, \beta, \gamma, \delta \in \mathbb{N}$, we set $\|u\|_{\alpha,\beta,\infty} = \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \|u^{(i,j)}\|_\infty$

$$\|K\|_{\alpha,\beta,\gamma,\delta,\infty} = \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \sum_{i'=0}^{\gamma} \sum_{j'=0}^{\delta} \|D^{(i,j,i',j')} K(s,t,\xi,\eta)\|_\infty,$$

Then we have

$$\|D^{(i,j)}(\mathcal{K}u)\|_\infty \leq (b-a)(d-c)\|K\|_{i,j,0,0,\infty}\|u\|_\infty, \tag{18}$$

and

$$\|\mathcal{K}u\|_{\alpha,\beta,\infty} = \sum_{i=0}^{\alpha}\sum_{j=0}^{\beta}\|D^{(i,j)}(\mathcal{K}u)\|_\infty. \tag{19}$$

In the following theorem we give the error bounds for the Nyström operator.

Theorem 2.1 ([15])**.** *Let $\mathcal{K}$ be an integral operator with a kernel $K(.,.,.,.) \in \mathcal{C}^{(2k',2l')}(D) \times \mathcal{C}^{(2k',2l')}(D)$ and $\mathcal{K}_{hk}$ be the Nyström operator defined by (14), then for any $u \in \mathcal{C}^{(2k',2l')}(D)$, the following holds*

$$\|(\mathcal{K} - \mathcal{K}_{hk})u\|_\infty \leq c\,(h^{2k'} + k^{2l'})\|u\|_{2k',2l',\infty}, \tag{20}$$

where c is independent of h and k.

Definition 2.2 ([1])**.** Let $\mathbb{X}$ be a Banach space and, $\mathcal{T}$ and $\mathcal{T}_n$ are bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$. Then $\{\mathcal{T}_n\}$ is said to be ν-convergent to $\mathcal{T}$, if

$$\|\mathcal{T}_n\| \leq c, \quad \|(\mathcal{T}_n - \mathcal{T})\mathcal{T}\| \to 0, \quad \|(\mathcal{T}_n - \mathcal{T})\mathcal{T}_n\| \to 0 \text{ as } n \to \infty.$$

We quote the following lemma which is useful in proving the existence of eigenvalue and eigenvectors in discrete and iterated discrete collocation methods.

Lemma 2.3 ([2])**.** *Let $\mathbb{X}$ be a Banach space and $S \subset \mathbb{X}$ is a relatively compact set. Assume that $\mathcal{T}$ and $\mathcal{T}_n$ are bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ satisfying $\|\mathcal{T}_n\| \leq c$ for all $n \in \mathbb{N}$, and for each $x \in S$, $\|\mathcal{T}_n x - \mathcal{T}x\| \to 0$ as $n \to \infty$, where c is a constant independent of n. Then $\|\mathcal{T}_n x - \mathcal{T}x\| \to 0$ uniformly for all $x \in S$.*

Theorem 2.4. *$\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}$ and $\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k$ are ν-convergent to $\mathcal{K}$.*

Proof. Since $\mathcal{K}_{hk}$, $\mathcal{P}_h$ and $\mathcal{P}_k$ are uniformly bounded, it follows that $\|\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}\|_\infty \leq \|\mathcal{P}_h\|_\infty\|\mathcal{P}_k\|_\infty\|\mathcal{K}_{hk}\|_\infty < \infty$ and $\|\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k\|_\infty \leq \|\mathcal{K}_{hk}\|_\infty\|\mathcal{P}_h\|_\infty\|\mathcal{P}_k\|_\infty < \infty$. Now using (8) and Theorem 2.1, we see that

$$\|(\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk} - \mathcal{K})u\|_\infty \leq \|\mathcal{P}_h\|_\infty\|\mathcal{P}_k\|_\infty\|(\mathcal{K}_{hk} - \mathcal{K})u\|_\infty + \|(\mathcal{P}_h\mathcal{P}_k - \mathcal{I})\mathcal{K}u\|_\infty \to 0,$$

This shows that $\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}$ point wise converges to $\mathcal{K}$.

Let $B = \{x \in \mathbb{X} : \|x\| \leq 1\}$ be a closed unit ball in $\mathbb{X}$. Since $\mathcal{K}$ is a compact operator, the set $S = \{\mathcal{K}x : x \in B\}$ is a relatively compact set in X. By Lemma 2.3, we have

$$\begin{aligned}\|(\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk} - \mathcal{K})\mathcal{K}\|_\infty &= \sup\{\|(\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk} - \mathcal{K})\mathcal{K}u\|_\infty : u \in B\}\\ &= \sup\{\|(\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk} - \mathcal{K})u\|_\infty : u \in S\} \to 0 \text{ as } h,k \to 0.\end{aligned}$$

Since $\mathcal{P}_h\mathcal{P}_k$ is bounded and $\mathcal{K}_{hk}$ compact, $S' = \{\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}x : x \in B\}$ is a relatively compact set. Thus

$$\begin{aligned}\|(\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk} - \mathcal{K})\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}\|_\infty &= \sup\{\|(\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk} - \mathcal{K})\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}u\|_\infty : u \in B\}\\ &= \sup\{\|(\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk} - \mathcal{K})u\|_\infty : u \in S'\} \to 0,\end{aligned}$$

as $h,k \to 0$. Combining all these results leads to the first result that $\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}$ is ν-convergent to $\mathcal{K}$. The proof of $\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k$ is ν-convergent to $\mathcal{K}$ follows by similar steps as in above. $\square$

Since $\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}$ and $\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k$ are ν-convergent to $\mathcal{K}$, the spectrum of both $\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}$ and $\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k$ inside Γ consists of m eigenvalues say $\tilde{\lambda}_{hk,1}, \tilde{\lambda}_{hk,2}, \ldots, \tilde{\lambda}_{hk,m}$ counted accordingly to their algebraic multiplicities inside Γ with ascent ℓ (cf., [3, 14]). Let

$$\hat{\tilde{\lambda}}_{hk} = \frac{\tilde{\lambda}_{hk,1} + \tilde{\lambda}_{hk,2} + \cdots + \tilde{\lambda}_{hk,m}}{m},$$

denote their arithmetic mean and we approximate λ by $\hat{\tilde{\lambda}}_{hk}$. Let

$$\mathcal{P}^S \quad = \quad -\frac{1}{2\pi i}\int_\Gamma (\mathcal{K} - z\mathcal{I})^{-1}\mathrm{d}z, \tag{21}$$

be the spectral projections of $\mathcal{K}$ associated with their corresponding spectra inside Γ. Similarly, $\mathcal{P}^S_{hk}$ and $\widetilde{\mathcal{P}}^S_{hk}$ be the spectral projections of $\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}$ and $\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k$, respectively. Let $\mathcal{R}(\mathcal{P}^S)$, $\mathcal{R}(\mathcal{P}^S_{hk})$ and $\mathcal{R}(\widetilde{\mathcal{P}}^S_{hk})$ be the ranges of the spectral projections $\mathcal{P}^S$, $\mathcal{P}^S_{hk}$ and $\widetilde{\mathcal{P}}^S_{hk}$, respectively.

To discuss the closeness of eigenfunctions of the integral operator $\mathcal{K}$ and those of the approximate operators, we recall (cf., [3]) the concept of gap between the spectral subspaces. For nonzero closed subspaces $\mathbb{Y}_1$ and $\mathbb{Y}_2$ of $\mathbb{X}$, let

$$\delta(\mathbb{Y}_1, \mathbb{Y}_2) = \sup\{\mathrm{dist}(y, \mathbb{Y}_2) : y \in \mathbb{Y}_1, \ \ \|y\|_\infty = 1\},$$

then

$$\hat{\delta}(\mathbb{Y}_1, \mathbb{Y}_2) = \ \max\{\delta(\mathbb{Y}_1, \mathbb{Y}_2),\, \delta(\mathbb{Y}_2, \mathbb{Y}_1)\},$$

is known as the gap between $\mathbb{Y}_1$ and $\mathbb{Y}_2$.
We quote the following three Lemmas, which give the error bounds for the eigenelements.

Theorem 2.5 ([1], [13])**.** *Let $\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}$ be ν-convergent to $\mathcal{K}$. Then for sufficiently large M, N, there exists a constant c independent of M, N, we have*

$$\hat{\delta}(\mathcal{R}(\mathcal{P}^S_{hk}), \mathcal{R}(\mathcal{P}^S)) \quad \leq \quad c\|(\mathcal{K} - \mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk})\mathcal{K}\|_\infty,$$

Theorem 2.6 ([13], [16])**.** *Let $\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k$ is ν-convergent to $\mathcal{K}$. Then for sufficiently large M, N, there exists a constant c independent of M, N, we have*

$$\hat{\delta}(\mathcal{R}(\widetilde{\mathcal{P}}^S_{hk}), \mathcal{R}(\mathcal{P}^S)) \quad \leq \quad c\|(\mathcal{K} - \mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k)\mathcal{K}\|_\infty.$$

Theorem 2.7 ([1], [13])**.** *If $\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k$ is ν-convergent to $\mathcal{K}$ then for sufficiently large M, N, there exists a constant c independent of M, N such that for $j = 1, 2...., m$,*

$$|\lambda - \hat{\lambda}_{hk}| \leq c\,\|(\mathcal{K} - \mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k)\mathcal{K}\|_\infty,$$
$$|\lambda - \lambda_{hk,j}|^\ell \leq c\,\|(\mathcal{K} - \mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k)\mathcal{K}\|_\infty.$$

3. Convergence Rates

In this section we discuss the convergence rates for the approximated eigenvalues and eigenvectors to the exact eigenvalues and exact eigenvectors of the integral operator $\mathcal{K}$. To do this, first we prove the following Lemma.

Lemma 3.1. *Let $\mathcal{K}_{hk}$ be the Nyström operator defined by (14) with a kernel $K(.,.,.,.) \in \mathcal{C}^{(2k',2l')}(D)\times\mathcal{C}^{(2k',2l')}(D)$, $k' \geq p$, $l' \geq q$. Then for $u \in \mathcal{C}^{(2p,2q)}(D)$, the following hold*

$$\|(\mathcal{I}-\mathcal{P}_h\mathcal{P}_k)u\|_\infty \leq \; c\max\{h^p, k^q\}\|u\|_{p,q,\infty}, \tag{22}$$

$$\|\mathcal{K}_{hk}(\mathcal{I}-\mathcal{P}_h\mathcal{P}_k)u\|_\infty \leq \; c\max\{h^{2p}, k^{2q}\}\|u\|_{2p,2q,\infty}. \tag{23}$$

Proof. Using the estimates (6), we have

$$\|(\mathcal{I}-\mathcal{P}_h)u\|_\infty \leq ch^p\|u^{(p,0)}\|_\infty, \;\; \|(\mathcal{I}-\mathcal{P}_k)u\|_\infty \leq ck^q\|u^{(0,q)}\|_\infty,$$

and

$$\|(\mathcal{I}-\mathcal{P}_h)(\mathcal{I}-\mathcal{P}_k)u\|_\infty \leq ch^p\|((\mathcal{I}-\mathcal{P}_k)u)^{(p,0)}\|_\infty \leq c\; h^pk^q\|u^{(p,q)}\|_\infty.$$

Combining these estimates with the identity (7), proof of (22) follows.

To prove the estimate (23), let us denote $H(t) = \prod_{i'=0}^{p-1}(t-\tau_{i'})$ and $\widetilde{H}(t) = \prod_{j'=0}^{q-1}(t-\theta_{j'})$. Since $H(t)$ and $\widetilde{H}(t)$ are orthogonal polynomials of degree p and q, respectively, and the numerical quadratures defined by (10) and (11) have degree of precision $2k'$ and $2l'$, respectively, it follows that, for $k' \geq p$ and $l' \geq q$,

$$R(t^\mu H(t)) = \sum_{i=0}^{k'-1} w_i c_i^\mu H(c_i) = \int_0^1 t^\mu H(t)dt = 0, \; \mu = 0,1,\ldots,p-1, \tag{24}$$

$$S(t^\nu \widetilde{H}(t)) = \sum_{j=0}^{l'-1} w_j d_j^\nu \widetilde{H}(d_j) = \int_0^1 t^\nu \widetilde{H}(t)dt = 0, \; \nu = 0,1,\ldots,q-1. \tag{25}$$

Since $\mathcal{P}_h$ is the interpolatory projection interpolating at $u(s,t)$ in the first variable s at the points $s_{m,0}, s_{m,1},\ldots,s_{m,p-1}$ in the subintervals $[x_m, x_{m+1}]$, $m = 0,1,\ldots,M-1$, we have for $s \in [x_m, x_{m+1}]$, $t \in [c,d]$,

$$(\mathcal{I}-\mathcal{P}_h)u(s,t) = h^p H\left(\frac{s-x_m}{h}\right)\delta^{(p,0)}u(s,t), \tag{26}$$

where $\delta^{(p,0)}u(s,t) = [s_{m,0}, s_{m,1},\ldots,s_{m,p-1}, s; t]u$ be the Newton divided difference of u in first variable. Similarly, since $\mathcal{P}_k$ is the interpolatory projection interpolating at $u(s,t)$ in the second variable t at the points $t_{n,0}, t_{n,1},\ldots,t_{n,q-1}$ in the subintervals $[y_n, y_{n+1}]$, $n = 0,1,\ldots,N-1$, we have for $t \in [y_n, y_{n+1}]$, $s \in [a,b]$,

$$(\mathcal{I}-\mathcal{P}_k)u(s,t) = k^q \widetilde{H}\left(\frac{t-y_n}{k}\right)\delta^{(0,q)}u(s,t), \tag{27}$$

where $\delta^{(0,q)}u(s,t) = [s; t_{n,0}, t_{n,1}, \dots, t_{n,q-1}, t]u$ be the Newton divided difference of u in second variable. Now using the identity (7), we have

$$\mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h\mathcal{P}_k)u = \mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h)u + \mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_k)u - \mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h)(\mathcal{I} - \mathcal{P}_k)u. \quad (28)$$

For the first term in the above, for any $(s,t) \in D$, using (26), we obtain

$$\mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h)u(s,t) = h^{p+1}k \sum_{i=0}^{k'-1}\sum_{j=0}^{l'-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} w_i\tilde{w}_j H(c_i) g_i(x_{m,i}, y_{n,j}), \quad (29)$$

where $g_i(x_{m,i}, y_{n,j}) = K_{s,t}(x_{m,i}, y_{n,j})\delta^{(p,0)}u(x_{m,i}, y_{n,j})$. The Taylor's expansion of $g_i(x_{m,i}, y_{n,j}) = g_i(x_m + c_i h, y_{n,j})$ at the point x_m is given by

$$g_i(x_{m,i}, y_{n,j}) = \sum_{\mu=0}^{p-1} \frac{1}{\mu!} h^\mu {c_i}^\mu g_i^{(\mu,0)}(x_m, y_{n,j}) + \frac{1}{p!} h^p c_i^p g_i^{(p,0)}(\xi_m, y_{n,j}), \quad (30)$$

where $\xi_m \in [x_m, x_{m+1}]$. Using (30) in the estimate (29), we obtain

$$\begin{aligned}
&\mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h)u(s,t) \\
&\quad = \sum_{\mu=0}^{p-1} \frac{1}{\mu!} h^\mu k \sum_{j=0}^{l'-1}\sum_{n=0}^{N-1} \tilde{w}_j h^p \sum_{m=0}^{M-1} h\Big(\sum_{i=0}^{k'-1} w_i {c_i}^\mu H(c_i)\Big) g_i^{(\mu,0)}(x_m, y_{n,j}) \\
&\quad + h^{2p}\frac{1}{p!} k \sum_{n=0}^{N-1}\sum_{j=0}^{l'-1} \tilde{w}_j \sum_{m=0}^{M-1} h\Big(\sum_{i=0}^{k'-1} w_i c_i^p H(c_i)\Big) g_i^{(p,0)}(\xi_m, y_{n,j}). \qquad (31)
\end{aligned}$$

Using the estimate (24) in (31), it follows that

$$|\mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h)u(s,t)| \le c\, h^{2p} \max_{\xi\in[a,b]} |g_i^{(p,0)}(\xi, y_{n,j})| \le c\, h^{2p} \|K\|_{0,0,p,0,\infty} \|u\|_{2p,0,\infty}.$$

On the similar mechanism, for the second term in (28), we can prove that

$$|\mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_k)u(s,t)| \quad \le \quad ck^{2q}\|K\|_{0,0,0,q,\infty}\|u\|_{0,2q,\infty}. \quad (32)$$

Let $\delta^{(p,q)}u(s,t) = [s_{m,0}, s_{m,1}, \dots, s_{m,p-1}, s; t_{n,0}, t_{n,1}, \dots, t_{n,q-1}, t]u$ be p and qth Newton divided difference of u in first and second variables, respectively. Then we have

$$(\mathcal{I} - \mathcal{P}_h)(\mathcal{I} - \mathcal{P}_k)u(s,t) = h^p k^q H\Big(\frac{s - x_m}{h}\Big)\widetilde{H}\Big(\frac{t - y_n}{k}\Big)\delta^{(p,q)}u(s,t). \quad (33)$$

Using this in the third term of (28), we have

$$\begin{aligned}
&\mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h)(\mathcal{I} - \mathcal{P}_k)u(s,t) \\
&\qquad = hk \sum_{i=0}^{k'-1}\sum_{j=0}^{l'-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h^p k^q w_i \tilde{w}_j H(c_i)\widetilde{H}(d_j)\; g_{i,j}(x_{m,i}, y_{n,j}),
\end{aligned}$$

where $g_{i,j}(x_{m,i}, y_{n,j}) = K_{s,t}(x_{m,i}, y_{n,j})\delta^{(p,q)}u(x_{m,i}, y_{n,j})$. The Taylor's series expansion for $g_{i,j}(x_{m,i}, y_{n,j}) = g_{i,j}(x_m + c_i h, y_n + d_j k)$ at the point x_m and y_n

is given by

$$\begin{aligned} g_{i,j}(x_{m,i}, y_{n,j}) = & \sum_{\mu=0}^{r_1-1} \frac{1}{\mu!} \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} (c_i h)^{\nu} (d_j k)^{\mu-\nu} g_{i,j}^{(\nu,\mu-\nu)}(x_m, y_n) \\ & + \frac{1}{r_1!} \sum_{\nu=0}^{r_1} \binom{r_1}{\nu} (c_i h)^{\nu} (d_j k)^{r_1-\nu} g_{i,j}^{(\nu,r_1-\nu)}(\xi, \eta), \end{aligned} \tag{34}$$

where $r_1 = p + q$. Using (34) in the above equation and by adjustment with the the estimates (24) and (25), we obtain

$$\begin{aligned} |\mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h)(\mathcal{I} - \mathcal{P}_k)u(s,t)| & \leq c\, h^{2p} k^{2q} \max_{(\xi,\eta)\in D} |g_{i,j}^{(p,q)}(\xi,\eta)| \\ & \leq c\, h^{2p} k^{2q} \|K\|_{0,0,p,q,\infty} \|u\|_{2p,2q,\infty}, \end{aligned} \tag{35}$$

where $c = \binom{p+q}{p}(b-a)(d-c)R(t^p H(t))S(t^q \widetilde{H}(t))$ is a constant independent of h and k. Combining the estimates (28), (32) and (35), the result (23) follows. This completes the proof. □

Theorem 3.2. *Let $\mathcal{K}$ be a compact integral operator with a kernel $K(.,.,.,.) \in \mathcal{C}^{(2k',2l')}(D) \times \mathcal{C}^{(2k',2l')}(D)$, $k' \geq p$, $l' \geq q$ and $\mathcal{K}_{hk}$ be the Nyström operator defined by (14). Then the following hold*

$$\|(\mathcal{K} - \mathcal{K}_{hk})\mathcal{K}\|_\infty = \mathcal{O}(\max\{h^{2k'}, k^{2l'}\}), \tag{36}$$

$$\|(\mathcal{I} - \mathcal{P}_h\mathcal{P}_k)\mathcal{K}\|_\infty = \mathcal{O}(\max\{h^p, k^q\}), \tag{37}$$

$$\|\mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h\mathcal{P}_k)\mathcal{K}\|_\infty = \mathcal{O}(\max\{h^{2p}, k^{2q}\}). \tag{38}$$

Proof. Replacing u by $\mathcal{K}u$ in (20), and using the estimate (19), we obtain

$$\begin{aligned} \|(\mathcal{K} - \mathcal{K}_{hk})\mathcal{K}u\|_\infty & \leq c\,(h^{2k'} + k^{2l'})\|\mathcal{K}u\|_{2k',2l',\infty} \\ & \leq c\,(h^{2k'} + k^{2l'})\|K\|_{2k',2l',0,0,\infty}\|u\|_\infty, \end{aligned}$$

where c is a constant independent of h and k. This completes the proof of (36). Now replacing u by $\mathcal{K}u$ in (22) and using the estimate (19), we see that

$$\|(\mathcal{I} - \mathcal{P}_h\mathcal{P}_k)\mathcal{K}u\|_\infty \leq c\,\max\{h^p, k^q\}\|\mathcal{K}u\|_{p,q,\infty} \leq c\,\max\{h^p, k^q\}\|K\|_{p,q,0,0\infty}\|u\|_\infty,$$

this proves the estimate (37). Again replacing u by $\mathcal{K}u$ in (23), then we obtain

$$\|\mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h\mathcal{P}_k)\mathcal{K}u\|_\infty \leq c\,\max\{h^{2p}, k^{2q}\}\|K\|_{2p,2q,0,0\infty}\|u\|_\infty,$$

this proves the estimate (38). □

Theorem 3.3. *Assume that all the conditions of theorem 3.2 hold. Then the following hold*

$$\|(\mathcal{K} - \mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk})\mathcal{K}\|_\infty = \mathcal{O}(\max\{h^{\min\{p,2k'\}}, k^{\min\{q,2l'\}}\}), \tag{39}$$

$$\|(\mathcal{K} - \mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k)\mathcal{K}\|_\infty = \mathcal{O}(\max\{h^{\min\{2p,2k'\}}, k^{\min\{2q,2l'\}}\}). \tag{40}$$

Proof. Since

$$\|(\mathcal{K}-\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk})\mathcal{K}\|_\infty \leq \|(\mathcal{I}-\mathcal{P}_h\mathcal{P}_k)\mathcal{K}\|_\infty\|\mathcal{K}\|_\infty+\|\mathcal{P}_h\|_\infty\|\mathcal{P}_k\|_\infty\|(\mathcal{K}-\mathcal{K}_{hk})\mathcal{K}\|_\infty,$$
$$\|(\mathcal{K}-\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k)\mathcal{K}\|_\infty \leq \|(\mathcal{K}-\mathcal{K}_{hk})\mathcal{K}\|_\infty+\|\mathcal{K}_{hk}(\mathcal{I}-\mathcal{P}_h\mathcal{P}_k)\mathcal{K}\|_\infty,$$

the proof follows from the above Theorem 3.2. □

In the following Theorem we give the superconvergence results for the eigenvalues and eigenvectors.

Theorem 3.4. *Suppose $\mathcal{K}$ is a compact integral operator with a kernel function $K(.,.,.,.)\in\mathcal{C}^{(2k',2l')}(D)\times\mathcal{C}^{(2k',2l')}(D)$, $k'\geq p$, $l'\geq q$, and suppose that λ be the eigenvalue of $\mathcal{K}$ with algebraic multiplicity m and ascent ℓ. Let $\{\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k\}$ and $\{\mathcal{P}_h\mathcal{P}_k\mathcal{K}_{hk}\}$ be a sequence of bounded operators on $\mathbb{X}$, which converges to $\mathcal{K}$ in $\nu-$convergence. Then*

$$\hat{\delta}(\mathcal{R}(\mathcal{P}^S_{hk}),\mathcal{R}(\mathcal{P}^S))=\mathcal{O}(\max\{h^{\min\{p,2k'\}},k^{\min\{q,2l'\}}\}),$$
$$\hat{\delta}(\mathcal{R}(\widetilde{\mathcal{P}}^S_{hk}),\mathcal{R}(\mathcal{P}^S))=\mathcal{O}(\max\{h^{\min\{2p,2k'\}},k^{\min\{2q,2l'\}}\}).$$

In particular, for any $\tilde{u}_{hk}\in\mathcal{R}(\mathcal{P}^S_{hk})$ and $\tilde{u}'_{hk}\in\mathcal{R}(\widetilde{\mathcal{P}}^S_{hk})$, we have

$$\begin{aligned}\|\tilde{u}_{hk}-\mathcal{P}^S\tilde{u}_{hk}\|_\infty &= \mathcal{O}(\max\{h^{\min\{p,2k'\}},k^{\min\{q,2l'\}}\}),\\ \|\tilde{u}'_{hk}-\mathcal{P}^S\tilde{u}'_{hk}\|_\infty &= \mathcal{O}(\max\{h^{\min\{2p,2k'\}},k^{\min\{2q,2l'\}}\}).\end{aligned}$$

For $j=1,2,\ldots,m$,

$$|\lambda-\hat{\tilde{\lambda}}_{hk}|=\mathcal{O}(\max\{h^{\min\{2p,2k'\}},k^{\min\{2q,2l'\}}\}),$$
$$|\lambda-\tilde{\lambda}_{hk,j}|^\ell=\mathcal{O}(\max\{h^{\min\{2p,2k'\}},k^{\min\{2q,2l'\}}\}).$$

Proof. The proof follows directly using the Theorems 2.5, 2.6, 2.7 and 3.3. □

Remark: From Theorem 3.4, we observe that discrete collocation eigenvectors converges with the order of convergence $\mathcal{O}(\max\{h^{\min\{p,2k'\}},k^{\min\{q,2l'\}}\})$ where as iterated discrete collocation eigenvectors and eigenvalues converges with the order of convergence $\mathcal{O}(\max\{h^{\min\{2p,2k'\}},k^{\min\{2q,2l'\}}\})$. This shows that iterated discrete eigenvectors gives superconvergence results over the discrete collocation eigenvectors. By choosing the degree of precisions of the numerical quadrature rules sufficiently large, i.e., $2k'\geq 2p$ and $2l'\geq 2q$ on $[a,b]$ and $[c,d]$, respectively, we obtain the superconvergence results for the eigenvalues and eigenvectors in the discrete collocation and iterated discrete collocation methods.

4. Richardson Extrapolation

In this section, we derive an asymptotic error expansions (cf., [10], [11]) for the iterated discrete collocation operator $\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k$ and an asymptotic error expansion of arithmetic mean of approximate eigenvalues. We then apply Richardson extrapolation to obtain improved error bounds for the eigenvalues.

Lemma 4.1. *(Euler-MacLaurin summation formulae)*([7]).
Let $f(x,y) \in \mathcal{C}^{r+1}(D),\ 0 \le \tau \le 1,\ 0 \le \theta \le 1$. *Then*

$$
\begin{aligned}
&hk \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(x_m + \tau h, y_n + \theta k) \\
&= \sum_{a=0}^{r} \sum_{b=0}^{r-a} h^a k^b \frac{B_a(\tau)}{a!} \frac{B_b(\theta)}{b!} \left[\frac{\partial^{a+b-2}}{\partial x^{a-1} \partial y^{b-1}} f(x,y) \right]_{x=a,y=c}^{b,\quad d} + \mathcal{O}(h^{r+1} + k^{r+1})
\end{aligned}
$$

where $B_i(t)$ *are Bernoulli polynomials of degree* i.

Theorem 4.2 ([15]). *Let* $\mathcal{K}$ *be a compact integral operator with a kernel* $K(.,.,.,.) \in \mathcal{C}(D) \times \mathcal{C}^{r+1}(D)$ *and* $\mathcal{K}_{hk}$ *be the Nyström operator defined by (14), then there holds*

$$
\mathcal{K}_{hk} - \mathcal{K} = \sum_{i=k}^{[\frac{r}{2}]} h^{2i} \mathcal{D}_{2i} + \sum_{j=l}^{[\frac{r}{2}]} k^{2j} \mathcal{E}_{2j} + \sum_{i=k}^{[\frac{r}{2}]-l} \sum_{j=l}^{[\frac{r}{2}]-i} h^{2i} h^{2j} \mathcal{F}_{2i,2j} + \mathcal{O}(h^{r+1} + k^{r+1}),
$$

where $\mathcal{D}_{2i}, \mathcal{E}_{2j}$, *and* $\mathcal{F}_{2i,2j}$ *are bounded linear operators independent of* h *and* k.

Lemma 4.3 ([7]). *Assume that* $u(x,y) \in \mathcal{C}^{r+1}(D)$. *Let* $\mathcal{P}_h$ *and* $\mathcal{P}_k$ *be the interpolatory projections defined by (4) and (5), respectively. Then for any* $(x,y) \in I_{mn}$, *the following holds*

$$
\begin{aligned}
(\mathcal{I} - \mathcal{P}_h \mathcal{P}_k) u(x,y) \quad &= \quad \sum_{\mu=p}^{r} h^{\mu} u^{(\mu,0)}(x,y) \Phi_{\mu}(\tau) + \sum_{\nu=q}^{r} k^{\nu} u^{(0,\nu)}(x,y) \Psi_{\nu}(\theta) \\
&- \quad \sum_{\mu=p}^{r-q} \sum_{\nu=q}^{r-\mu} h^{\mu} k^{\nu} u^{(\mu,\nu)}(x,y) \Phi_{\mu}(\tau) \Psi_{\nu}(\theta) + \mathcal{O}(h^{r+1} + k^{r+1}),
\end{aligned}
$$

where

$$
\Phi_{\mu}(\tau) = \prod_{i=0}^{p-1} (\tau - \tau_i) [\tau_0, \tau_1, \dots, \tau_{p-1}, x] \frac{(. - \tau)^{\mu}}{\mu!}, \tag{41}
$$

$$
\Psi_{\nu}(\theta) = \prod_{j=0}^{q-1} (\theta - \theta_j) [\theta_0, \theta_1, \dots, \theta_{q-1}, x] \frac{(. - \theta)^{\nu}}{\nu!}. \tag{42}
$$

Proposition 4.4. *Assume that the kernel* $K(.,.,.,.) \in \mathcal{C}^{(r+1)}(D) \times \mathcal{C}^{(r+1)}(D)$ *and* $u \in \mathcal{C}^{r+1}(D)$, *then the following holds*

$$
\begin{aligned}
\mathcal{K}_{hk}(\mathcal{I} - \mathcal{P}_h \mathcal{P}_k) &= \sum_{i=p}^{[\frac{r}{2}]} h^{2i} \mathcal{R}_{2i,0} + \sum_{j=q}^{[\frac{r}{2}]} k^{2j} \mathcal{S}_{0,2j} + \sum_{i=p}^{[\frac{r}{2}]-l'} \sum_{j=l'}^{[\frac{r}{2}]-i} h^{2i} k^{2j} \mathcal{R}_{2i,2j} \\
&+ \sum_{i=q}^{[\frac{r}{2}]-k'} \sum_{j=k'}^{[\frac{r}{2}]-i} h^{2i} k^{2j} \mathcal{S}_{2i,2j} - \sum_{i=p}^{[\frac{r}{2}]-q} \sum_{j=q}^{[\frac{r}{2}]-i} h^{2i} k^{2j} \mathcal{T}_{2i,2j} + \mathcal{O}(h^{r+1} + k^{r+1}),
\end{aligned}
$$

where $\mathcal{R}_{2i,2j}, \mathcal{S}_{2i,2j}$ *and* $\mathcal{T}_{2i,2j}$ *are bounded linear operators on* $\mathcal{C}^{r+1}(D)$.

Proof. Using the definition of $\mathcal{K}_{hk}$ defined by (14) and the Lemma 4.3, we have

$$\begin{aligned}
&\mathcal{K}_{hk}(\mathcal{I}-\mathcal{P}_h\mathcal{P}_k)u(s,t)\\
&= hk\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{i=0}^{k'-1}\sum_{j=0}^{l'-1} w_i\tilde{w}_j K_{s,t}(x_{m,i},y_{n,j})\times\\
&\quad\Bigg[\sum_{\mu=p}^{r} h^{\mu}u^{(\mu,0)}(x_{m,i},y_{n,j})\Phi_\mu(c_i)+\sum_{\nu=q}^{r} k^{\nu}u^{(0,\nu)}(x_{m,i},y_{n,j})\Psi_\nu(d_j)\\
&\qquad-\sum_{\mu=p}^{r-q}\sum_{\nu=q}^{r-\mu} h^{\mu}k^{\nu}u^{(\mu,\nu)}(x_{m,i},y_{n,j})\Phi_\mu(c_i)\Psi_\nu(d_j)+\mathcal{O}(h^{r+1}+k^{r+1})\Bigg]\\
&= I_1+I_2-I_3+\mathcal{O}(h^{r+1}+k^{r+1}). \qquad (43)
\end{aligned}$$

Now we consider I_1,

$$I_1=\sum_{\mu=p}^{r} h^{\mu}\sum_{i=0}^{k'-1} w_i\Phi_\mu(c_i)\sum_{j=0}^{l'-1}\tilde{w}_j\Bigg[hk\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} K_{s,t}(x_{m,i},y_{n,j})u^{(\mu,0)}(x_{m,i},y_{n,j})\Bigg].$$

Using Euler-Maclaurin summation formula in I_1, we obtain

$$\begin{aligned}
I_1=&\sum_{\mu=p}^{r}\sum_{a=0}^{r-\mu}\sum_{b=0}^{r-a-\mu}\frac{h^{\mu+a}}{a!}\frac{k^b}{b!}A_{\mu,a}S(B_b)\Bigg[\frac{\partial^{a+b-2}}{\partial x^{a-1}\partial y^{b-1}}K_{s,t}(x,y)u^{(\mu,0)}(x,y)\Bigg]\\
&+\mathcal{O}(h^{r+1}+k^{r+1}), \qquad (44)
\end{aligned}$$

where $A_{\mu,a}=\sum_{i=0}^{k'-1} w_i\Phi_\mu(c_i)B_a(c_i)$ and $S(B_b)$ is the numerical quadrature of B_a defined as in (11). Since $\tau_0,\tau_1,\ldots,\tau_{p-1}$ are symmetric points in the interval $[0,1]$, i.e., $\tau_j=1-\tau_{p-1-j}$ for $0\le j\le p-1$, we have

$$[\tau_0,\tau_1,\ldots,\tau_{p-1},1-c_i]\frac{(.-(1-c_i))^{\mu}}{\mu!}=(-1)^{\mu-p}[\tau_0,\tau_1,\ldots,\tau_{p-1},c_i]\frac{(.-c_i)^{\mu}}{\mu!},$$

and

$$\prod_{j=0}^{p-1}(1-c_i-\tau_j)=(-1)^p\prod_{j=0}^{p-1}(c_i-\tau_j).$$

Using these estimates we obtain $\Phi_\mu(1-c_i)=(-1)^{\mu}\Phi_\mu(c_i),\ i=0,1,2,\ldots,k'-1$. Also note that $B_a(c_i)=(-1)^aB_a(1-c_i),\quad c_i=1-c_{k'-1-i}$ and $w_{k'-i-1}=w_i$, for $0\le i\le k'-1$. Hence we have

$$A_{\mu,a}=\sum_{i=0}^{k'-1} w_i\Phi_\mu(c_i)B_a(c_i)=(-1)^{\mu+a}\sum_{i=0}^{k'-1} w_i\,\Phi_\mu(c_i)B_a(c_i)=(-1)^{\mu+a}A_{\mu,a}.$$

From this, it follows that $A_{\mu,a} = 0$, when $\mu + a = \text{odd}$. Since the quadrature rule (10) is exact for polynomial of degree less than $2k'$ and $\prod_{j=0}^{p-1}(c_i - \tau_j)$ is orthogonal to all polynomial of degree less than p and $[\tau_0, \tau_1, \ldots, \tau_{p-1}, c_i]\frac{(.-c_i)^\mu}{\mu!}$ is a polynomial of degree $\mu - p$, then for $\mu + a < 2p$, $k' \geq p$, we have

$$A_{\mu,a} = \sum_{i=0}^{k'-1} w_i \Phi_\mu(c_i) B_a(c_i) = \int_0^1 \prod_{j=0}^{p-1}(t-\tau_j)[\tau_0, \tau_1, \ldots, \tau_{p-1}, t]\frac{(.-t)^\mu}{\mu!} B_a(t) = 0,$$

and $A_{0,0} = \int_0^1 \Phi_0(t) B_0(t) dt = \int_0^1 -1.dt = -1$. Thus we obtain

$$A_{\mu,a} = \begin{cases} -1, \text{ if } \mu = a = 0, \\ 0, \text{ if } 1 \leq \mu + a \leq 2p-1, \\ 0, \text{ if } \mu + a = \text{odd}. \end{cases} \tag{45}$$

Since d_j, $j = 0, 1, \ldots, l'-1$ are the Gauss points in the interval $(0,1)$ and the quadrature rule (3) is an symmetric quadrature rule, we have $d_{l'-j-1} = 1 - d_j$ and $\tilde{w}_j = \tilde{w}_{l'-j-1}$ for $j = 0, 1, \ldots, l'-1$. Noting that the Bernoulli polynomials have the property that $B_b(1-d) = (-1)^b B_b(d)$, we have

$$S(B_b) = \sum_{j=0}^{l'-1} \tilde{w}_{l'-j-1} B_b(1 - d_{l'-j-1}) = (-1)^b \sum_{j=0}^{l'-1} \tilde{w}_j B_b(d_j) = (-1)^b S(B_b).$$

From this, it follows that $S(B_b)$ is zero when b is odd. Also we have $S(B_0) = 1$. Now since the degree of precision of this quadrature rule (10) is $2l'-1$, it follows that for $1 \leq b \leq 2l'-1$,

$$S(B_b) \;=\; \sum_{j=0}^{l'-1} \tilde{w}_j B_b(d_j) = \int_0^1 B_b(t)\,\mathrm{d}t = \left[\frac{1}{b+1} B_{b+1}(t)\right]_{t=0}^1 = 0.$$

Thus we have

$$S(B_b) = \begin{cases} 1, \text{ if } b = 0, \\ 0, \text{ if } 1 \leq b \leq 2l'-1, \\ 0, \text{ if } b = \text{odd}. \end{cases} \tag{46}$$

Combining the estimates (46) and (45) with (44) we obtain

$$I_1 = \sum_{i=p}^{[\frac{r}{2}]} h^{2i} \mathcal{R}_{2i,0} u + \sum_{i=p}^{[\frac{r}{2}]-l'} \sum_{j=l'}^{[\frac{r}{2}]-i} h^{2i} k^{2j} \mathcal{R}_{2i,2j} u + \mathcal{O}(h^{r+1} + k^{r+1}),$$

where,

$$\mathcal{R}_{2i,2j} u = \sum_{(\mu,a)\in Z(i)} \frac{A_{\mu,a}}{a!} \frac{S(B_{2j})}{2j!} \left[\frac{\partial^{a+2j-2}}{\partial x^{a-1} \partial y^{2j-1}} K_{s,t}(x,y) u^{(\mu,0)}(x,y)\right]_{x=a,\ y=c}^{b,\quad d},$$

$$Z(i) = \left\{(\mu, a),\ p \leq \mu \leq r,\ 0 \leq a \leq r - \mu,\ \mu + a = 2i\right\}.$$

Similarly we can prove that

$$I_2 = \sum_{j=q}^{[\frac{r}{2}]} k^{2j}\mathcal{S}_{0,2j}u + \sum_{i=k'}^{[\frac{r}{2}]-q}\sum_{j=q}^{[\frac{r}{2}]-i} h^{2i}k^{2j}\mathcal{S}_{2i,2j}u + \mathcal{O}(h^{r+1}+k^{r+1}),$$

where,

$$\mathcal{S}_{2i,2j}u = \sum_{(\nu,b)\in Z(j)} \frac{R(B_{2i})}{2i!}\frac{B_{\nu,b}}{b!}\left[\frac{\partial^{2i+b-2}}{\partial x^{2i-1}\partial y^{b-1}}K_{s,t}(x,y)u^{(0,\nu)}(x,y)\right]_{x=a,\ y=c}^{b,\quad d},$$

$$Z(j) = \left\{(\nu,b),\ q\le\nu\le r,\ 0\le b\le r-\nu,\ \nu+b=2j\right\}.$$

Similarly for I_3, we can prove that

$$I_3 = \sum_{i=p}^{[\frac{r}{2}]-q}\sum_{j=q}^{[\frac{r}{2}]-i} h^{2i}k^{2j}\mathcal{T}_{2i,2j}u + \mathcal{O}(h^{r+1}+k^{r+1})$$

where,

$$\mathcal{T}_{2i,2j}u = \sum_{(\mu,a,\nu,b)\in Z(i,j)} \frac{A_{\mu,a}}{a!}\frac{B_{\nu,b}}{b!}\left[\frac{\partial^{a+b-2}}{\partial x^{a-1}\partial y^{b-1}}K_{s,t}(x,y)u^{(\mu,\nu)}(x,y)\right]_{x=a,\ y=c}^{b,\quad d},$$

$$Z(i,j) = \Big\{(\mu,a,\nu,b),\ \ p\le\mu\le r-q,\ 0\le a\le r-\mu-\nu,$$
$$q\le\nu\le r-\mu,\ 0\le b\le r-a-\mu-\nu,\ \mu+a=2i,\ \nu+b=2j\Big\}.$$

Now combining the estimates for $I_1,\ I_2$ and I_3 with (43), we obtain the following asymptotic expansion This completes the proof. □

Theorem 4.5. *Let $\mathcal{K}$ be a compact integral operator with the kernel $K(.,.,.,.)\in\mathcal{C}^{r+1}(D)\times\mathcal{C}^{r+1}(D)$ with $p=k'$ and $q=l'$. Then the following holds*

$$\mathcal{K}-\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k = \sum_{i=p}^{[\frac{r}{2}]} h^{2i}\mathcal{U}_{2i} + \sum_{j=q}^{[\frac{r}{2}]} k^{2j}\mathcal{V}_{2j} + \sum_{i=p}^{[\frac{r}{2}]-q}\sum_{j=q}^{[\frac{r}{2}]-i} h^{2i}k^{2j}\mathcal{W}_{2i,2j} + \mathcal{O}(h^{r+1}+k^{r+1}),$$

where $\mathcal{U}_{2i},\mathcal{V}_{2j}$ and $\mathcal{W}_{2i,2j}$ are bounded linear operators on $\mathcal{C}^{r+1}(D)$.

Proof. Combining Theorems 4.4 and 4.2, we have

$$\begin{aligned}\mathcal{K}-\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k &= \mathcal{K}-\mathcal{K}_{hk}+\mathcal{K}_{hk}-\mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k\\ &= \sum_{i=p}^{[\frac{r}{2}]} h^{2i}\mathcal{U}_{2i} + \sum_{j=q}^{[\frac{r}{2}]} k^{2j}\mathcal{V}_{2j} + \sum_{i=p}^{[\frac{r}{2}]-l'}\sum_{j=l'}^{[\frac{r}{2}]-i} h^{2i}k^{2j}\mathcal{W}_{2i,2j} + \mathcal{O}(h^{r+1}+k^{r+1}),\end{aligned}$$

where $\mathcal{U}_{2i}=\mathcal{R}_{2i,0}-\mathcal{D}_{2i}$, $\mathcal{V}_{2j}=\mathcal{S}_{0,2j}-\mathcal{E}_{2j}$, and $\mathcal{W}_{2i,2j}=\mathcal{R}_{2i,2j}+\mathcal{S}_{2i,2j}-\mathcal{T}_{2i,2j}-\mathcal{F}_{2i,2j}$ are bounded linear operators on $\mathcal{C}^{r+1}(D)$. This completes the proof. □

In rest of the paper, we choose the domain $D = [a, b] \times [a, b]$ and the partition $\triangle_{M,N} = \triangle_M^{(1)} \times \triangle_M^{(1)}$, i.e., $h = k$. Also we choose the numerical quadratures (10) and (11) to be same, i.e., $k' = l'$. Then we have the following corollary.

Corollary 4.6. *Let $\mathcal{K}_{hk}$ be the Nyström operator defined by (14) with $p = q = k' = l'$. Assume that the kernel $K(.,.,.,.) \in \mathcal{C}^{2p+2}(D) \times \mathcal{C}^{2p+2}(D)$. Then the following holds*

$$\mathcal{K} - \mathcal{K}_{hk}\mathcal{P}_h\mathcal{P}_k = h^{2p}\mathcal{C}_{2p} + \mathcal{O}(h^{2p+2}),$$

where $\mathcal{C}_{2p} = \mathcal{U}_{2p} + \mathcal{V}_{2p}$ is a bounded linear operators on $\mathcal{C}^{2p+2}(D)$.

Richardson Extrapolation For Eigenvalue problem: By the similar way as followed in [15], we obtain the following theorem which gives an asymptotic error expansion of the arithmetic mean of eigenvalues by iterated discrete collocation method.

Theorem 4.7 ([15]). *Let λ be the eigenvalue of $\mathcal{K}$ with algebraic multiplicity m and $\hat{\tilde{\lambda}}_{hk}$ be the arithmetic mean of the eigenvalues $\tilde{\lambda}_{hk,j}$, $j = 1, 2, ..., m$. Then the following holds*

$$\hat{\tilde{\lambda}}_{hk} - \lambda = \frac{1}{m} h^{2p} trace(\mathcal{Q}_{2p}) + \mathcal{O}(h^{2p+2}), \tag{47}$$

where $\mathcal{Q}_{2p} = \mathcal{C}_{2p}\mathcal{P}^S - \mathcal{K}\mathrm{U}_{2p}$ is a bounded linear operator independent of h.

According to the asymptotic expansion (47), the Richardson extrapolation for eigenvalue approximation should be the following. We first divide each subinterval with respect to the partitions of $\Delta_M^{(1)}$ and $\Delta_M^{(2)}$ into two halves which makes up a new partitions denoted by $\Delta_{2M}^{(1)}$ and $\Delta_{2M}^{(2)}$,

$$\Delta_{2M}^{(1)} = \Delta_{2M}^{(2)} : a = x_0 < x_{\frac{1}{2}} < x_1 < \cdots < x_{M-\frac{1}{2}} < x_M = b.$$

Here $h = k = \frac{1}{2M}$ and $D = [a, b] \times [a, b]$. We then have following asymptotic expansion for eigenvalue approximation with respect to this new partitions,

$$\hat{\tilde{\lambda}}_{h/2,\,k/2} = \lambda + \frac{1}{m}\Big(\frac{h}{2}\Big)^{2p} trace(\mathcal{Q}_{2p}) + \mathcal{O}(h^{2p+2}). \tag{48}$$

From the asymptotic expansions (47) and (48), the Richardson extrapolation for the eigenvalue approximation is defined by

$$\hat{\tilde{\lambda}}_{hk}^E = \frac{2^{2p}\hat{\tilde{\lambda}}_{h/2,\,k/2} - \hat{\tilde{\lambda}}_{hk}}{2^{2p} - 1}. \tag{49}$$

In the following Theorem we give the superconvergence rates for the eigenvalue approximation using Richardson extrapolation.

Theorem 4.8. *Assume that conditions of Theorem 4.7 hold and the Richardson extrapolation $\hat{\tilde{\lambda}}_{hk}^E$ is defined by (49). Then the following error estimate holds*

$$|\hat{\tilde{\lambda}}_{hk}^E - \lambda| = \mathcal{O}(h^{2p+2}).$$

5. Numerical Example

Consider the eigenvalue problem (2), for the integral operator $\mathcal{K}$ (3) for various smooth kernels $K(s,t,x,y)$.

Let $\mathbb{X}_{MM}$ be the space of piecewise constant functions (p=q=1) on $[0,1]\times[0,1]$ with respect to the initial uniform partitions $\Delta_M^{(1)} = \Delta_M^{(2)} : 0 < \frac{1}{M} < \frac{2}{M} < \cdots < \frac{M-1}{M} < 1$, $\Delta_{MN} = \Delta_M^{(1)} \times \Delta_M^{(2)} = \{(\frac{i}{M}, \frac{j}{M}) : 0 \le i \le M,\ 0 \le j \le M\}$ with $h = k = \frac{1}{M}$. We choose numerical quadrature as the one-point composite Gaussian quadrature formula which is exact for all polynomials of degree less than 2, that is $k = l = 1$.
The quadrature points and weights are given by

$$x_{m,i} = y_{m,i} = \frac{2m+1}{2M},\ i = 1,\ m = 0,1,\ldots,M-1$$

and $w_i = \tilde{w}_i = \frac{1}{M},\ i = 1,2,\ldots M$, respectively.

For different kernels and for different values of M, we compute the discrete collocation eigen vector $\tilde{u}_{hk}$, iterated eigen vector $\tilde{u}'_{hk}$ and eigenvalue $\hat{\tilde{\lambda}}_{hk}$, and approximated eigenvalue in Richardson extrapolation $\hat{\tilde{\lambda}}^E_{hk}$. Denote

$$|\lambda - \hat{\tilde{\lambda}}_{hk}| = \mathcal{O}(h^\alpha),\quad \|\tilde{u}_{hk} - \mathcal{P}^S\tilde{u}_{hk}\|_\infty = \mathcal{O}(h^\beta),$$
$$\|\tilde{u}'_{hk} - \mathcal{P}^S\tilde{u}'_{hk}\|_\infty = \mathcal{O}(h^\gamma),\quad |\lambda - \hat{\tilde{\lambda}}^E_{hk}| = \mathcal{O}(h^\delta),$$

where $h = k = \frac{1}{M}$ is the step length. For $M = 2,4,8,16$, we compute α,β,γ and δ which are listed in the following Table.

Since $k' = l' = 1$, we get the theoretical convergence of the order of eigenvector is 1, the orders of the iterated eigenvector and eigenvalues are 2, and the order of eigenvalue in Richardson extrapolation is 4. In the following Table 1 and Table 2, the numerical results agrees with the theoretical results.

Example. $K(s,t,x,y) = s\sin(t) + xe^y$, $[a,b] = [0,1]$, $[c,d] = [0,1]$.

Table 1: Eigenvector error bounds

M	$\|\tilde{u}_{hk} - \mathcal{P}^S\tilde{u}_{hk}\|_\infty$	$\|\tilde{u}'_{hk} - \mathcal{P}^S\tilde{u}'_{hk}\|_\infty$	β	γ
2	1.955787e-01	1.738201e-02	*	*
4	9.343799e-02	4.254548e-03	1.065668	2.030517
8	4.617936e-02	1.058051e-03	1.016761	2.007596
16	2.302235e-02	2.641661e-04	1.004212	2.001891

Table 2: Eigenvalue error bounds

M	$\|\lambda - \hat{\tilde{\lambda}}_{hk}\|$	α	$\|\lambda - \hat{\tilde{\lambda}}^E_{hk}\|$	δ
2	1.955475e-02	*	*	*
4	4.858064e-03	2.009065	4.083223e-05	*
8	1.212553e-03	2.002332	2.616340e-06	3.964086
16	3.030157.e-04	2.000584	1.636705e-07	3.998683

References

1. M. Ahues, A. Largillier and B.V. Limaye, *Spectral computations for bounded operators*, Chapman and Hall/CRC, New York, 2001.
2. K.E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge University Press, Cambridge, UK, 1997.
3. F. Chatelin, *Spectral approximation of linear operators*, Academic Press, New York, 1983.
4. Z. Chen, G. Long and G. Nelakanti, *Richardson extrapolation of iterated discrete projection methods for eigenvalue appoximation*, J. Comp. and Appl. Math. **223** (2009), 48-61.
5. S.D. Conte and C. de Boor, *Elementary numerical analysis*, Tata McGraw-Hill, 2005.
6. G. Han, *Extrapolation of a discrete collocation-type method of Hammerstein equations*, J. Comp. Appl. Math. **61** (1995), 73-86.
7. G. Han and L. Zhang, *Asymptotic Error Expansion of Two-Dimensional Volterra Integral Equation by Iterated Collocation*, Appl. Math. Comp. **61** (1994), 269-285.
8. E. Isaacson and H.B. Keller, *Analysis of Numerical Methods*, John Wiley and Sons, 1966.
9. R.P. Kulkarni and G. Nelakanti, *Spectral approximation using iterated discrete Galerkin method*, Numer. Funct. Opt. **23** (2002), 91-104.
10. Q. Lin, I.H. Sloan, R. Xie, *Extrapolation of the iterated-collocation method for integral equations of the second kind*, SIAM J. Numer. Anal. **27** (1990), 1535-1541.
11. W. McLean, *Asymptotic error expansions for numerical solutions of integral equations*, IMA J. Numer. Anal. **9** (1989), 373-384.
12. G. Nelakanti, *A degenerate kernel method for eigenvalue problems of compact integral operators*, Adv. Comput. Math. **27** (2007) 339-354.
13. G. Nelakanti, *Spectral approximation for integral operators*, Ph.D Thesis, Indian Insitute of Technology, Bombay, India, 2003.
14. J.E. Osborn, *Spectral approximation for compact operators*, Math. Comp. **29** (1975) 712-725.
15. B.L. Panigrahi and G. Nelakanti, *Richardson extrapolation of iterated discrete Galerkin method for eigenvalue problem of a two dimensional compact integral operator*, J. Sci. Comp. **51** (2012), 421-448.
16. I.H. Sloan, *Iterated Galerkin method for eigenvalue problems*, SIAM J. Numer. Anal. **13** (1976), 753-760.

Bijaya Laxmi Panigrahi received M.Sc. from Sambalpur University, India, and Ph.D from Indian Institute of Technology Kharagpur, India. She is currently working as a Lecturer in the Department of Mathematics, Sambalpur University. Her research interests include spectral approximations of integral operators.

Department of Mathematics, Sambalpur University, Jyoti Vihar, Odisha-768019, India.
e-mail: bijayalaxmi_rs@maths.iitkgp.ernet.in

Gnaneshwar Nelakanti received Ph.D. from Indian Institute of Technology Bombay, India. He is currently working as Asst. Professor in Indian Institute of Technology Kharagpur, India. His research interests include Inverse and ill-posed problems and Approximate solutions of operator equations.

Department of Mathematics, Indian Institute of Technology Kharagpur, West Bengal - 721302, India.
e-mail: gnanesh@maths.iitkgp.ernet.in