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WEYL'S THEOREM, TENSOR PRODUCT, FUGLEDE-PUTNAM THEOREM AND CONTINUITY SPECTRUM FOR k-QUASI CLASS $\mathcal{A}_n^*$ OPERATORS

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Abstract. An operator $T \in L(H)$, is said to belong to k-quasi class $\mathcal{A}_n^*$ operator if

$$T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k \geq O$$

for some positive integer n and some positive integer k.

First, we will see some properties of this class of operators and prove Weyl's theorem for algebraically k-quasi class $\mathcal{A}_n^*$. Second, we consider the tensor product for k-quasi class $\mathcal{A}_n^*$, giving a necessary and sufficient condition for $T \otimes S$ to be a k-quasi class $\mathcal{A}_n^*$, when T and S are both non-zero operators. Then, the existence of a nontrivial hyperinvariant subspace of k-quasi class $\mathcal{A}_n^*$ operator will be shown, and it will also be shown that if X is a Hilbert-Schmidt operator, A and $(B^*)^{-1}$ are k-quasi class $\mathcal{A}_n^*$ operators such that $AX = XB$, then $A^*X = XB^*$. Finally, we will prove the spectrum continuity of this class of operators.

1. Introduction

Throughout this paper, let H and K be infinite dimensional separable complex Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$. We denote by $L(H, K)$ the set of all bounded operators from H into K. To simplify, we put $L(H) := L(H, H)$. For $T \in L(H)$, we denote by $\ker T$ the null space and by $T(H)$ the range of T. The null operator and the identity on H will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$. We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all non-negative integers by $\mathbb{N}$ and the complex conjugate of a complex number λ by $\overline{\lambda}$. The closure of a set M will be denoted by $\overline{M}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. An operator $T \in L(H)$, is a positive operator, $T \geq O$, if $\langle Tx, x \rangle \geq 0$ for all $x \in H$. We write $\sigma(T)$, $\sigma_p(T)$, $\sigma_s(T)$ and $\sigma_a(T)$ for the spectrum, point spectrum, surjective spectrum and approximate point spectrum, respectively.

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Sets of isolated points and accumulation points of $\sigma(T)$ are denoted by iso$\sigma(T)$ and acc$\sigma(T)$, respectively. We write $r(T)$ for the spectral radius. It is well known that $r(T) \leq \|T\|$. The operator T is called normaloid if $r(T) = \|T\|$.

For an operator $T \in L(H)$, as usual, $|T| = (T^*T)^{\frac{1}{2}}$. An operator T is said to be normal operator if $T^*T = TT^*$ and T is said to be hyponormal, if $|T|^2 \geq |T^*|^2$. The operator T is said to be a p-hyponormal operator if and only if $(T^*T)^p \geq (TT^*)^p$ for a positive number p [3]. The operator T is said to be (p,k)-quasihyponormal operator if $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq O$ for some positive integer k and $p > 0$. An operator $T \in L(H)$, is said to be paranormal [16], if $\|Tx\|^2 \leq \|T^2x\|$ for any unit vector x in H. Further, T is said to be $*$-paranormal [5], if $\|T^*x\|^2 \leq \|T^2x\|$ for any unit vector x in H. T is said to be n-paranormal operator if $\|Tx\|^{n+1} \leq \|T^{n+1}x\|\|x\|^n$ for all $x \in H$, and T is said to be n-$*$-paranormal operator if $\|T^*x\|^{n+1} \leq \|T^{n+1}x\|\|x\|^n$ for all $x \in H$. An operator T is said to be (n,k)-quasi-$*$-paranormal [43] if

$$\|T^*T^kx\| \leq \|T^{1+n+k}x\|^{\frac{1}{1+n}}\|T^kx\|^{\frac{n}{n+1}} \quad \text{for all } x \in H.$$

T. Furuta, M. Ito and T. Yamazaki [18] introduced a very interesting class of bounded linear Hilbert space operators: class $\mathcal{A}$ defined by $|T^2| \geq |T|^2$, and they showed that the class $\mathcal{A}$ is a subclass of paranormal operators. An operator is said to be quasi class $(\mathcal{A},k)$ operator if $T^{*k}(|T^2| - |T|^2)T^k \geq O$ for a positive integer k. B. P. Duggal, I. H. Jeon, and I. H. Kim [13], introduced $*$-class $\mathcal{A}$ operator. An operator $T \in L(H)$ is said to be a $*$-class $\mathcal{A}$ operator, if $|T^2| \geq |T^*|^2$. A $*$-class $\mathcal{A}$ is a generalization of a hyponormal operator, [13, Theorem 1.2], and $*$-class $\mathcal{A}$ is a subclass of the class of $*$-paranormal operators, [13, Theorem 1.3]. We denote the set of $*$-class $\mathcal{A}$ by $\mathcal{A}^*$. An operator $T \in L(H)$, is said to be a quasi-$*$-class $\mathcal{A}$ operator, if $T^*|T^2|T \geq T^*|T^*|^2T$, [39]. We denote the set of quasi-$*$-class $\mathcal{A}$ by $\mathcal{Q}(\mathcal{A}^*)$. In [42], is defined class $\mathcal{A}_n$ operator: an operator $T \in L(H)$, is said to be $\mathcal{A}_n$ operator if $|T^{n+1}|^{\frac{2}{n+1}} \geq |T|^2$ for some positive integer n. An operator $T \in L(H)$, is said to be $\mathcal{A}_n^*$ operator if $|T^{n+1}|^{\frac{2}{n+1}} \geq |T^*|^2$ for some positive integer n. If $T \in \mathcal{A}_n^*$, then T is n-$*$-paranormal operator, thus T is normaloid [36]. An operator $T \in L(H)$, is said to belong to k-quasi class $\mathcal{A}_n^*$ operator if

$$T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k \geq 0$$

for some positive integer n and some positive integer k [23].

If $n = 1$ and $k = 1$, then k-quasi class $\mathcal{A}_n^*$ operators coincide with $\mathcal{Q}(\mathcal{A}^*)$ operators.

Since $S \geq O$ implies $T^*ST \geq O$, then: If T belongs to class $\mathcal{A}_n^*$ for some positive integer $n \geq 1$, then T belongs k-quasi class $\mathcal{A}_n^*$ for every positive integer k.

Obviously,

$$1\text{-quasi class } \mathcal{A}_n^* \subseteq 2\text{-quasi class } \mathcal{A}_n^* \subseteq 3\text{-quasi class } \mathcal{A}_n^* \subseteq \cdots.$$

We say that $T \in L(H)$ is an algebraically k-quasi class $\mathcal{A}_n^*$ operator if there exists a nonconstant complex polynomial p such that $p(T)$ is a k-quasi class $\mathcal{A}_n^*$ operator. We have the following implications:

$$k\text{-quasi class } \mathcal{A}_n^* \Rightarrow \text{algebraically } k\text{-quasi class } \mathcal{A}_n^*$$

Lemma 1.1 ([9, Holder-McCarthy inequality])**.** *Let T be a positive operator. Then the following inequalities hold for all $x \in H$:*

(1) $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for* $0 < r < 1$,
(2) $\langle T^r x, x\rangle \geq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for* $r \geq 1$.

2. Weyl's type theorems for k-quasi class $\mathcal{A}_n^*$ operator

Theorem 2.1 ([23])**.** *Let $T \in L(H)$ be a k-quasi class $\mathcal{A}_n^*$ operator, T^k does not have a dense range, and let T have the following representation*

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad on \quad H = \overline{T^k(H)} \oplus \ker T^{*k}.$$

Then A is a class $\mathcal{A}_n^*$ on $\overline{T^k(H)}$, $C^k = O$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

A subspace M of space H is said to be nontrivial invariant (alternatively, T-invariant) under T, if $\{0\} \neq M \neq H$ and $T(M) \subseteq M$. A closed subspace $M \subseteq H$ is said to be a nontrivial hyperinvariant subspace for T, if $\{0\} \neq M \neq H$ and is invariant under every operator $S \in L(H)$ which fulfills $TS = ST$.

Theorem 2.2 ([23])**.** *If T is a k-quasi class $\mathcal{A}_n^*$ and $\mathcal{M}$ is a closed T-invariant subspace, then the restriction $T_{|\mathcal{M}}$ is also a k-quasi class $\mathcal{A}_n^*$ operator.*

Corollary 2.3. *If $T \in L(H)$, is a k-quasi class $\mathcal{A}_n^*$ and the restriction A on $\overline{T^k(H)}$ is invertible, then T is similar to a direct sum of a class $\mathcal{A}_n^*$ and a nilpotent operator.*

Proof. Suppose that $T = \left(\begin{smallmatrix} A & B \\ O & C \end{smallmatrix}\right)$ on $H = \overline{T^k(H)} \oplus \ker T^{*k}$. By Theorem 2.1, A is a class $\mathcal{A}_n^*$ and C is a nilpotent operator with nilpotency k. Since $0 \notin \sigma(A)$ by assumption, we have $\sigma(A) \cap \sigma(C) = \emptyset$. Hence by Rosenblum's Corollary [35], there exists an operator S for which $AS - SC = B$. Therefore

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} = \begin{pmatrix} I & S \\ O & I \end{pmatrix}^{-1} \begin{pmatrix} A & O \\ O & C \end{pmatrix} \begin{pmatrix} I & S \\ O & I \end{pmatrix}$$

which completes the proof. □

Lemma 2.4. *Let $T \in L(H, K)$ operator, defined as*

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix}.$$

If A belongs to class $\mathcal{A}_n^*$ operator, surjective and $C^k = O$, then T is similar to (n, k)-quasi-$*$-paranormal operator.

Proof. Since A is surjective and $C^k = O$ we have $\sigma_s(A) \cap \sigma_a(C) = \emptyset$. Hence by [26, Theorem 3.5.1], there exists an operator S for which $AS - SC = B$. Therefore

$$\begin{pmatrix} I & S \\ O & I \end{pmatrix} \begin{pmatrix} A & B \\ O & C \end{pmatrix} = \begin{pmatrix} A & O \\ O & C \end{pmatrix} \begin{pmatrix} I & S \\ O & I \end{pmatrix},$$

so T is similar to $R = \left(\begin{smallmatrix} A & O \\ O & C \end{smallmatrix} \right)$.

Let $x = x_1 + x_2 \in H \oplus K$. Since A is a class $\mathcal{A}_n^*$, then A is n-$*$-paranormal operator, and since $C^k = O$ we have

$$\begin{aligned} \|R^*(R^k x)\|^2 &= \|R^*(R^k(x_1 + x_2))\|^2 \\ &= \|A^*(A^k(x_1))\|^2 \\ &\leq \|A^{1+n}(A^k(x_1))\|^{\frac{2}{1+n}} \|A^k(x_1)\|^{\frac{2n}{n+1}} \\ &= \|R^{1+n}(R^k(x_1 + x_2))\|^{\frac{2}{1+n}} \|R^k(x_1 + x_2)\|^{\frac{2n}{n+1}} \\ &= \|R^{1+n}(R^k x)\|^{\frac{2}{1+n}} \|R^k(x)\|^{\frac{2n}{n+1}} \end{aligned}$$

so, R is (n, k)-quasi-$*$-paranormal operator. □

Lemma 2.5. *Let T be a class $\mathcal{A}_n^*$ operator, and assume that $\sigma(T) = \{0\}$. Then $T = O$.*

Proof. Since T is class $\mathcal{A}_n^*$, T is normaloid, therefore $T = O$. □

Corollary 2.6. *Let T be a k-quasi class $\mathcal{A}_n^*$ operator. If T is a quasinilpotent operator, then T is a nilpotent operator.*

Proof. Suppose that T is a k-quasi class $\mathcal{A}_n^*$ operator. Consider two cases:

Case I: If the range of T^k has dense range, then it is a class $\mathcal{A}_n^*$ operator. Hence by above lemma T is nilpotent operator.

Case II: If T does not have dense range, by Theorem 2.1 we can represent T as the upper triangular matrix

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad \text{on } H = \overline{T^k(H)} \oplus \ker T^{*k}.$$

Since T is quasinilpotent operator, $\sigma(T) = \{0\}$. From Theorem 2.1 we have $\sigma(A) = \{0\}$. Since A belongs to class $\mathcal{A}_n^*$, $A = O$ and we have

$$T^{k+1} = \begin{pmatrix} O & BC^k \\ O & C^{k+1} \end{pmatrix} = O.$$

□

Lemma 2.7. *Let $T \in L(H)$ be an algebraically k-quasi class $\mathcal{A}_n^*$ operator and $\sigma(T) = \{\lambda_0\}$. Then $T - \lambda_0$ is nilpotent.*

Proof. Assume $p(T)$ is a k-quasi class $\mathcal{A}_n^*$ operator for some non-constant polynomial $p(z)$. Since $\sigma(p(T)) = p(\sigma(T)) = \{p(\lambda_0)\}$, the operator $p(T) - p(\lambda_0)$ is nilpotent by Corollary 2.6.

Let

$$p(z) - p(\lambda_0) = a(z - \lambda_0)^k (z - \lambda_1)^{k_1} \cdot \dots \cdot (z - \lambda_n)^{k_n},$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$. Then

$$O = (p(T) - p(\lambda_0))^m = a^m (T - \lambda_0)^{mk} (T - \lambda_1)^{mk_1} \cdot \cdots \cdot (T - \lambda_n)^{mk_n}$$

and hence $(T - \lambda_0)^{mk} = O$. □

An operator $T \in L(H)$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T, while an operator $T \in L(H)$ is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. In general, if T is polaroid, then T is isoloid. However, the converse is not true. For $T \in L(H)$, the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$ is called the ascent of T and is denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. We say that $T \in L(H)$ is of finite ascent (finitely ascensive) if $p(T) < \infty$. For $T \in L(H)$, the smallest nonnegative integer q, such that $T^q(H) = T^{q+1}(H)$, is called the descent of T and is denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$. We say that $T \in L(H)$ is of finite descent if $q(T - \lambda) < \infty$ for all $\lambda \in \mathbb{C}$.

Lemma 2.8. *Let T be an algebraically k-quasi class $\mathcal{A}_n^*$ operator. Then T is polaroid.*

Proof. Suppose T is an algebraically k-quasi class $\mathcal{A}_n^*$ operator. Then $p(T)$ is a k-quasi class $\mathcal{A}_n^*$ for some non-constant polynomial p. Let $\lambda \in \text{iso}\sigma(T)$. Using the spectral projection $P = \frac{1}{2\pi i}\int_{\partial D}(T - \lambda)^{-1} d\lambda$, where D is an open disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = T_1 \oplus T_2, \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T_1 is an algebraically k-quasi class $\mathcal{A}_n^*$ operator, $T_1 - \lambda$ is also algebraically k-quasi class $\mathcal{A}_n^*$. But $\sigma(T_1 - \lambda) = \{0\}$, it follows from Lemma 2.7 that $T_1 - \lambda$ is nilpotent operator. Therefrom $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, clearly it has finite ascent and descent. Therefore $T - \lambda$ has finite ascent and descent. Thus λ is a pole of the resolvent of T. Hence T is polaroid, which means that T is isoloid. □

We write $\alpha(T) = \dim\ker T$, $\beta(T) = \dim(H/T(H))$. An operator $T \in L(H)$ is called an upper semi-Fredholm, if it has a closed range and $\alpha(T) < \infty$, while T is called a lower semi-Fredholm if $\beta(T) < \infty$. However, T is called a semi-Fredholm operator if T is either an upper or a lower semi-Fredholm, and T is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in L(H)$ is semi-Fredholm, then the index is defined by

$$\text{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in L(H)$ is said to be upper semi-Weyl operator if it is upper semi-Fredholm and $\text{ind}(T) \leq 0$, while $T \in L(H)$ is said to be lower semi-Weyl operator if it is lower semi-Fredholm and $\text{ind}(T) \geq 0$. An operator is said to be Weyl operator if it is Fredholm of index zero. The Weyl spectrum and the essential approximate spectrum are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_{uw}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Weyl}\}.$$

An operator $T \in L(H)$ is said to be upper semi-Browder operator, if it is upper semi-Fredholm and $p(T) < \infty$. An operator $T \in L(H)$ is said to be lower semi-Browder operator, if it is lower semi-Fredholm and $q(T) < \infty$. An operator $T \in L(H)$ is said to be Browder operator, if it is Fredholm of finite ascent and descent. The Browder spectrum and the upper semi-Browder spectrum are defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}$$

and

$$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Browder}\}.$$

For $T \in L(H)$ we will denote $p_{00}(T)$ the set of all poles of finite rank of T. We have $\sigma(T) \setminus \sigma_b(T) = p_{00}(T)$ and we say that T satisfies Browder's theorem if

$$\sigma_w(T) = \sigma_b(T) \text{ or } \sigma(T) \setminus \sigma_w(T) = p_{00}(T).$$

For $T \in L(H)$ we write $\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$ for the isolated eigenvalues of finite multiplicity. We say that T satisfies Weyl's theorem, if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$

Let $\pi^a_{00}(T) = \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T-\lambda) < \infty\}$ be the set of all eigenvalues of T of finite multiplicity, which are isolated in the approximate point spectrum. We say that T satisfies a-Weyl's theorem, if

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi^a_{00}(T).$$

We will denote $p^a_{00}(T)$ the set of all left poles of finite rank of T. We have

$$\sigma_a(T) \setminus \sigma_{ub}(T) = p^a_{00}(T)$$

and we say that T satisfies a-Browder's theorem, if

$$\sigma_{uw}(T) = \sigma_{ub}(T) \text{ or } \sigma_a(T) \setminus \sigma_{uw}(T) = p^a_{00}(T).$$

Let $Hol(\sigma(T))$ be the space of all analytic functions in an open neighborhood of $\sigma(T)$. We say that $T \in L(H)$ has the single valued extension property at $\lambda \in \mathbb{C}$, if for every open neighborhood U of λ the only analytic function $f : U \to \mathbb{C}$ which satisfies equation $(T - \lambda)f(\lambda) = 0$, is the constant function $f \equiv 0$. The operator T is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$.

Theorem 2.9. *If T or T^* is an algebraically k-quasi class $\mathcal{A}^*_n$, then Weyl's theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$.*

Proof. Suppose T is an algebraically k-quasi class $\mathcal{A}_n^*$. From Lemma 2.8 we have T is polaroid. Since T is an algebraically k-quasi class $\mathcal{A}_n^*$, $p(T)$ is a k-quasi class $\mathcal{A}_n^*$ operator for some non-constant polynomial p. From [23, Corollary 3.9] $p(T)$ has SVEP. Therefore T has SVEP by [26, Theorem 3.3.9]. Then, from [2, Theorem 3.3], T satisfies Weyl's theorem.

Since T is isoloid from [27] we have

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T)).$$

Then, by [23, Theorem 3.10] we have

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T)),$$

which implies that Weyl's theorem holds for $f(T)$.

Suppose T^* is an algebraically k-quasi class $\mathcal{A}_n^*$, then T^* is polaroid. From [2, Theorem 2.11] T is polaroid as well as isoloid. Since T^* has SVEP, from [1, Theorem 4.23] T satisfies Browder's theorem, and since T is polaroid then T satisfies Weyl's theorem. Since T is isoloid, as in the proof of the firs part, we have that Weyl's theorem holds for $f(T)$. □

Theorem 2.10. *If T^* is an algebraically k-quasi class $\mathcal{A}_n^*$ operator, then T satisfies a-Weyl's theorem.*

Proof. Let T^* be an algebraically k-quasi class $\mathcal{A}_n^*$ operator. T^* has SVEP and from [1, Theorem 4.34], T satisfies a-Browder theorem. We use the fact [1, Theorem 4.51]: T satisfies a-Weyl's theorem, if and only if, T satisfies a-Browder's theorem and $\pi_{00}^a(T) = p_{00}^a(T)$. We show that $\pi_{00}^a(T) = p_{00}^a(T)$. Since $\pi_{00}^a(T) \supseteq p_{00}^a(T)$ holds for every operator T, it would suffice to prove the inclusion $\pi_{00}^a(T) \subseteq p_{00}^a(T)$. Let λ be an arbitrary point of $\pi_{00}^a(T)$. Then $\lambda \in \text{iso}\sigma_a(T)$ and $0 < \alpha(T-\lambda) < \infty$. Thus $\lambda \in \sigma_a(T)$. But T^* has SVEP, hence $\sigma(T) = \sigma_a(T)$ by [1, Corollary 2.28]. Therefore λ is an isolated point of $\sigma(T)$. So, $\lambda \in \pi_{00}(T)$. Since Weyl's theorem holds for T, $\lambda \notin \sigma_w(T)$. Since $T-\lambda$ is Fredholm operator and T has SVEP in λ, then $p(T-\lambda) < \infty$, by [1, Theorem 2.45.]. Therefore, $T-\lambda$ is semi-upper Browder operator, and hence $\lambda \in p_{00}^a(T)$. □

Let $T \in L(H)$. It is well known that the inclusion $\sigma_{uw}(f(T)) \subseteq f(\sigma_{uw}(T))$ holds for every $f \in Hol(\sigma(T))$ with no restriction on T [33, Theorem 3.3.].

Lemma 2.11. *If T^* is an algebraically k-quasi class $\mathcal{A}_n^*$ operator, then*

$$\sigma_{uw}(f(T)) = f(\sigma_{uw}(T))$$

holds for every $f \in Hol(\sigma(T))$.

Proof. Let T^* be an algebraically k-quasi class $\mathcal{A}_n^*$ and let $f \in Hol(\sigma(T))$. It suffices to show that $f(\sigma_{uw}(T)) \subseteq \sigma_{uw}(f(T))$. Suppose that $\lambda \notin \sigma_{uw}(f(T))$. Then $f(T)-\lambda$ is semi-upper Weyl operator and

$$f(T) - \lambda = c(T-\lambda_1)(T-\lambda_2) \cdot \cdots \cdot (T-\lambda_n)g(T), \tag{1}$$

where $c, \lambda_1, \lambda_2, \dots \lambda_n \in \mathbb{C}$, and $g(T)$ is invertible. Since T^* is an algebraically k-quasi class $\mathcal{A}_n^*$, T^* has SVEP. It follows from [1, Corollary 2.48] that $\text{ind}(T - \lambda_i) \geq 0$ for each $i = 1, 2, \dots, n$. Since

$$0 \leq \sum_{i=1}^{n} \text{ind}(T - \lambda_i) = \text{ind}(f(T) - \lambda) \leq 0,$$

$T - \lambda_i$ is Weyl for each $i = 1, 2, \dots, n$. Hence $\lambda \notin f(\sigma_{uw}(T))$, and so $f(\sigma_{uw}(T)) \subseteq \sigma_{uw}(f(T))$. This completes the proof. □

An operator $T \in L(H)$ is called a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. Clearly, if T is a-isoloid, then T is isoloid. However, the converse is not true.

Lemma 2.12. *If T^* is an algebraically k-quasi class $\mathcal{A}_n^*$ operator, then T is a-isoloid.*

Proof. Let λ be an isolated point of $\sigma_a(T)$. Since T^* has SVEP, by [1, Corollary 2.28] λ is an isolated point of $\sigma(T)$. But T^* is polaroid, hence T is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$. Thus T is a-isoloid. □

Theorem 2.13. *If T^* is an algebraically k-quasi class $\mathcal{A}_n^*$ operator, then a-Weyl's theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$.*

Proof. Let $f \in Hol(\sigma(T))$. From Theorem 2.10, T satisfies a-Weyl's theorem and we have $\sigma_{uw}(T) = \sigma_{ub}(T)$. It follows

$$\sigma_{ub}(f(T)) = f(\sigma_{ub}(T)) = f(\sigma_{uw}(T)) = \sigma_{uw}(f(T))$$

and hence $f(T)$ satisfies a-Browders theorem.

It is sufficient to show $\pi_{00}^a(f(T)) \subseteq p_{00}^a(f(T))$. Suppose $\lambda \in \pi_{00}^a(f(T))$. Then λ is an isolated point of $\sigma_a(f(T))$ and $0 < \alpha(f(T) - \lambda) < \infty$. Thus $\lambda \in \sigma_a(f(T))$ and equation (1) is fulfilled.

Since λ is an isolated point of $f(\sigma_a(T))$, if $\lambda_i \in \sigma_a(T)$, then λ_i is an isolated point of $\sigma_a(T)$ by (1). Since T is a-isoloid, $0 < \alpha(T - \lambda_i) < \infty$ for each $i = 1, 2, \dots, n$. Thus $\lambda_i \in \pi_{00}^a(T)$ for each $i = 1, 2, \dots, n$. Since T satisfies a-Weyl's theorem, $T - \lambda_i$ is upper-semi Fredholm and $\text{ind}(T - \lambda_i) \leq 0$ for each $i = 1, 2, \dots, n$. Therefore $f(T) - \lambda$ is upper semi-Fredholm. Since $\lambda \in \text{iso}\sigma_a(f(T))$, $f(T)$ has SVEP in λ, thus by [1, Theorem 2.45] $p(f(T) - \lambda) < \infty$, so $f(T) - \lambda$ is semi-upper Browder operator. Therefore $\lambda \in p_{00}^a(f(T))$. □

3. Tensor products for k-quasi class $\mathcal{A}_n^*$

Let H and K denote the Hilbert spaces. For given non zero operators $T \in L(H)$ and $S \in L(K)$, $T \otimes S$ denotes the tensor product on the product space $H \otimes K$. The normaloid property is invariant under tensor products, [37]. There exist paranormal operators T and S, such that $T \otimes S$ is not paranormal, [4]. In [40], Stochel proved that $T \otimes S$ is normal, if and only if, T and S are normal. This result was extended to class $\mathcal{A}$ operators, $*$-class $\mathcal{A}$ operators, class $\mathcal{A}_n$ operators and quasi class $\mathcal{A}_n$ operators in [24], [13], [30], and [31] respectively. In this section, we prove an analogues result for k-quasi class $\mathcal{A}_n^*$ operators.

Let $T \in L(H)$ and $S \in L(K)$ be non zero operators. Then $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$ holds. By the uniqueness of positive square roots, we have $|T \otimes S|^r = |T|^r \otimes |S|^r$ for any positive rational number r. From the density of the rationales in the real, we obtain $|T \otimes S|^p = |T|^p \otimes |S|^p$ for any positive real number p.

Theorem 3.1. *Let $T \in L(H)$ and $S \in L(K)$ be non zero operators. Then $T \otimes S$ is a k-quasi class $\mathcal{A}_n^*$ operator, if and only if, one of the following holds*:
(1) *T and S are k-quasi class $\mathcal{A}_n^*$,*
(2) *$T^{k+1} = O$ or $S^{k+1} = O$.*

Proof. We have

$$\begin{aligned} &(T \otimes S)^{*k}\left(|(T \otimes S)^{n+1}|^{\frac{2}{n+1}} - |(T \otimes S)^*|^2\right)(T \otimes S)^k \\ &= (T \otimes S)^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} \otimes |S^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \otimes |S^*|^2\right)(T \otimes S)^k \\ &= T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k \otimes S^{*k}|S^{n+1}|^{\frac{2}{n+1}}S^k \\ &\quad + T^{*k}|T^*|^2T^k \otimes S^{*k}\left(|S^{n+1}|^{\frac{2}{n+1}} - |S^*|^2\right)S^k. \end{aligned}$$

Hence, if either (1) T and S are k-quasi class $\mathcal{A}_n^*$ operators or (2) $T^{k+1} = O$ or $S^{k+1} = O$, then $T \otimes S$ is a k-quasi class $\mathcal{A}_n^*$ operator.

Conversely, suppose that $T \otimes S$ is a k-quasi class $\mathcal{A}_n^*$ operator. Then, for $x \in H$, $y \in K$ we get

$$\begin{aligned} &\left\langle T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^kx, x\right\rangle \left\langle S^{*k}|S^{n+1}|^{\frac{2}{n+1}}S^ky, y\right\rangle \\ &+ \left\langle T^{*k}|T^*|^2T^kx, x\right\rangle \left\langle S^{*k}(|S^{n+1}|^{\frac{2}{n+1}} - |S^*|^2)S^ky, y\right\rangle \geq 0. \end{aligned}$$

It suffices to show that if the statement (1) does not hold, then the statement (2) holds. Thus, assume to the contrary that neither T^{k+1} nor S^{k+1} is the zero operator, and T is not a k-quasi class $\mathcal{A}_n^*$ operator. Then, there exists $x_0 \in H$, such that:

$$\begin{aligned} &\left\langle T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^kx_0, x_0\right\rangle = \alpha < 0 \quad \text{and} \\ &\left\langle T^{*k}|T^*|^2T^kx_0, x_0\right\rangle = \beta > 0. \end{aligned}$$

From the above relation, we have

$$(\alpha + \beta)\left\langle S^{*k}|S^{n+1}|^{\frac{2}{n+1}}S^ky, y\right\rangle \geq \beta\left\langle S^{*k}|S^*|^2S^ky, y\right\rangle.$$

Thus, S is a k-quasi class $\mathcal{A}_n^*$ operator, because $\alpha + \beta < \beta$.

We have,

$$\left\langle S^{*k}|S^*|^2S^ky, y\right\rangle = \left\langle |S^*|^2S^ky, S^ky\right\rangle = \langle S^*S^ky, S^*S^ky\rangle = \|S^*S^ky\|^2$$

and using the Holder McCarthy inequality, we get

$$\begin{aligned}\left\langle S^{*k}|S^{n+1}|^{\frac{2}{n+1}}S^ky,y\right\rangle &= \left\langle \left(S^{*(n+1)}S^{n+1}\right)^{\frac{1}{n+1}} S^ky, S^ky\right\rangle \\ &\le \left\langle S^{*(n+1)}S^{n+1}S^ky, S^ky\right\rangle^{\frac{1}{n+1}} \|S^ky\|^{\frac{2n}{n+1}} \\ &= \|S^{n+k+1}y\|^{\frac{2}{n+1}}\|S^ky\|^{\frac{2n}{n+1}}.\end{aligned}$$

Then

$$(\alpha+\beta)\|S^{n+k+1}y\|^{\frac{2}{n+1}}\|S^ky\|^{\frac{2n}{n+1}} \ge \beta\|S^*S^ky\|^2.$$

Since S is a k-quasi class $\mathcal{A}_n^*$ operator, from Theorem 2.1 S has decomposition of the form

$$S = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \text{ on } H = \overline{S^k(H)} \oplus kerS^{*k},$$

where $A = S\mid_{\overline{S^k(H)}}$ is $\mathcal{A}_n^*$ operator, we have

$$(\alpha+\beta)\|A^{n+1}\mu\|^{\frac{2}{n+1}}\|\mu\|^{\frac{2n}{n+1}} \ge \beta\|S^*\mu\|^2 \ge \beta\|A^*\mu\|^2$$

for all $\mu \in \overline{S^k(H)}$.

Since $A \in \mathcal{A}_n^*$, A is normaloid. Thus, taking supremum on both sides of the above inequality, we have

$$(\alpha+\beta)\|A\|^2 \ge \beta\|A^*\|^2 = \beta\|A\|^2.$$

This inequality makes $A = O$. From Corollary 2.6, we have $S^{k+1} = O$. This is a contradiction to that S^{k+1} is not a zero operator. So T must be a k-quasi class $\mathcal{A}_n^*$ operator. A similar argument shows that S is also a k-quasi class $\mathcal{A}_n^*$ operator, which completes the proof. □

4. Fuglede-Putnam theorem for k-quasi class $\mathcal{A}_n^*$

The famous Fuglede-Putnam's theorem is as follows:

Theorem 4.1. *Let A and B be normal operators, and X be an operator so that $AX = XB$. Then, $A^*X = XB^*$.*

The Fuglede-Putnam's theorem is very useful in operators' theory, thanks to its numerous applications. In fact, the Fuglede-Putnam's theorem was first proved in the $A = B$ case by B. Fuglede [15], and then a proof in the general case by C. R. Putnam [32]. A lot of researchers have worked on it since the papers of Fuglede and Putnam.

Suppose $\{e_n\}$ is an orthonormal bases in H. We define the Hilbert-Schmidt norm of T to be $\|T\|_2 = (\sum_{n=1}^{\infty}\|Te_n\|^2)^{\frac{1}{2}}$. This definition is independent of the choice of basis (see [10]). If $\|T\|_2 < \infty$, T is said to be a Hilbert-Schmidt operator. The set of all Hilbert-Schmidt operators will be denoted by $\mathcal{C}_2(H)$.

In the past several years, many authors have extended this theorem for several classes of nonnormal operators. In [6], S. Berberian has extended the result by assuming A and B^* are hyponormal operators and X is a Hilbert-Schmidt operator. In [17], Furuta extended the result by assuming A and B^* are subnormal operators and X is a Hilbert-Schmidt operator. A. Uchiyama and K. Tanahashi [41] showed that Fuglede-Putnam's theorem holds for p-hyponormal and log-hyponormal operators. If let $X \in L(H)$ be Hilbert-Schmidt class, S. Mecheri and A. Uchiyama [28] showed that normality in Fuglede-Putnam's theorem can be replaced by A and B^* class $\mathcal{A}$ operators. Recently M. H. M. Rashid and M. S. M. Noorani [34] showed that the above result of S. Mecheri and A. Uchiyama holds for A and B^* quasi-class $\mathcal{A}$ operators with the additional condition $\| \, |A^*| \, \| \, \| \, |B|^{-1} \, \| \leq 1$. In this paper, we show that if X is a Hilbert-Schmidt operator, A and $(B^*)^{-1}$ are k-quasi class $\mathcal{A}_n^*$ operators such that $AX = XB$, then $A^*X = XB^*$.

For each pair of operators $A, B \in L(H)$, there is an operator $\Gamma_{A,B}$ defined on $\mathcal{C}_2(H)$ via the formula $\Gamma_{A,B}(X) = AXB$.

Let $\mathcal{C}_1(H)$ be the set $\{C = AB : A, B \in \mathcal{C}_2(H)\}$. Then, operators belonging to $\mathcal{C}_1(H)$ are called trace class operators. We define the linear functional

$$tr : \mathcal{C}_1(H) \longrightarrow \mathbb{C} \text{ by } tr(C) = \sum_{n=1}^{\infty} \langle Ce_n, e_n \rangle$$

for an orthonormal basis $\{e_n\}$ for H. In this case, the definition of $tr(C)$ does not depend on the choice of an orthonormal basis, and $tr(C)$ is called the trace of C.

Lemma 4.2 ([10]). *If $\langle A, B \rangle = \sum_{n=1}^{\infty} \langle Ae_n, Be_n \rangle = tr(B^*A) = tr(AB^*)$ for A and B in $\mathcal{C}_2(H)$, and for any orthonormal basis $\{e_n\}$ for H, then $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{C}_2(H)$, and $\mathcal{C}_2(H)$ is a Hilbert-Schmidt space with respect to this inner product.*

From the above lemma, we have:

$$\begin{aligned} \langle \Gamma^* X, Y \rangle &= \langle X, \Gamma Y \rangle = \langle X, AYB \rangle = tr((AYB)^* X) \\ &= tr(XB^*Y^*A^*) = tr(A^*XB^*Y^*) = \langle A^*XB^*, Y \rangle. \end{aligned}$$

So, the adjoint of Γ is given by the formula $\Gamma^* X = A^*XB^*$.

Theorem 4.3. *Let A and $B \in L(H)$. Then $\Gamma_{A,B}$ is a k-quasi class $\mathcal{A}_n^*$ operator on $\mathcal{C}_2(H)$ if and only if one of the following assertions holds:*

(1) $A^{k+1} = O$ *or* $B^{k+1} = O$;

(2) *A and B^* are k-quasi class $\mathcal{A}_n^*$ operators.*

Proof. The unitary operator $U : \mathcal{C}_2(H) \to H \otimes H$ by a map $x \otimes y^* \to x \otimes y$ induces the $*$-isomorphism $\Psi : L(\mathcal{C}_2(H)) \to L(H \otimes H)$ by a map $X \to UXU^*$. Then we can obtain $\Psi(\Gamma_{A,B}) = A \otimes B^*$ [8]. The complete proof comes from Theorem 3.1. □

Lemma 4.4 ([23]). *Let $T \in L(H)$ be a k-quasi class $\mathcal{A}_n^*$ operator for a positive integer k. If $\lambda \neq 0$ and $(T - \lambda)x = 0$ for some $x \in H$, then $(T - \lambda)^* x = 0$.*

Now we are ready to extend Fuglede-Putnam's theorem to k-quasi class $\mathcal{A}_n^*$ operators.

Theorem 4.5. *Let A and $(B^*)^{-1}$ be k-quasi class $\mathcal{A}_n^*$ operators. If $AX = XB$ for $X \in \mathcal{C}_2(H)$, then $A^*X = XB^*$.*

Proof. Let Γ be defined on $\mathcal{C}_2(H)$ by $\Gamma Y = AYB^{-1}$. Since A and $(B^*)^{-1}$ are k-quasi class $\mathcal{A}_n^*$ operators, Γ is a k-quasi class $\mathcal{A}_n^*$ operator on $\mathcal{C}_2(H)$, by Theorem 4.3. Since $AX = XB$, $\Gamma X = AXB^{-1} = X$, so X is an eigenvector of Γ. By Lemma 4.4 we have $\Gamma^* X = A^*X(B^{-1})^* = X$, which implies $A^*X = XB^*$. $\square$

5. Hyperinvariant subspace

Let $\sigma_T(x) \subseteq \mathbb{C}$ denote the local spectral of T at the point $x \in H$, i.e., the complement of the set $\rho_T(x)$ of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U of λ in $\mathbb{C}$ and an analytic function $f : U \to H$ such that $(T - \mu)f(\mu) = x$ holds for all $\mu \in U$. Moreover, $\sigma_T(x) \subseteq \sigma(T)$. For every closed subset F of $\mathbb{C}$, let $H_T(F) = \{x \in H : \sigma_T(x) \subseteq F\}$ denote the corresponding analytic spectral subspace of T.

An operator $T \in L(H)$ is said to be decomposable if, for any open covering $\{U, V\}$ of the complex plane $\mathbb{C}$ there are two closed T-invariant subspaces Y and Z of H such that $H = Y + Z$, $\sigma(T_{|Y}) \subseteq U$ and $\sigma(T_{|Z}) \subseteq V$. For every decomposable operator T the identity $H = H_T(\overline{U}) + H_T(\overline{V})$ holds for every open cover $\{U, V\}$ of $\mathbb{C}$ [26, Theorem 1.2.23].

An operator $A \in L(H, K)$ is called quasi-affine if it has trivial kernel and has dense range. An operator $S \in L(H)$ is said to be a quasi-affine transform of $T \in L(K)$ if there exists a quasi-affine $A \in L(H, K)$ such that $AS = TA$.

Theorem 5.1. *Let $T \in L(H)$ be a k-quasi class $\mathcal{A}_n^*$ operator such that $T \neq zI$ for all $z \in \mathbb{C}$. If S is a decomposable quasi-affine transform of T, then T has a nontrivial hyperinvariant subspace.*

Proof. If S is a decomposable quasi-affine transform of T, then there exists a quasi-affine A such that $AS = TA$, where S is decomposable. Assume that T has no nontrivial hyperinvariant subspace. From [25, Lemma 3.6.1] $\sigma_p(T) = \emptyset$ and $H_T(F) = \{0\}$ for each closed set F proper in $\sigma(T)$. Let $\{U, V\}$ be an open cover of $\mathbb{C}$ such that $\sigma(T) \setminus \overline{U} \neq \emptyset$ and $\sigma(T) \setminus \overline{V} \neq \emptyset$.

Now, if $x \in H_S(\overline{U})$, then $\sigma_S(x) \subset \overline{U}$. Hence there exists an analytic H-valued function f defined on $\mathbb{C} \setminus \overline{U}$ such that $(S - z)f(z) = x$ for all $z \in \mathbb{C} \setminus \overline{U}$. So $(T - z)Af(z) = A(S - z)f(z) = Ax$. Hence $\mathbb{C} \setminus \overline{U} \subset \rho_T(Ax)$, this implies $Ax \in H_T(\overline{U})$. Thus $A(H_S(\overline{U})) \subseteq H_T(\overline{U})$, similar $A(H_S(\overline{V})) \subseteq H_T(\overline{V})$.

Therefore, since S is decomposable then $H = H_S(\overline{U}) + H_S(\overline{V})$, and finally

$$A(H) = A(H_S(\overline{U})) + A(H_S(\overline{V})) \subseteq H_T(\overline{U}) + H_T(\overline{V}) = \{0\}.$$

This is a contradiction. Hence, T has a nontrivial hyperinvariant subspace. $\square$

Theorem 5.2. *Let $T \in L(H \oplus K)$ be a k-quasi class $\mathcal{A}_n^*$ operator. If there exists a nonzero vector $x \in H \oplus K$ such that $\sigma_T(x) \subsetneqq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.*

Proof. Let's set $M = H_T(\sigma_T(x)) = \{y \in H \oplus K : \sigma_T(y) \subseteq \sigma_T(x)\}$. From [26, Theorem 1.2.16] M is a T-hyperinvariant subspace. Since $x \in M$, $M \neq \{0\}$. Suppose $M = H \oplus K$. Since T is a k-quasi class $\mathcal{A}_n^*$ operator, from [23, Corollary 3.11], T has SVEP. From [26, Theorem 1.3.2]

$$\sigma(T) = \cup\{\sigma_T(y) : y \in H \oplus K\} \subseteq \sigma_T(x) \subsetneqq \sigma(T),$$

which is contradiction. Hence M is a nontrivial T-hyperinvariant subspace. □

6. Spectrum continuity on the set of k-quasi class $\mathcal{A}_n^*$ operator

Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{C}$. Let's define the inferior and superior limits of $\{E_n\}_{n\in\mathbb{N}}$, denoted respectively by $\liminf_{n\to\infty}\{E_n\}$ and $\limsup_{n\to\infty}\{E_n\}$ as it follows:

1) $\liminf_{n\to\infty}\{E_n\} = \{\lambda \in \mathbb{C}$: for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $B(\lambda, \epsilon) \cap E_n \neq \emptyset$ for all $n > N\}$,
2) $\limsup_{n\to\infty}\{E_n\} = \{\lambda \in \mathbb{C}$: for every $\epsilon > 0$, there exists $J \subseteq \mathbb{N}$ infinite such that $B(\lambda, \epsilon) \cap E_n \neq \emptyset$ for all $n \in J\}$.

If $\liminf_{n\to\infty}\{E_n\}=\limsup_{n\to\infty}\{E_n\}$, then $\lim_{n\to\infty}\{E_n\}$ is defined by this common limit.

A mapping p, defined on $L(H)$, whose values are compact subsets on $\mathbb{C}$ is said to be upper semi-continuous at T, if $T_n \to T$, then $\limsup_{n\to\infty} p(T_n) \subset p(T)$, and lower semi-continuous at T, if $T_n \to T$, then $p(T) \subset \liminf_{n\to\infty} p(T_n)$. If p is both upper and lower semi-continuous at T, then it is said to be continuous at T and in this case $\lim_{n\to\infty} p(T_n) = p(T)$.

The spectrum $\sigma : T \to \sigma(T)$ is upper semi-continuous by [21, Problem 102], but it is not continuous in general as shown in the next example.

Example 6.1. Let U be the unilateral shift on $l^2(\mathbb{N})$ and let T and T_n, be operators defined on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ as

$$T = \begin{pmatrix} U & O \\ O & U^* \end{pmatrix} \text{ and } T_n = \begin{pmatrix} U & \frac{1}{n}(I - UU^*) \\ O & U^* \end{pmatrix}.$$

Observe that $T_n \to T$, but $\sigma(T_n) \nrightarrow \sigma(T)$. Indeed, each T_n is similar to T_1 and T_1 is an unitary operator, so for every n, $\sigma(T_n) = \sigma(T_1) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

It has been proved that σ is continuous in the set of normal operators and hyponormal operators by Halmos in [21]. And this result has been extended to quasihyponormal operators by S. V. Djordjević in [11], to p-hyponormal operators by Hwang and Lee in [22], to (p, k)-quasihyponomal operators and paranormal operators by Duggal, Jeon and Kim in [14], to quasi-clas $(\mathcal{A}, k)$ operators by Gao and Fang in [19], to k-quasi-$*$-class $\mathcal{A}$ by Gao and Li in [20].

The Berberian extension theorem [7] says that for a given operator $T \in L(H)$ there exists a Hilbert space Y such that $H \subset Y$ and a map $\varphi : L(H) \to L(Y)$ such that $\varphi : T \to \varphi(T) = T^0$ preserving order such that $\sigma_a(T) = \sigma_a\left(T^0\right) = \sigma_p\left(T^0\right)$ and $\sigma(T) = \sigma(T^0)$. If T is a k-quasi class $\mathcal{A}_n^*$ operator, then T^0 is a k-quasi class $\mathcal{A}_n^*$ operator too, [23, Theorem 3.7].

Lemma 6.2 ([29])**.** *If $\{T_n\} \subset L(H)$ and $T \in L(H)$ are such that T_n converges, according to the operator norm topology to T, then* $\mathrm{iso}\sigma(T) \subseteq \liminf_{n\to\infty} \sigma(T_n)$.

Theorem 6.3. *The spectrum σ is continuous on the set of k-quasi class $\mathcal{A}_n^*$ for a positive integer k.*

Proof. Let $\{T_n\}$ be a sequence of operators so that it belongs to k-quasi class $\mathcal{A}_n^*$ operators and $\lim_{n\to\infty} \|T_n - T\| = 0$, where T is a k-quasi class $\mathcal{A}_n^*$ operator. Since the function σ is upper semi-continuous, $\limsup_{n\to\infty} \sigma(T_n) \subset \sigma(T)$. Therefore, to prove the theorem, it will be sufficient to prove that $\sigma(T) \subset \liminf_{n\to\infty} \sigma(T_n)$. From [38, Proposition 4.9] it will be sufficient to prove $\sigma_a(T) \subset \liminf_{n\to\infty} \sigma(T_n)$. Since $\sigma(T) = \sigma(T^0)$, $\sigma(T_n) = \sigma(T_n^0)$ and $\sigma_a(T) = \sigma_a(T^0)$ we have

$$\sigma_a(T) \subset \liminf_{n\to\infty} \sigma(T_n) \Longleftrightarrow \sigma_a(T^0) \subset \liminf_{n\to\infty} \sigma(T_n^0).$$

Let $\lambda \in \sigma_a(T^0)$. Then $\lambda \in \sigma_p(T^0)$. By [23, Theorem 3.5] T^0 has a representation

$$T^0 = \lambda \oplus A \text{ on } H = \ker(T^0 - \lambda) \oplus (\ker(T^0 - \lambda))^\perp \text{ and } \ker(A - \lambda) = \{0\}.$$

Therefore $A-\lambda$ is an upper semi-Fredholm operator and $\alpha(A-\lambda) = 0$. There exists an $\epsilon > 0$ such that $A-(\lambda-\mu_0)$ is an upper semi-Fredholm operator with $\mathrm{ind}(A-(\lambda-\mu_0)) = \mathrm{ind}(A-\lambda)$ and $\alpha(A-(\lambda-\mu_0)) = 0$ for every μ_0 such that $0 < |\mu_0| < \epsilon$. Let's set $\mu = \lambda - \mu_0$, and we have $T^0 - \mu = (\lambda - \mu) \oplus (A - \mu)$ is upper semi-Fredholm operator, $\mathrm{ind}(T^0 - \mu) = \mathrm{ind}(A - \mu)$ and $\alpha(T^0 - \mu) = 0$.

Suppose the contrary, $\lambda \notin \liminf_{n\to\infty} \sigma(T_n^0)$. Then, there exists a $\delta > 0$, a neighborhood $\mathcal{D}_\delta(\lambda)$ of λ and a subsequence $\{T_{n_k}^0\}$ of $\{T_n^0\}$ such that $\sigma(T_{n_k}^0) \cap \mathcal{D}_\delta(\lambda) = \emptyset$ for every $k \geq 1$. This implies that $T_{n_k}^0 - \mu$ is a Fredholm operator and $\mathrm{ind}(T_{n_k}^0 - \mu) = 0$ for every $\mu \in \mathcal{D}_\delta(\lambda)$ and

$$\lim_{n\to\infty} \|(T_{n_k}^0 - \mu) - (T^0 - \mu)\| = 0.$$

It follows from the continuity of the index that $\mathrm{ind}(T^0 - \mu) = 0$ and $T^0 - \mu$ is a Fredholm operator. Since $\alpha(T^0 - \mu) = 0$, $\mu \notin \sigma(T^0)$ for every μ in a ϵ-neighborhood of λ. This contradicts Lemma 6.2, therefore we must have $\lambda \in \liminf_{n\to\infty} \sigma(T_n^0)$. □

Corollary 6.4. *The spectrum σ_w is continuous on the set of a k-quasi class $\mathcal{A}_n^*$ for a positive integer k.*

Proof. Since Weyl's theorem holds for k-quasi class $\mathcal{A}_n^*$ operators, then σ_w is continuous from Theorem 6.3 and [12, Theorem 2.1]. □

Corollary 6.5. *The spectrum σ_b is continuous on the set of a k-quasi class $\mathcal{A}_n^*$ for a positive integer k.*

Proof. Since Weyl's theorem holds for k-quasi class $\mathcal{A}_n^*$ operators, then σ_b is continuous from Theorem 6.3 and [12, Theorem 2.2]. □

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References

[1] P. Aiena, *Semi-Fredholm Operators, Perturbations Theory and Localized SVEP*, Merida, Venezuela, 2007.

[2] P. Aiena, E. Aponte, and E. Bazan, *Weyl type theorems for left and right polaroid operators*, Integral Equations Operator Theory **66** (2010), no. 1, 1–20.

[3] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory **13** (1990), no. 3, 307–315.

[4] T. Ando, *Operators with a norm condition*, Acta Sci. Math. (Szeged), **33** (1972), 169–178.

[5] S. C. Arora and J. K. Thukral, *On a class of operators*, Glas. Math. Ser. III**21(41)** (1986), no. 2, 381–386.

[6] S. K. Berberian, *Note on a theorem of Fuglede and Putnam*, Proc. Amer. Math. Soc. **10** (1959), 175–182.

[7] ———, *Approximate proper vectors*, Proc. Amer. Math. Soc. **13** (1962), 111–114.

[8] A. Brown and C. Pearcy, *Spectra of tensor products of operators*, Proc. Amer. Math. Soc. **17** (1966), 162–166.

[9] C. A. Mc Carthy, *Cp*, Israel J. Math. **5** (1967), 249–271.

[10] J. B. Conway, *A Course in Functional Analysis*, Second Edition, Springer-Verlag, New York, 1990.

[11] S. V. Djordjević, *Continuity of the essential spectrum in the class of quasihyponormal operators*, Mat. Vesnik **50** (1998), no. 3-4, 71–74.

[12] S. V. Djordjević and D. S. Djordjević, *Weyl's theorems, continuity of the spectrum and quasihyponormal operators*, Acta Sci. Math. (Szeged) **64** (1998), no. 1-2, 259–269.

[13] B. P. Duggal, I. H. Jeon, and I. H. Kim, *On ∗-paranormal contractions and properties for ∗-class $\mathcal{A}$ operators*, Linear Algebra Appl. **436** (2012), no. 5, 954–962.

[14] ———, *Continuity of the spectrum on a class of upper triangular operator matrices*, J. Math. Anal. Appl. **370** (2010), no. 2, 584–587.

[15] B. Fuglede, *A Commutativity theorem for normal operator*, Proc. Natl. Acad. Sci. USA **36** (1950), 35–40.

[16] T. Furuta, *On the class of paranormal operators*, Proc. Japan Acad. **43** (1967), 594–598.

[17] ———, *An extension of the Fuglede-Putnam theorem to subnormal operators using a Hilbert-Schmidt norm inequality*, Proc. Amer. Math. Soc. **81** (1981), no. 2, 240–242.

[18] T. Furuta, M. Ito, and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math. **1** (1998), no. 3, 389–403.

[19] F. Gao and X. C. Fang, *Generalized Weyl's theorem and spectral continuity for quasi-class $(\mathcal{A}, k)$ operators*, Acta Sci. Math. (Szeged) **78** (2012), no. 1-2, 241–250.

[20] F. Gao and X. Li, *Tensor products and the spectral continuity for k-quasi-∗-class $\mathcal{A}$ Operators*, Banach J. Math. Anal. **8** (2014), no. 1, 47–54.

[21] P. R. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag, New York, 1982.

[22] I. S. Hwang and W. Y. Lee, *The spectrum is continuous on the set of p-hyponormal operators*, Math. Z. **235** (2000), no. 1, 151–157.

[23] I. Hoxha and N. L. Braha, *On k-quasi class $\mathcal{A}_n^*$ operators*, Bull. Math. Anal. Appl. **6** (2014), no. 1, 23–33.

[24] I. H. Kim, *Weyl's theorem and tensor product for operators satisfying* $T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k$, J. Korean Math. Soc. **47** (2010), no. 2, 351–361.
[25] R. Lange and S. Wang, *New Approaches in Spectral Decomposition*, Contemp. Math. **128**, Amer. Math. Society, 1992.
[26] K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, The Clarendon Press, Oxford University Press, New York, 2000.
[27] W. Y. Lee and S. H. Lee, *A spectral mapping theorem for the Weyl spectrum*, Glasgow Math. J. **38** (1996), no. 1, 61–64.
[28] S. Mecheri and A. Uchiyama, *An extension of the Fuglede-Putnam's theorem to class $\mathcal{A}$ operators*, Math. Inequal. Appl. **13** (2010), no. 1, 57–61.
[29] J. D. Newburgh, *The variation of Spectra*, Duke Math. J. **18** (1951), 165–176.
[30] S. Panayappan, N. Jayanthi, and D. Sumathi, *Weyl's theorem and tensor product for class $\mathcal{A}_k$ operators*, Pure Mathematical Sciences **1** (2012), no. 1, 13–23.
[31] ———, *Weyl's theorem and tensor product for quasi class $\mathcal{A}_k$ operators*, Pure Mathematical Sciences **1** (2012), no. 1, 33–41.
[32] C. R. Putnam, *On normal operators in Hilbert space*, Amer. J. Math. **73** (1951), 357–362.
[33] V. Rakočević, *Approximate point spectrum and commuting compact perturbations*, Glasgow Math. J. **28** (1986), no. 2, 193–198.
[34] M. H. M. Rashid and M. S. M. Noorani, *On relaxation normality in the Fuglede-Putnam's theorem for a quasi-class $\mathcal{A}$ operators*, Tamkang. J. Math. **40** (2009), no. 3, 307–312.
[35] M. Rosenblum, *On the operator equation $BX - XA = Q$*, Duke Math. J. **23** (1956), 263–269.
[36] C. S. Ryoo and P. Y. Sik, *k^*-paranormal operators*, Pusan Kyongnam Math. J. **11** (1995), no. 2, 243–248.
[37] T. Saito, *Hyponormal Operators and Related Topics*, Lecture notes in Mathematics, **247**, Springer-Verlag, 1971.
[38] S. Sanchez-Perales and V. A. Cruz-Barriguete, *Continuity of approximate point spectrum on the algebra $B(X)$*, Commun. Korean Math. Soc. **28** (2013), no. 3, 487–500.
[39] J. L. Shen, F. Zuo, and C. S. Yang, *On operators satisfying* $T^*|T^2|T \geq T^*|T^*|^2T$, Acta Math. Sin. (Engl. Ser.) **26** (2010), no. 11, 2109–2116.
[40] J. Stochel, *Seminormality of operators from their tensor products*, Proc. Amer. Math. **124** (1996), no. 1, 435–440.
[41] A. Uchiyama and K. Tanahashi, *Fuglede-Putnam's theorem for p-hyponormal or log-hyponormal operators*, Glasgow Math. J. **44** (2002), no. 3, 397–410.
[42] J. Yuan and Z. Gao, *Weyl spectrum of class $A(n)$ and n-paranormal operators*, Integr. Equ. Oper. Theory **60** (2008), no. 2, 289–298.
[43] Q. Zeng and H. Zhong, *On (n, k)-quasi-$*$-paranormal operators*, arXiv 1209.5050v1 [math. FA], 2012.

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