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IDENTITIES OF SYMMETRY FOR THE HIGHER ORDER q-BERNOULLI POLYNOMIALS

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Abstract. The study of the identities of symmetry for the Bernoulli polynomials arises from the study of Gauss's multiplication formula for the gamma function. There are many works in this direction. In the sense of p-adic analysis, the q-Bernoulli polynomials are natural extensions of the Bernoulli and Apostol-Bernoulli polynomials (see the introduction of this paper). By using the N-fold iterated Volkenborn integral, we derive serval identities of symmetry related to the q-extension power sums and the higher order q-Bernoulli polynomials. Many previous results are special cases of the results presented in this paper, including Tuenter's classical results on the symmetry relation between the power sum polynomials and the Bernoulli numbers in [A symmetry of power sum polynomials and Bernoulli numbers, Amer. Math. Monthly 108 (2001), no. 3, 258–261] and D. S. Kim's eight basic identities of symmetry in three variables related to the q-analogue power sums and the q-Bernoulli polynomials in [Identities of symmetry for q-Bernoulli polynomials, Comput. Math. Appl. 60 (2010), no. 8, 2350–2359].

1. Introduction

The Bernoulli numbers B_m and the Bernoulli polynomials $B_m(x)$ may be defined by the exponential generating functions

$$\frac{t}{e^t-1}=\sum_{m=0}^{\infty}B_m\frac{t^m}{m!}\quad(|t|<2\pi) \tag{1.1}$$

and

$$\left(\frac{t}{e^t-1}\right)e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}\quad(|t|<2\pi). \tag{1.2}$$

The Bernoulli numbers and polynomials were first introduced by J. Bernoulli to express the power sums, that is, for

$$k\in\mathbb{N}=\{0,1,2,\ldots\}\text{ and }n\in\mathbb{Z}^+=\{1,2,3,\ldots\},$$

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we set $S_k(n) = \sum_{r=0}^{n-1} r^k$, then

$$S_k(n) = \frac{B_{k+1}(n) - B_{k+1}}{k+1}. \tag{1.3}$$

For details, we refer to Zhi-Wei Sun's lecture [29].

The exponential generating function for $S_k(n)$ is given by

$$\sum_{k=0}^{\infty} S_k(n)\frac{t^k}{k!} = \sum_{i=0}^{n} e^{it} = \frac{1-e^{(n+1)t}}{1-e^t}. \tag{1.4}$$

The study of the identities of symmetry for Bernoulli polynomials arises from the study of Gauss's multiplication formula for the gamma function. Legendre's duplication formula

$$B_m = \frac{1}{2(1-2^m)} \sum_{j=0}^{m-1} 2^j \binom{m}{j} B_j \tag{1.5}$$

and the triplication formula

$$B_m = \frac{1}{3(1-3^m)} \sum_{j=0}^{m-1} 3^j \binom{m}{j} (1+2^{m-j}) B_j \tag{1.6}$$

are special cases of Gauss's multiplication formula (see [22]). Namias [22] conjectured that infinitely many recurrence relations can be obtained at first from Gauss's multiplication formula. Subsequently, Belinfante [3] posed a problem in the MONTHLY for a generalization of (1.5) and (1.6) as follows:

$$B_m = \frac{1}{a(1-a^m)} \sum_{j=0}^{m-1} a^j \binom{m}{j} B_j \sum_{i=1}^{a-1} i^{m-j}, \quad a \geq 2. \tag{1.7}$$

This has been proved by Belinfante and Gessel [4], Deeba and Rodriguez [6], Howard [7]. In fact, this recurrence (1.7) is a consequence of the following symmetry relation between the power sum polynomials and the Bernoulli numbers which was given by Tuenter in [32].

Theorem 1.1 (Tuenter)**.** *For every pair of positive integers a and b, and all nonnegative integers m*

$$\sum_{j=0}^{m} \binom{m}{j} a^{j-1} B_j b^{m-j} S_{m-j}(a-1) = \sum_{j=0}^{m} \binom{m}{j} b^{j-1} B_j a^{m-j} S_{m-j}(b-1). \tag{1.8}$$

Tao and Sun [31] generalized the above identities of symmetry (1.8) to the Bernoulli polynomials

$$\sum_{j=0}^{m} \binom{m}{j} a^{j-1} B_j(bx) b^{m-j} S_{m-j}(a-1) = \sum_{j=0}^{m} \binom{m}{j} b^{j-1} B_j(ax) a^{m-j} S_{m-j}(b-1).$$

Young [35] generalized the above identities of symmetry (1.8) to the degenerate Bernoulli polynomials. Yang [34] generalized the above identities of symmetry (1.8) to the higher order Bernoulli polynomials.

The p-adic integrals are powerful tools for studying the symmetry properties of Bernoulli and Euler polynomials. In p-adic analysis, the Bernoulli and Euler polynomials can be represented by Volkenborn and fermionic p-adic integrals (see the formulas (1.14) below). So we can obtain a lot more identities of symmetry of the Bernoulli and Euler polynomials by considering certain symmetric properties of p-adic integrals. For example, T. Kim [15] derived results on the Bernoulli polynomials and the sums of integer powers expressions on $\mathbb{Z}_p$ and he also obtained several results on the Euler polynomials and the alternating sums of integer powers by considering certain symmetry of fermionic p-adic integral expressions on $\mathbb{Z}_p$. Following the idea of [15], D. S. Kim and Park [10] got many new symmetry identities in three variables related to Euler polynomials and alternating power sums. The derivations of these identities are based on the p-adic integral expression of the generating function for the Euler polynomials and on the quotient of certain p-adic integrals that can be expressed as the exponential generating function for the alternating power sums. By using the fermionic p-adic integral, D. S. Kim, Lee, Na and Park [9] derived several basic identities of symmetry in three variables related to the alternating power sums and the Euler polynomials.

The Apostol-Bernoulli polynomials $B_m(x, \lambda)$ are natural generalizations of the Bernoulli polynomials, they were first introduced by Apostol [1] in order to study the Lipschitz-Lerch zeta functions. Their definitions are as follows,

$$\left(\frac{t}{\lambda e^t - 1}\right) e^{xt} = \sum_{m=0}^{\infty} B_m(x, \lambda) \frac{t^m}{m!}, \tag{1.9}$$

where $|t| \leq 2\pi$ when $\lambda = 1$; $|t| \leq |\log \lambda|$ when $\lambda \neq 1$ (see [19, 21]).

In particular, $B_m(\lambda) = B_m(0, \lambda)$ are the Apostol-Bernoulli numbers. Luo and Srivastava [21] generalized this definition to obtain the generalized Apostol-Bernoulli and Euler polynomials, and they also studied them systematically. Recently, Luo [20], Bayad [2], Navas, Francisco and Varona [23] investigated Fourier expansions for the Apostol-Bernoulli and Apostol-Euler polynomials. Kim and Hu [12] obtained the sums of product identity for the Apostol-Bernoulli numbers which is an analogue of the classical sums of product identity for the Bernoulli numbers dating back to Euler.

Letting $\lambda = 1$ and $x = 0$ in (1.9), we obtain the classical Bernoulli polynomials and numbers $B_m(x)$, B_m, respectively.

The higher order Apostol-Bernoulli polynomials, denoted $B_m^{(N)}(x, \lambda)$, are defined as follows:

$$\left(\frac{t}{\lambda e^t - 1}\right)^N e^{xt} = \sum_{m=0}^{\infty} B_m^{(N)}(x, \lambda) \frac{t^m}{m!}, \tag{1.10}$$

where $|t| \leq 2\pi$ when $\lambda = 1$; $|t| \leq |\log \lambda|$ when $\lambda \neq 1$ (see [21]). When $N = 1$, $B_m^{(1)}(x, \lambda)$ becomes the Apostol-Bernoulli polynomial $B_m(x, \lambda)$ defined above.

The Apostol-Bernoulli polynomials can also be represented by p-adic integrals. Before introducing this, we need some notations.

Let p be a fixed odd prime number. Let $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ be the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $|\cdot|_p$ be the p-adic valuation on $\mathbb{Q}$ with $|p|_p = p^{-1}$. The extended valuation on $\mathbb{C}_p$ is denoted by the same symbol $|\cdot|_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly (or strictly) differentiable function on $\mathbb{Z}_p$. Then the Volkenborn integral of f is defined by

$$\int_{\mathbb{Z}_p} f(z)d\mu(z) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{a=0}^{p^N-1} f(a) \tag{1.11}$$

and this limit always exists when $f \in UD(\mathbb{Z}_p)$ (see [5] and [26, Section 55]). For such functions we have

$$\int_{\mathbb{Z}_p} f(z+1)d\mu(z) - \int_{\mathbb{Z}_p} f(z)d\mu(z) = f'(0), \tag{1.12}$$

where $f'(0) = (\mathrm{d}f(z)/\mathrm{d}z)|_{z=0}$. Also, T. Kim [13] defined the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{N\to\infty} \sum_{a=0}^{p^N-1} f(a)(-1)^a = \int_{\mathbb{Z}_p} f(z)d\mu_{-1}(z). \tag{1.13}$$

It is interesting that

$$\int_{\mathbb{Z}_p} (x+a)^n d\mu(a) = B_n(x) \text{ and } \int_{\mathbb{Z}_p} (x+a)^n d\mu_{-1}(a) = E_n(x), \tag{1.14}$$

where $B_n(x)$ and $E_n(x)$ are the Bernoulli and Euler polynomials, respectively, for details of the above formulas, we refer to [11, 13, 15, 17, 25]. The multiple Volkenborn integrals considered in this paper are all computable by iterated integrations.

The N-fold iterated Volkenborn integral

$$\int_{\mathbb{Z}_p^N} F(z_1, \ldots, z_N)d\mu(z_1)\cdots d\mu(z_N) \tag{1.15}$$

is denoted by

$$\int_{\mathbb{Z}_p^N} F(\overline{z})d\mu(\overline{z}), \quad \text{where } \overline{z} = (z_1, \ldots, z_N). \tag{1.16}$$

Notice that, interchanging the Volkenborn integral and infinite sum is allowed by Proposition 55.4 in [26].

Let C_{p^n} be the cyclic group consisting of all p^nth roots of unity in $\mathbb{C}_p$ for each $n \geq 0$ and $\mathbb{T}_p$ be the direct limit of C_{p^n} with respect to the natural homomorphisms, i.e.,

$$\mathbb{T}_p = \{\lambda \in \mathbb{C}_p \mid \lambda^{p^n} = 1 \text{ for some } n \geq 0\}.$$

Hence $\mathbb{T}_p$ is the union of all C_{p^n} with discrete topology (see [17, 26]).

For each $\lambda \in \mathbb{T}_p$, by Proposition 7.1 (1) in [26], $\log \lambda = 0$, thus for $t \in D$, $x \in \mathbb{Z}_p$ and a positive integer ω, comparing with (1.10), we have

$$\int_{\mathbb{Z}_p^N} \lambda^{\omega(z_1+\cdots+z_N)} e^{(z_1+\cdots+z_N+x)t} d\mu(\overline{z}) = \left(\frac{\omega \log \lambda + t}{\lambda^\omega e^t - 1}\right)^N e^{xt} = \left(\frac{t}{\lambda^\omega e^t - 1}\right)^N e^{xt} = \sum_{m=0}^{\infty} B_m^{(N)}(x, \lambda^\omega) \frac{t^m}{m!}, \tag{1.17}$$

which is the p-adic integral representation of the higher order Apostol-Bernoulli polynomials.

Now we go to a more general case, that is, denote D by

$$D = \left\{t \in \mathbb{C}_p \mid |t|_p < p^{-\frac{1}{p-1}}\right\}, \tag{1.18}$$

we extend the value of λ from $\mathbb{T}_p$ to D in the above formula, and get an extension of the higher order Apostol-Bernoulli polynomials in the p-adic sense, so called q-Bernoulli polynomials.

The detail of this process is as follows:

Assume that $q, t \in \mathbb{C}_p$, with $q-1, t \in D$, so that $q^z = \exp(z \log q)$ and e^{zt} are, as functions of z, analytic functions on $\mathbb{Z}_p$. By applying N-fold iterated Volkenborn integral (1.15) to F, with $F(z) = q^{\omega(z_1+\cdots+z_N)} e^{(z_1+\cdots+z_N+x)t}$, we get the p-adic definition for the higher order q-Bernoulli polynomials $B_m^{(N)}(x, q^\omega)$ as follows

$$\int_{\mathbb{Z}_p^N} q^{\omega(z_1+\cdots+z_N)} e^{(z_1+\cdots+z_N+x)t} d\mu(\overline{z}) = \left(\frac{\omega \log q + t}{q^\omega e^t - 1}\right)^N e^{xt} = \sum_{m=0}^{\infty} B_m^{(N)}(x, q^\omega) \frac{t^m}{m!} \tag{1.19}$$

and

$$B_m^{(N)}(x, q^\omega) = \int_{\mathbb{Z}_p^N} q^{\omega(z_1+\cdots+z_N)} (z_1 + \cdots + z_N + x)^m d\mu(\overline{z}), \tag{1.20}$$

where $d\mu(\overline{z}) = d\mu(z_1) \cdots d\mu(z_N)$ (see [8, 14, 15]). Notice that

$$\lim_{q \to 1} B_m^{(N)}(x, q^\omega) = B_m^{(N)}(x),$$

where $B_m^{(N)}(x)$ is the mth higher order Bernoulli polynomials. And if we restrict the values of q in $\mathbb{T}_p$, $B_m^{(N)}(x,q^\omega)$ becomes the mth higher order Apostol-Bernoulli polynomials as in (1.17). Thus, the study of the symmetry properties of Bernoulli and Aopstol polynomials can be involved in the study of the symmetry properties of q-Bernoulli polynomials. The higher order q-Bernoull numbers and polynomials including their many applications in number theory, mathematical analysis and statistics have been extensively studied by several authors (see [9, 14, 24, 27, 28, 30, 33, 34]). Recently, D. S. Kim [8] derived eight basic identities of symmetry in three variables related to the q-extension power sums and the q-Bernoulli polynomials, thus generalized Tuenter's classical identity (1.8) in various forms.

As in Tuenter's Theorem (Theorem 1.4 above), in order to obtain the identities of symmetry related to the power sums and the q-Bernoulli polynomials, we need the following q-extension of the sum of integer powers. The q-extension of the sum of integer powers $S_{k,q}(n)$ is defined by

$$S_{k,q}(n)=\sum_{i=0}^{n} i^k q^i = 0^k q^0 + 1^k q^1 + \cdots + n^k q^n, \tag{1.21}$$

where $0^0 := 1$. When $q\to 1$, $S_{k,q}(n)$ becomes the sum of integer powers $S_k(n) = \sum_{i=0}^{n} i^k$ (see (1.4)). The exponential generating function for $S_{k,q}(n)$ is given by

$$\sum_{k=0}^{\infty} S_{k,q}(n)\frac{t^k}{k!} = \sum_{i=0}^{n} q^i e^{it} = \frac{1-q^{n+1}e^{(n+1)t}}{1-qe^t}. \tag{1.22}$$

In particular, we have

$$S_{0,q}(n)=\sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q}, \quad S_{k,q}(0)=\begin{cases}1, & \text{for } k=0\\ 0, & \text{for } k\in\mathbb{Z}^+.\end{cases} \tag{1.23}$$

Let ω be any positive integer. By using (1.20) (when $N=1$) and (1.22), we see

$$\frac{\omega\int_{\mathbb{Z}_p} q^z e^{zt}d\mu(z)}{\int_{\mathbb{Z}_p} q^{\omega z}e^{\omega zt}d\mu(z)} = \sum_{l=0}^{\omega-1} q^l e^{lt} = \sum_{k=0}^{\infty} S_{k,q}(\omega-1)\frac{t^k}{k!} \quad (q-1, t\in D) \tag{1.24}$$

(see [8, 15]).

In this paper, by using the N-fold iterated Volkenborn integral, we derive several identities of symmetry related to the q-extension power sums and the higher order q-Bernoulli polynomials. It should be noted that Tuenter's [32] and other authors' results on identities of symmetry [8, 14, 15, 16, 18, 33, 34] are all special cases of the identities given in this paper when $N\in\mathbb{Z}^+$ or $q\to 1$ (see Remark 6.4).

Our paper is organized as follows:

In Section 2, we prove four kinds of quotient type identities for the N-fold iterated Volkenborn integral.

In Sections 3–6, we shall generalize D. S. Kim's idea in [8] to prove the corresponding identities of symmetry in three variables related to the q-extension power sums and the higher order q-Bernoulli polynomials by using each kind of quotient type identities for the N-fold iterated Volkenborn integral given in Section 2, respectively.

2. $I^{(N)}(\Lambda_{23}^i)$-type for $i = 0, 1, 2, 3$

Let ω_1, ω_2 and ω_3 be positive integers. In this section, we will introduce four kinds of quotient type identities for the N-fold iterated Volkenborn integrals on $\mathbb{Z}_p$ or $\mathbb{Z}_p^N$ by building symmetries of ω_1, ω_2, ω_3 from three fundamental identities.

From the definition (1.11) of the Volkenborn integral on $\mathbb{Z}_p$, we have

$$\begin{aligned}\frac{\omega\int_{\mathbb{Z}_p} q^{\omega_1\omega_2 z} e^{\omega_1\omega_2 zt} d\mu(z)}{\int_{\mathbb{Z}_p} q^{\omega_1\omega_2\omega z} e^{\omega_1\omega_2\omega zt} d\mu(z)} &= \sum_{l=0}^{\omega-1} q^{\omega_1\omega_2 l} e^{\omega_1\omega_2 lt} \\ &= \sum_{k=0}^{\infty} S_{k,q^{\omega_1\omega_2}}(\omega-1)\frac{(\omega_1\omega_2 t)^k}{k!}.\end{aligned} \tag{2.1}$$

The p-adic integral expression for the generating function of higher order q-Bernoulli numbers $B_m^{(N)}(q^\omega)$ can be derived from the expansion as

$$\begin{aligned} I_q^{(N)}(\omega, t) &= \int_{\mathbb{Z}_p^N} q^{\omega(z_1+\cdots+z_N)} e^{(z_1+\cdots+z_N)t} d\mu(\overline{z}) \\ &= \left(\frac{\omega\log q + t}{q^\omega e^t - 1}\right)^N \\ &= \sum_{m=0}^{\infty} B_m^{(N)}(q^\omega)\frac{t^m}{m!}.\end{aligned} \tag{2.2}$$

Then we get the following p-adic integral expression for the generating function of higher order q-Bernoulli polynomials $B_m^{(N)}(x, q^\omega)$ as follows (see (1.19)):

$$I_q^{(N)}(\omega, x, t) = I_q^{(N)}(\omega, t)e^{xt} = \sum_{m=0}^{\infty} B_m^{(N)}(x, q^\omega)\frac{t^m}{m!}. \tag{2.3}$$

From (2.1), (2.2), and (2.3), we have the four kinds of identity symmetry in $\omega_1, \omega_2, \omega_3$ as follows:

$$\begin{aligned} I^{(N)}(\Lambda_{23}^i) &= I_q^{(N)}(\omega_2\omega_3, \omega_2\omega_3 t) \\ &\quad\times I_q^{(N)}(\omega_1\omega_3, \omega_1\omega_3 t) \\ &\quad\times I_q^{(N)}(\omega_1\omega_2, \omega_1\omega_2 t) \\ &\quad\times \frac{e^{\omega_1\omega_2\omega_3\left(\sum_{j=1}^3 x_j\right)t}}{\left(\int_{\mathbb{Z}_p} q^{\omega_1\omega_2\omega_3 z} e^{\omega_1\omega_2\omega_3 zt} d\mu(z)\right)^i},\end{aligned} \tag{2.4}$$

which is equivalent to

$$(2.5)\quad I^{(N)}(\Lambda_{23}^{i}) = \frac{(\omega_1\omega_2\omega_3)^{2N-i}(\log q+t)^{3N-i}e^{\omega_1\omega_2\omega_3(\sum_{j=1}^{3}x_j)t}(q^{\omega_1\omega_2\omega_3}e^{\omega_1\omega_2\omega_3 t}-1)^i}{(q^{\omega_2\omega_3}e^{\omega_2\omega_3 t}-1)^N(q^{\omega_1\omega_3}e^{\omega_1\omega_3 t}-1)^N(q^{\omega_1\omega_2}e^{\omega_1\omega_2 t}-1)^N}.$$

Remark 2.1. From (2.3) and (2.4), we see that the expressions $I^{(N)}(\Lambda_{23}^{i})$, for each $i=0,1,2,3$, are invariant under any permutation of $\omega_1,\omega_2,\omega_3$. So, we can obtain identities of symmetry in three variables $\omega_1,\omega_2,\omega_3$ related to the q-extension power sums and the higher order q-Bernoulli polynomials.

2.1. $I^{(N)}(\Lambda_{23}^{0})$-type

From (2.2), (2.3), and (2.4), we have the $I^{(N)}(\Lambda_{23}^{0})$-type identity for the higher order q-Bernoulli polynomials as follows:

$$\text{(a)}\quad \begin{aligned} I^{(N)}(\Lambda_{23}^{0}) &= I_q^{(N)}(\omega_2\omega_3,\omega_2\omega_3 t)e^{\omega_1x_1\cdot\omega_2\omega_3 t} \\ &\quad\times I_q^{(N)}(\omega_1\omega_3,\omega_1\omega_3 t)e^{\omega_2x_2\cdot\omega_1\omega_3 t} \\ &\quad\times I_q^{(N)}(\omega_1\omega_2,\omega_1\omega_2 t)e^{\omega_3x_3\cdot\omega_1\omega_2 t} \\ &= I_q^{(N)}(\omega_2\omega_3,\omega_1x_1,\omega_2\omega_3 t) \\ &\quad\times I_q^{(N)}(\omega_1\omega_3,\omega_2x_2,\omega_1\omega_3 t) \\ &\quad\times I_q^{(N)}(\omega_1\omega_2,\omega_3x_3,\omega_1\omega_2 t). \end{aligned}$$

2.2. $I^{(N)}(\Lambda_{23}^{1})$-type

From (2.4), we have the $I^{(N)}(\Lambda_{23}^{1})$-type identity for the higher order q-Bernoulli polynomials as follows:

$$\text{(b)}\quad I^{(N)}(\Lambda_{23}^{1}) = \frac{I^{(N)}(\Lambda_{23}^{0})}{\int_{\mathbb{Z}_p} q^{\omega_1\omega_2\omega_3 z}e^{\omega_1\omega_2\omega_3 zt}d\mu(z)}.$$

From (2.3), we also have

$$(2.6)\quad \frac{I_q^{(N)}(\omega_1\omega_2,\omega_3x_3,\omega_1\omega_2 t)}{\int_{\mathbb{Z}_p} q^{\omega_1\omega_2 z}e^{\omega_1\omega_2 zt}d\mu(z)} = I_q^{(N-1)}(\omega_1\omega_2,\omega_3x_3,\omega_1\omega_2 t).$$

Thus combining (2.1), (2.2) with (b), we can write $I^{(N)}(\Lambda_{23}^{1})$ in following two different ways:

$$\text{(b-1)}\quad \begin{aligned} I^{(N)}(\Lambda_{23}^{1}) &= I_q^{(N)}(\omega_2\omega_3,\omega_1x_1,\omega_2\omega_3 t) \\ &\quad\times I_q^{(N)}(\omega_1\omega_3,\omega_2x_2,\omega_1\omega_3 t) \\ &\quad\times I_q^{(N-1)}(\omega_1\omega_2,\omega_3x_3,\omega_1\omega_2 t) \\ &\quad\times \frac{1}{\omega_3}\sum_{l=0}^{\infty} S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\frac{(\omega_1\omega_2 t)^l}{l!} \end{aligned}$$

and

$$\text{(b-2)} \qquad \begin{aligned} I^{(N)}(\Lambda_{23}^1) = {} & I_q^{(N)}(\omega_2\omega_3, \omega_1 x_1, \omega_2\omega_3 t) \\ & \times I_q^{(N)}(\omega_1\omega_3, \omega_2 x_2, \omega_1\omega_3 t) \\ & \times I_q^{(N-1)}(\omega_1\omega_2, \omega_3 x_3, \omega_1\omega_2 t) \\ & \times \frac{1}{\omega_3}\sum_{l=0}^{\omega_3-1} q^{\omega_1\omega_2 l} e^{\omega_1\omega_2 t l}. \end{aligned}$$

2.3. $I^{(N)}(\Lambda_{23}^2)$-type

From (2.4), we have the $I^{(N)}(\Lambda_{23}^2)$-type identity for the higher order q-Bernoulli polynomials as follows:

$$\text{(c)} \qquad I^{(N)}(\Lambda_{23}^2) = \frac{I^{(N)}(\Lambda_{23}^0)}{\left(\int_{\mathbb{Z}_p} q^{\omega_1\omega_2\omega_3 z} e^{\omega_1\omega_2\omega_3 z t} d\mu(z)\right)^2}.$$

Form (2.1), (2.6), (a), and (c), $I^{(N)}(\Lambda_{23}^2)$ can be written in the following three different ways:

$$\text{(c-1)} \qquad \begin{aligned} I^{(N)}(\Lambda_{23}^2) = {} & I_q^{(N)}(\omega_2\omega_3, \omega_1 x_1, \omega_2\omega_3 t) \\ & \times I_q^{(N-1)}(\omega_1\omega_3, \omega_2 x_2, \omega_1\omega_3 t) \\ & \times \frac{1}{\omega_2}\sum_{k=0}^{\infty} S_{k,q^{\omega_1\omega_3}}(\omega_2-1)\frac{(\omega_1\omega_3 t)^k}{k!} \\ & \times I_q^{(N-1)}(\omega_1\omega_2, \omega_3 x_3, \omega_1\omega_2 t) \\ & \times \frac{1}{\omega_3}\sum_{m=0}^{\infty} S_{m,q^{\omega_1\omega_2}}(\omega_3-1)\frac{(\omega_1\omega_2 t)^m}{m!}, \end{aligned}$$

$$\text{(c-2)} \qquad \begin{aligned} I^{(N)}(\Lambda_{23}^2) = {} & I_q^{(N)}(\omega_2\omega_3, \omega_1 x_1, \omega_2\omega_3 t) \\ & \times \frac{1}{\omega_2}\sum_{m=0}^{\omega_2-1} q^{\omega_1\omega_3 m} e^{\omega_1\omega_3 t m} \\ & \times I_q^{(N-1)}(\omega_1\omega_3, \omega_2 x_2, \omega_1\omega_3 t) \\ & \times I_q^{(N-1)}(\omega_1\omega_2, \omega_3 x_3, \omega_1\omega_2 t) \\ & \times \frac{1}{\omega_3}\sum_{l=0}^{\infty} S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\frac{(\omega_1\omega_2 t)^l}{l!}, \end{aligned}$$

and

$$\text{(c-3)} \qquad I^{(N)}(\Lambda_{23}^2) = I_q^{(N)}(\omega_2\omega_3, \omega_1 x_1, \omega_2\omega_3 t)$$

$$\times \frac{1}{\omega_2}\sum_{l=0}^{\omega_2-1} q^{\omega_1\omega_3 l} e^{\omega_1\omega_3 tl}$$
$$\times \frac{1}{\omega_3}\sum_{m=0}^{\omega_3-1} q^{\omega_1\omega_2 m} e^{\omega_1\omega_2 tm}$$
$$\times I_q^{(N-1)}(\omega_1\omega_3, \omega_2 x_2, \omega_1\omega_3 t)$$
$$\times I_q^{(N-1)}(\omega_1\omega_2, \omega_3 x_3, \omega_1\omega_2 t).$$

2.4. $I^{(N)}(\Lambda_{23}^3)$-type

From (2.1), (2.4), (a), and (2.6), we have the $I^{(N)}(\Lambda_{23}^3)$-type identity for the higher order q-Bernoulli polynomials as follows:

$$\text{(d)} \qquad \begin{aligned} I^{(N)}(\Lambda_{23}^3) &= \frac{I^{(N)}(\Lambda_{23}^0)}{\left(\int_{\mathbb{Z}_p} q^{\omega_1\omega_2\omega_3 z} e^{\omega_1\omega_2\omega_3 zt} d\mu(z)\right)^3} \\ &= I_q^{(N-1)}(\omega_2\omega_3, \omega_1 x_1, \omega_2\omega_3 t) \\ &\quad \times \frac{1}{\omega_1}\sum_{i=0}^{\infty} S_{i,q^{\omega_2\omega_3}}(\omega_1-1)\frac{(\omega_2\omega_3 t)^i}{i!} \\ &\quad \times I_q^{(N-1)}(\omega_1\omega_3, \omega_2 x_2, \omega_1\omega_3 t) \\ &\quad \times \frac{1}{\omega_2}\sum_{k=0}^{\infty} S_{k,q^{\omega_1\omega_3}}(\omega_2-1)\frac{(\omega_1\omega_3 t)^k}{k!} \\ &\quad \times I_q^{(N)}(\omega_1\omega_2, \omega_3 x_3, \omega_1\omega_2 t) \\ &\quad \times \frac{1}{\omega_3}\sum_{m=0}^{\infty} S_{m,q^{\omega_1\omega_2}}(\omega_3-1)\frac{(\omega_1\omega_2 t)^m}{m!}. \end{aligned}$$

3. Identities of symmetry in $I^{(N)}(\Lambda_{23}^0)$-type

In this section, by using the N-fold iterated Volkenborn integral given in Subsection 2.1, we will prove the corresponding identities of symmetry in three variables $\omega_1, \omega_2, \omega_3$ related to the higher order q-Bernoulli polynomials.

3.1. $I^{(N)}(\Lambda_{23}^0)$-type

From (2.3) and (a), we have

$$\text{(3.1)} \qquad \begin{aligned} I^{(N)}(\Lambda_{23}^0) &= I_q^{(N)}(\omega_2\omega_3, \omega_1 x_1, \omega_2\omega_3 t) \\ &\quad \times I_q^{(N)}(\omega_1\omega_3, \omega_2 x_2, \omega_1\omega_3 t) \\ &\quad \times I_q^{(N)}(\omega_1\omega_2, \omega_3 x_3, \omega_1\omega_2 t) \\ &= \sum_{k=0}^{\infty} B_k^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3})\frac{(\omega_2\omega_3 t)^k}{k!} \end{aligned}$$

$$\times \sum_{l=0}^{\infty} B_l^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) \frac{(\omega_1\omega_3 t)^l}{l!}$$
$$\times \sum_{m=0}^{\infty} B_m^{(N)}(\omega_3 x_3, q^{\omega_1\omega_2}) \frac{(\omega_1\omega_2 t)^m}{m!}.$$

Then by (3.1), we obtain

(3.2)
$$I^{(N)}(\Lambda_{23}^0) = \sum_{n=0}^{\infty} \Bigg(\sum_{k+l+m=n} \binom{n}{k,l,m} B_k^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) B_l^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) \times B_m^{(N)}(\omega_3 x_3, q^{\omega_1\omega_2}) \omega_1^{l+m} \omega_2^{k+m} \omega_3^{k+l} \Bigg) \frac{t^n}{n!},$$

where the inner sum is taken over all nonnegative integers k, l, m with $k+l+m = n$, and
$$\binom{n}{k,l,m} = \frac{n!}{k!l!m!}.$$

For $i = 0, 1, 2, 3$, the $I^{(N)}(\Lambda_{23}^i)$-type for N-fold iterated Volkenborn integrals is invariant under any permutation of $\omega_1, \omega_2, \omega_3$. So the corresponding expressions in (3.2) are also invariant under any permutation of $\omega_1, \omega_2, \omega_3$. Thus our results concerning identities of symmetry immediately follow from this observation.

Theorem 3.1. *Let $\omega_1, \omega_2, \omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1, \omega_2, \omega_3$, and we obtain the following six symmetries*

$$\sum_{k+l+m=n} \binom{n}{k,l,m} B_k^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) B_l^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) B_m^{(N)}(\omega_3 x_3, q^{\omega_1\omega_2}) \times \omega_1^{l+m} \omega_2^{k+m} \omega_3^{k+l}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_k^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) B_l^{(N)}(\omega_3 x_2, q^{\omega_1\omega_2}) B_m^{(N)}(\omega_2 x_3, q^{\omega_1\omega_3}) \times \omega_1^{l+m} \omega_3^{k+m} \omega_2^{k+l}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_k^{(N)}(\omega_2 x_1, q^{\omega_1\omega_3}) B_l^{(N)}(\omega_1 x_2, q^{\omega_2\omega_3}) B_m^{(N)}(\omega_3 x_3, q^{\omega_1\omega_2}) \times \omega_2^{l+m} \omega_1^{k+m} \omega_3^{k+l}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_k^{(N)}(\omega_2 x_1, q^{\omega_1\omega_3}) B_l^{(N)}(\omega_3 x_2, q^{\omega_1\omega_2}) B_m^{(N)}(\omega_1 x_3, q^{\omega_2\omega_3}) \times \omega_2^{l+m} \omega_3^{k+m} \omega_1^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_k^{(N)}(\omega_3 x_1, q^{\omega_1\omega_2}) B_l^{(N)}(\omega_1 x_2, q^{\omega_2\omega_3}) B_m^{(N)}(\omega_2 x_3, q^{\omega_1\omega_3})$$
$$\times \omega_3^{l+m}\omega_1^{k+m}\omega_2^{k+l}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_k^{(N)}(\omega_3 x_1, q^{\omega_1\omega_2}) B_l^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) B_m^{(N)}(\omega_1 x_3, q^{\omega_2\omega_3})$$
$$\times \omega_3^{l+m}\omega_2^{k+m}\omega_1^{k+l}.$$

Remark 3.2. Letting further $N = 1$ in Theorem 3.1, we get Theorem 4.1 in [8].

4. Identities of symmetry in $I^{(N)}(\Lambda_{23}^1)$-type

In this section, by using quotient type identities for the N-fold iterated Volkenborn integral given in Subsection 2.2, we prove the corresponding identities of symmetry in three variables related to the q-extension power sums and the higher order q-Bernoulli polynomials.

4.1. Symmetric identities from (b-1)

From (2.1), (2.3), and (b-1), we have

$$I^{(N)}(\Lambda_{23}^1) = \sum_{i=0}^{\infty} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) \frac{(\omega_2\omega_3 t)^i}{i!} \times \sum_{j=0}^{\infty} B_j^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) \frac{(\omega_1\omega_3 t)^j}{j!} \times \sum_{k=0}^{\infty} B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) \frac{(\omega_1\omega_2 t)^k}{k!} \times \frac{1}{\omega_3}\sum_{l=0}^{\infty} S_{l,q^{\omega_1\omega_2}}(\omega_3 - 1) \frac{(\omega_1\omega_2 t)^l}{l!}. \tag{4.1}$$

Then we obtain

(4.2)
$$I^{(N)}(\Lambda_{23}^1) = \sum_{n=0}^{\infty}\left(\sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) B_j^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) \times B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) S_{l,q^{\omega_1\omega_2}}(\omega_3 - 1)\omega_1^{j+k+l}\omega_2^{i+k+l}\omega_3^{i+j-1}\right)\frac{t^n}{n!},$$

where the inner sum is taken over all nonnegative integers i, j, k, l with $i + j + k + l = n$, and

$$\binom{n}{i,j,k,l} = \frac{n!}{i!j!k!l!}.$$

Theorem 4.1. *Let $\omega_1, \omega_2, \omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1, \omega_2, \omega_3$, and we obtain the following six symmetries*

$$\begin{aligned}
&\sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) B_j^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) \\
&\quad \times B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) S_{l,q^{\omega_1\omega_2}}(\omega_3 - 1)\omega_1^{j+k+l}\omega_2^{i+k+l}\omega_3^{i+j-1} \\
&= \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) B_j^{(N)}(\omega_3 x_2, q^{\omega_1\omega_2}) \\
&\quad \times B_k^{(N-1)}(\omega_2 x_3, q^{\omega_1\omega_3}) S_{l,q^{\omega_1\omega_3}}(\omega_2 - 1)\omega_1^{j+k+l}\omega_3^{i+k+l}\omega_2^{i+j-1} \\
&= \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_2 x_1, q^{\omega_1\omega_3}) B_j^{(N)}(\omega_1 x_2, q^{\omega_2\omega_3}) \\
&\quad \times B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) S_{l,q^{\omega_1\omega_2}}(\omega_3 - 1)\omega_2^{j+k+l}\omega_1^{i+k+l}\omega_3^{i+j-1} \\
&= \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_2 x_1, q^{\omega_1\omega_3}) B_j^{(N)}(\omega_3 x_2, q^{\omega_1\omega_2}) \\
&\quad \times B_k^{(N-1)}(\omega_1 x_3, q^{\omega_2\omega_3}) S_{l,q^{\omega_2\omega_3}}(\omega_1 - 1)\omega_2^{j+k+l}\omega_3^{i+k+l}\omega_1^{i+j-1} \\
&= \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_3 x_1, q^{\omega_1\omega_2}) B_j^{(N)}(\omega_1 x_2, q^{\omega_2\omega_3}) \\
&\quad \times B_k^{(N-1)}(\omega_2 x_3, q^{\omega_1\omega_3}) S_{l,q^{\omega_1\omega_3}}(\omega_2 - 1)\omega_3^{j+k+l}\omega_1^{i+k+l}\omega_2^{i+j-1} \\
&= \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_3 x_1, q^{\omega_1\omega_2}) B_j^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) \\
&\quad \times B_k^{(N-1)}(\omega_1 x_3, q^{\omega_2\omega_3}) S_{l,q^{\omega_2\omega_3}}(\omega_1 - 1)\omega_3^{j+k+l}\omega_2^{i+k+l}\omega_1^{i+j-1}.
\end{aligned}$$

Letting $x_3 = 0$ and $N = 1$ in Theorem 4.1. Since

$$B_k^{(0)}(0, q) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbb{Z}^+, \end{cases} \tag{4.3}$$

we get the following corollary. This has also been obtained in [8, Theorem 4.2].

Corollary 4.2. *Let $\omega_1, \omega_2, \omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1, \omega_2, \omega_3$, and we obtain the following six symmetries*

$$\begin{aligned}
&\sum_{i+j+l=n} \binom{n}{i,j,l} B_i(\omega_1 x_1, q^{\omega_2\omega_3}) B_j(\omega_2 x_2, q^{\omega_1\omega_3}) S_{l,q^{\omega_1\omega_2}}(\omega_3 - 1)\omega_1^{j+l}\omega_2^{i+l}\omega_3^{i+j-1} \\
&= \sum_{i+j+l=n} \binom{n}{i,j,l} B_i(\omega_1 x_1, q^{\omega_2\omega_3}) B_j(\omega_3 x_2, q^{\omega_1\omega_2}) S_{l,q^{\omega_1\omega_3}}(\omega_2 - 1)\omega_1^{j+l}\omega_3^{i+l}\omega_2^{i+j-1}
\end{aligned}$$

$$= \sum_{i+j+l=n} \binom{n}{i,j,l} B_i(\omega_2 x_1, q^{\omega_1\omega_3}) B_j(\omega_1 x_2, q^{\omega_2\omega_3}) S_{l,q^{\omega_1\omega_2}}(\omega_3 - 1)\omega_2^{j+l}\omega_1^{i+l}\omega_3^{i+j-1}$$
$$= \sum_{i+j+l=n} \binom{n}{i,j,l} B_i(\omega_2 x_1, q^{\omega_1\omega_3}) B_j(\omega_3 x_2, q^{\omega_1\omega_2}) S_{l,q^{\omega_2\omega_3}}(\omega_1 - 1)\omega_2^{j+l}\omega_3^{i+l}\omega_1^{i+j-1}$$
$$= \sum_{i+j+l=n} \binom{n}{i,j,l} B_i(\omega_3 x_1, q^{\omega_1\omega_2}) B_j(\omega_1 x_2, q^{\omega_2\omega_3}) S_{l,q^{\omega_1\omega_3}}(\omega_2 - 1)\omega_3^{j+l}\omega_1^{i+l}\omega_2^{i+j-1}$$
$$= \sum_{i+j+l=n} \binom{n}{i,j,l} B_i(\omega_3 x_1, q^{\omega_1\omega_2}) B_j(\omega_2 x_2, q^{\omega_1\omega_3}) S_{l,q^{\omega_2\omega_3}}(\omega_1 - 1)\omega_3^{j+l}\omega_2^{i+l}\omega_1^{i+j-1}.$$

Letting $\omega_3 = 1$ in Corollary 4.2, we get Corollary 4.3 in [8] and $\omega_2 = \omega_3 = 1$ in Corollary 4.2, we get Corollary 4.4 in [8].

4.2. Symmetric identities from (b-2)

From (2.1), (2.3), and (b-2), we have

$$I^{(N)}(\Lambda_{23}^1) = \sum_{i=0}^{\infty} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) \frac{(\omega_2\omega_3 t)^i}{i!} \times \sum_{j=0}^{\infty} B_j^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) \frac{(\omega_1\omega_3 t)^j}{j!} \times \frac{1}{\omega_3} \sum_{l=0}^{\omega_3-1} q^{\omega_1\omega_2 l} e^{\omega_1\omega_2 t l} \times \sum_{k=0}^{\infty} B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) \frac{(\omega_1\omega_2 t)^k}{k!}. \tag{4.4}$$

Here we write $I^{(N)}(\Lambda_{23}^1)$ in two different ways:

(4.5)
$$I^{(N)}(\Lambda_{23}^1) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\omega_3-1} q^{\omega_1\omega_2 l} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) \times B_j^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) B_k^{(N-1)}(\omega_3 x_3 + l, q^{\omega_1\omega_2}) \omega_1^{j+k}\omega_2^{i+k}\omega_3^{i+j-1} \right) \frac{t^n}{n!}$$

and

(4.6)
$$I^{(N)}(\Lambda_{23}^1) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\omega_3-1} q^{\omega_1\omega_2 l} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) \times B_j^{(N)}\left(\omega_2 x_2 + \frac{\omega_2}{\omega_3} l, q^{\omega_1\omega_3}\right) B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) \omega_1^{j+k}\omega_2^{i+k}\omega_3^{i+j-1} \right) \frac{t^n}{n!}.$$

The corresponding expressions in (4.5) and (4.6) are invariant under any permutation of $\omega_1, \omega_2, \omega_3$. Thus our results concerning identities of symmetry immediately follow from this observation.

From (4.5), we obtain the following result.

Theorem 4.3. *Let* $\omega_1, \omega_2, \omega_3$ *be any positive integers. Then the following expression is invariant under any permutation of* $\omega_1, \omega_2, \omega_3$, *and we obtain the following six symmetries*

$$\begin{aligned}
&\sum_{l=0}^{\omega_3-1} q^{\omega_1\omega_2 l} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) \\
&\quad \times B_j^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) B_k^{(N-1)}(\omega_3 x_3 + l, q^{\omega_1\omega_2}) \omega_1^{j+k}\omega_2^{i+k}\omega_3^{i+j-1} \\
=& \sum_{l=0}^{\omega_2-1} q^{\omega_1\omega_3 l} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) \\
&\quad \times B_j^{(N)}(\omega_3 x_2, q^{\omega_1\omega_2}) B_k^{(N-1)}(\omega_2 x_3 + l, q^{\omega_1\omega_3}) \omega_1^{j+k}\omega_3^{i+k}\omega_2^{i+j-1} \\
=& \sum_{l=0}^{\omega_3-1} q^{\omega_1\omega_2 l} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}(\omega_2 x_1, q^{\omega_1\omega_3}) \\
&\quad \times B_j^{(N)}(\omega_1 x_2, q^{\omega_2\omega_3}) B_k^{(N-1)}(\omega_3 x_3 + l, q^{\omega_1\omega_2}) \omega_2^{j+k}\omega_1^{i+k}\omega_3^{i+j-1} \\
=& \sum_{l=0}^{\omega_1-1} q^{\omega_2\omega_3 l} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}(\omega_2 x_1, q^{\omega_1\omega_3}) \\
&\quad \times B_j^{(N)}(\omega_3 x_2, q^{\omega_1\omega_2}) B_k^{(N-1)}(\omega_1 x_3 + l, q^{\omega_2\omega_3}) \omega_2^{j+k}\omega_3^{i+k}\omega_1^{i+j-1} \\
=& \sum_{l=0}^{\omega_2-1} q^{\omega_1\omega_3 l} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}(\omega_3 x_1, q^{\omega_1\omega_2}) \\
&\quad \times B_j^{(N)}(\omega_1 x_2, q^{\omega_2\omega_3}) B_k^{(N-1)}(\omega_2 x_3 + l, q^{\omega_1\omega_3}) \omega_3^{j+k}\omega_1^{i+k}\omega_2^{i+j-1} \\
=& \sum_{l=0}^{\omega_1-1} q^{\omega_2\omega_3 l} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}(\omega_3 x_1, q^{\omega_1\omega_2}) \\
&\quad \times B_j^{(N)}(\omega_2 x_2, q^{\omega_1\omega_3}) B_k^{(N-1)}(\omega_1 x_3 + l, q^{\omega_2\omega_3}) \omega_3^{j+k}\omega_2^{i+k}\omega_1^{i+j-1}.
\end{aligned}$$

From (4.6), we obtain the following result.

Theorem 4.4. *Let* $\omega_1, \omega_2, \omega_3$ *be any positive integers. Then the following expression is invariant under any permutation of* $\omega_1, \omega_2, \omega_3$, *and we obtain the following six symmetries*

$$\sum_{l=0}^{\omega_3-1} q^{\omega_1\omega_2 l} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3})$$

$$\times B_j^{(N)}\left(\omega_2 x_2+\frac{\omega_2}{\omega_3}l, q^{\omega_1\omega_3}\right) B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2})\omega_1^{j+k}\omega_2^{i+k}\omega_3^{i+j-1}$$

$$= \sum_{l=0}^{\omega_2-1} q^{\omega_1\omega_3 l} \sum_{i+j+k=n}\binom{n}{i,j,k} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3})$$

$$\times B_j^{(N)}\left(\omega_3 x_2+\frac{\omega_3}{\omega_2}l, q^{\omega_1\omega_2}\right) B_k^{(N-1)}(\omega_2 x_3, q^{\omega_1\omega_3})\omega_1^{j+k}\omega_3^{i+k}\omega_2^{i+j-1}$$

$$= \sum_{l=0}^{\omega_3-1} q^{\omega_1\omega_2 l} \sum_{i+j+k=n}\binom{n}{i,j,k} B_i^{(N)}(\omega_2 x_1, q^{\omega_1\omega_3})$$

$$\times B_j^{(N)}\left(\omega_1 x_2+\frac{\omega_1}{\omega_3}l, q^{\omega_2\omega_3}\right) B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2})\omega_2^{j+k}\omega_1^{i+k}\omega_3^{i+j-1}$$

$$= \sum_{l=0}^{\omega_1-1} q^{\omega_2\omega_3 l} \sum_{i+j+k=n}\binom{n}{i,j,k} B_i^{(N)}(\omega_2 x_1, q^{\omega_1\omega_3})$$

$$\times B_j^{(N)}\left(\omega_3 x_2+\frac{\omega_3}{\omega_1}l, q^{\omega_1\omega_2}\right) B_k^{(N-1)}(\omega_1 x_3, q^{\omega_2\omega_3})\omega_2^{j+k}\omega_3^{i+k}\omega_1^{i+j-1}$$

$$= \sum_{l=0}^{\omega_2-1} q^{\omega_1\omega_3 l} \sum_{i+j+k=n}\binom{n}{i,j,k} B_i^{(N)}(\omega_3 x_1, q^{\omega_1\omega_2})$$

$$\times B_j^{(N)}\left(\omega_1 x_2+\frac{\omega_1}{\omega_2}l, q^{\omega_2\omega_3}\right) B_k^{(N-1)}(\omega_2 x_3, q^{\omega_1\omega_3})\omega_3^{j+k}\omega_1^{i+k}\omega_2^{i+j-1}$$

$$= \sum_{l=0}^{\omega_1-1} q^{\omega_2\omega_3 l} \sum_{i+j+k=n}\binom{n}{i,j,k} B_i^{(N)}(\omega_3 x_1, q^{\omega_1\omega_2})$$

$$\times B_j^{(N)}\left(\omega_2 x_2+\frac{\omega_2}{\omega_1}l, q^{\omega_1\omega_3}\right) B_k^{(N-1)}(\omega_1 x_3, q^{\omega_2\omega_3})\omega_3^{j+k}\omega_2^{i+k}\omega_1^{i+j-1}.$$

Letting $x_3 = 0$ and $N = 1$ in Theorem 4.4, by (4.3), we get the following corollary. This has also been obtained in [8, Theorem 4.5].

Corollary 4.5. *Let ω_1, ω_2 be any positive integers. Then*

$$\sum_{l=0}^{\omega_3-1} q^{\omega_1\omega_2 l} \sum_{i+j=n}\binom{n}{i,j} B_i(\omega_1 x_1, q^{\omega_2\omega_3}) B_j\left(\omega_2 x_2+\frac{\omega_2}{\omega_3}l, q^{\omega_1\omega_3}\right)\omega_1^{j}\omega_2^{i}\omega_3^{i+j-1}$$

$$= \sum_{l=0}^{\omega_2-1} q^{\omega_1\omega_3 l} \sum_{i+j=n}\binom{n}{i,j} B_i(\omega_1 x_1, q^{\omega_2\omega_3}) B_j\left(\omega_3 x_2+\frac{\omega_3}{\omega_2}l, q^{\omega_1\omega_2}\right)\omega_1^{j}\omega_3^{i}\omega_2^{i+j-1}$$

$$= \sum_{l=0}^{\omega_3-1} q^{\omega_1\omega_2 l} \sum_{i+j=n}\binom{n}{i,j} B_i(\omega_2 x_1, q^{\omega_1\omega_3}) B_j\left(\omega_1 x_2+\frac{\omega_1}{\omega_3}l, q^{\omega_2\omega_3}\right)\omega_2^{j}\omega_1^{i}\omega_3^{i+j-1}$$

$$= \sum_{l=0}^{\omega_1-1} q^{\omega_2\omega_3 l} \sum_{i+j=n} \binom{n}{i,j} B_i(\omega_2 x_1, q^{\omega_1\omega_3}) B_j\left(\omega_3 x_2 + \frac{\omega_3}{\omega_1} l, q^{\omega_1\omega_2}\right) \omega_2^j \omega_3^i \omega_1^{i+j-1}$$

$$= \sum_{l=0}^{\omega_2-1} q^{\omega_1\omega_3 l} \sum_{i+j=n} \binom{n}{i,j} B_i(\omega_3 x_1, q^{\omega_1\omega_2}) B_j\left(\omega_1 x_2 + \frac{\omega_1}{\omega_2} l, q^{\omega_2\omega_3}\right) \omega_3^j \omega_1^i \omega_2^{i+j-1}$$

$$= \sum_{l=0}^{\omega_1-1} q^{\omega_2\omega_3 l} \sum_{i+j=n} \binom{n}{i,j} B_i(\omega_3 x_1, q^{\omega_1\omega_2}) B_j\left(\omega_2 x_2 + \frac{\omega_2}{\omega_1} l, q^{\omega_1\omega_3}\right) \omega_3^j \omega_2^i \omega_1^{i+j-1},$$

where the inner sum is taken over all nonnegative integers i, j with $i + j = n$, and

$$\binom{n}{i,j} = \frac{n!}{i!j!}.$$

5. Identities of symmetry in $I^{(N)}(\Lambda_{23}^2)$-type

In this section, by using quotient type identities for the N-fold iterated Volkenborn integral given in subsection 2.3, we prove the corresponding identities of symmetry in three variables related to the q-extension power sums and the higher order q-Bernoulli polynomials.

5.1. Symmetric identities from (c-1)

From (2.1), (2.3), and (c-1), we have

$$\begin{aligned} I^{(N)}(\Lambda_{23}^2) = &\sum_{i=0}^{\infty} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) \frac{(\omega_2\omega_3 t)^i}{i!} \\ &\times \sum_{j=0}^{\infty} B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3}) \frac{(\omega_1\omega_3 t)^j}{j!} \\ &\times \frac{1}{\omega_2} \sum_{k=0}^{\infty} S_{k,q^{\omega_1\omega_3}}(\omega_2 - 1) \frac{(\omega_1\omega_3 t)^k}{k!} \\ &\times \sum_{l=0}^{\infty} B_l^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) \frac{(\omega_1\omega_2 t)^l}{l!} \\ &\times \frac{1}{\omega_3} \sum_{m=0}^{\infty} S_{m,q^{\omega_1\omega_2}}(\omega_3 - 1) \frac{(\omega_1\omega_2 t)^m}{m!}. \end{aligned} \tag{5.1}$$

Then we obtain

(5.2)

$$\begin{aligned} I^{(N)}(\Lambda_{23}^2) = &\sum_{n=0}^{\infty} \Bigg(\sum_{i+j+k+l+m=n} \binom{n}{i,j,k,l,m} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) \\ &\times B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3}) S_{k,q^{\omega_1\omega_3}}(\omega_2 - 1) B_l^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) \end{aligned}$$

$$\times S_{m,q^{\omega_1\omega_2}}(\omega_3-1)\omega_1^{j+k+l+m}\omega_2^{i+l+m-1}\omega_3^{i+j+k-1}\Bigg)\frac{t^n}{n!},$$

where the inner sum is taken over all nonnegative integers i, j, k, l, m with $i+j+k+l+m=n$, and

$$\binom{n}{i,j,k,l,m}=\frac{n!}{i!j!k!l!m!}.$$

Therefore, we get the following theorem.

Theorem 5.1. *Let $\omega_1, \omega_2, \omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1, \omega_2, \omega_3$, and we obtain the following six symmetries*

$$\begin{aligned}
&\sum_{i+j+k+l+m=n}\binom{n}{i,j,k,l,m}B_i^{(N)}(\omega_1x_1,q^{\omega_2\omega_3})B_j^{(N-1)}(\omega_2x_2,q^{\omega_1\omega_3})S_{k,q^{\omega_1\omega_3}}(\omega_2-1)\\
&\quad\times B_l^{(N-1)}(\omega_3x_3,q^{\omega_1\omega_2})S_{m,q^{\omega_1\omega_2}}(\omega_3-1)\omega_1^{j+k+l+m}\omega_2^{i+l+m-1}\omega_3^{i+j+k-1}\\
=&\sum_{i+j+k+l+m=n}\binom{n}{i,j,k,l,m}B_i^{(N)}(\omega_1x_1,q^{\omega_2\omega_3})B_j^{(N-1)}(\omega_3x_2,q^{\omega_1\omega_2})S_{k,q^{\omega_1\omega_2}}(\omega_3-1)\\
&\quad\times B_l^{(N-1)}(\omega_2x_3,q^{\omega_1\omega_3})S_{m,q^{\omega_1\omega_3}}(\omega_2-1)\omega_1^{j+k+l+m}\omega_3^{i+l+m-1}\omega_2^{i+j+k-1}\\
=&\sum_{i+j+k+l+m=n}\binom{n}{i,j,k,l,m}B_i^{(N)}(\omega_2x_1,q^{\omega_1\omega_3})B_j^{(N-1)}(\omega_1x_2,q^{\omega_2\omega_3})S_{k,q^{\omega_2\omega_3}}(\omega_1-1)\\
&\quad\times B_l^{(N-1)}(\omega_3x_3,q^{\omega_1\omega_2})S_{m,q^{\omega_1\omega_2}}(\omega_3-1)\omega_2^{j+k+l+m}\omega_1^{i+l+m-1}\omega_3^{i+j+k-1}\\
=&\sum_{i+j+k+l+m=n}\binom{n}{i,j,k,l,m}B_i^{(N)}(\omega_2x_1,q^{\omega_1\omega_3})B_j^{(N-1)}(\omega_3x_2,q^{\omega_1\omega_2})S_{k,q^{\omega_1\omega_2}}(\omega_3-1)\\
&\quad\times B_l^{(N-1)}(\omega_1x_3,q^{\omega_2\omega_3})S_{m,q^{\omega_2\omega_3}}(\omega_1-1)\omega_2^{j+k+l+m}\omega_3^{i+l+m-1}\omega_1^{i+j+k-1}\\
=&\sum_{i+j+k+l+m=n}\binom{n}{i,j,k,l,m}B_i^{(N)}(\omega_3x_1,q^{\omega_1\omega_2})B_j^{(N-1)}(\omega_1x_2,q^{\omega_2\omega_3})S_{k,q^{\omega_2\omega_3}}(\omega_1-1)\\
&\quad\times B_l^{(N-1)}(\omega_2x_3,q^{\omega_1\omega_3})S_{m,q^{\omega_1\omega_3}}(\omega_2-1)\omega_3^{j+k+l+m}\omega_1^{i+l+m-1}\omega_2^{i+j+k-1}\\
=&\sum_{i+j+k+l+m=n}\binom{n}{i,j,k,l,m}B_i^{(N)}(\omega_3x_1,q^{\omega_1\omega_2})B_j^{(N-1)}(\omega_2x_2,q^{\omega_1\omega_3})S_{k,q^{\omega_1\omega_3}}(\omega_2-1)\\
&\quad\times B_l^{(N-1)}(\omega_1x_3,q^{\omega_2\omega_3})S_{m,q^{\omega_2\omega_3}}(\omega_1-1)\omega_3^{j+k+l+m}\omega_2^{i+l+m-1}\omega_1^{i+j+k-1}.
\end{aligned}$$

Letting $x_2 = x_3 = 0$ and $N = 1$ in Theorem 5.1, by (4.3), we get the following corollary. This has also been obtained in [8, Theorem 4.8].

Corollary 5.2. *Let $\omega_1, \omega_2, \omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1, \omega_2, \omega_3$, and we obtain the following three symmetries*

$$\sum_{i+k+m=n}\binom{n}{i,k,m}B_i(\omega_1x_1,q^{\omega_2\omega_3})S_{k,q^{\omega_1\omega_3}}(\omega_2-1)S_{m,q^{\omega_1\omega_2}}(\omega_3-1)$$

$$
\begin{aligned}
&\times \omega_1^{k+m}\omega_2^{i+m-1}\omega_3^{i+k-1}\\
&= \sum_{i+k+m=n}\binom{n}{i,k,m}B_i(\omega_2 x_1, q^{\omega_1\omega_3})S_{k,q^{\omega_2\omega_3}}(\omega_1-1)S_{m,q^{\omega_1\omega_2}}(\omega_3-1)\\
&\quad\times \omega_2^{k+m}\omega_1^{i+m-1}\omega_3^{i+k-1}\\
&= \sum_{i+k+m=n}\binom{n}{i,k,m}B_i(\omega_3 x_1, q^{\omega_1\omega_2})S_{k,q^{\omega_2\omega_3}}(\omega_1-1)S_{m,q^{\omega_1\omega_3}}(\omega_2-1)\\
&\quad\times \omega_3^{k+m}\omega_1^{i+m-1}\omega_2^{i+k-1}.
\end{aligned}
$$

Letting $\omega_2=\omega_3=1$ in Corollary 5.2, we have the following corollary. This has also been obtained in [14, (2.22)] and [8, Corollary 4.10].

Corollary 5.3. *Let ω_1 be any positive integer. Then*

$$B_n(\omega_1 x_1, q) = \sum_{i+k=n}\binom{n}{i,k}B_i(x_1, q^{\omega_1})S_{k,q}(\omega_1-1)\omega_1^{i-1}.$$

5.2. Symmetric identities from (c-2)

From (2.1), (2.3), and (c-2), we have

$$
\begin{aligned}
I^{(N)}(\Lambda_{23}^2) &= \sum_{i=0}^{\infty}B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3})\frac{(\omega_2\omega_3 t)^i}{i!}\\
&\times \frac{1}{\omega_2}\sum_{m=0}^{\omega_2-1}q^{\omega_1\omega_3 m}e^{\omega_1\omega_3 t m}\\
&\times \sum_{j=0}^{\infty}B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3})\frac{(\omega_1\omega_3 t)^j}{j!}\\
&\times \sum_{k=0}^{\infty}B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2})\frac{(\omega_1\omega_2 t)^k}{k!}\\
&\times \frac{1}{\omega_3}\sum_{l=0}^{\infty}S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\frac{(\omega_1\omega_2 t)^l}{l!}.
\end{aligned}
\tag{5.3}
$$

Here we write $I^{(N)}(\Lambda_{23}^2)$ in two different ways. From (5.3), we obtain

$$
\begin{aligned}
I^{(N)}(\Lambda_{23}^2) = \sum_{n=0}^{\infty}\Bigg(&\sum_{m=0}^{\omega_2-1}q^{\omega_1\omega_3 m}\sum_{i+j+k+l=n}\binom{n}{i,j,k,l}\\
&\times B_i^{(N)}\left(\omega_1 x_1+\frac{\omega_1}{\omega_2}m, q^{\omega_2\omega_3}\right)\\
&\times B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3})B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2})\\
&\times S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\omega_1^{j+k+l}\omega_2^{i+k+l-1}\omega_3^{i+j-1}\Bigg)\frac{t^n}{n!}
\end{aligned}
\tag{5.4}
$$

and

$$I^{(N)}(\Lambda_{23}^2)=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\omega_2-1}q^{\omega_1\omega_3 m}\sum_{i+j+k+l=n}\binom{n}{i,j,k,l} \times B_i^{(N)}(\omega_1x_1,q^{\omega_2\omega_3})B_j^{(N-1)}(\omega_2x_2,q^{\omega_1\omega_3}) \times B_k^{(N-1)}\left(\omega_3x_3+\frac{\omega_3}{\omega_2}m,q^{\omega_1\omega_2}\right) \times S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\omega_1^{j+k+l}\omega_2^{i+k+l-1}\omega_3^{i+j-1}\Big)\frac{t^n}{n!}, \tag{5.5}$$

where the inner sum is taken over all nonnegative integers i,j,k,l with $i+j+k+l=n$.

From (5.4), we get the following theorem.

Theorem 5.4. *Let $\omega_1,\omega_2,\omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1,\omega_2,\omega_3$, and we obtain the following six symmetries*

$$\begin{aligned}
&\sum_{m=0}^{\omega_2-1}q^{\omega_1\omega_3 m}\sum_{i+j+k+l=n}\binom{n}{i,j,k,l}B_i^{(N)}\left(\omega_1x_1+\frac{\omega_1}{\omega_2}m,q^{\omega_2\omega_3}\right)\\
&\quad\times B_j^{(N-1)}(\omega_2x_2,q^{\omega_1\omega_3})B_k^{(N-1)}(\omega_3x_3,q^{\omega_1\omega_2})S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\omega_1^{j+k+l}\omega_2^{i+k+l-1}\omega_3^{i+j-1}\\
&=\sum_{m=0}^{\omega_3-1}q^{\omega_1\omega_2 m}\sum_{i+j+k+l=n}\binom{n}{i,j,k,l}B_i^{(N)}\left(\omega_1x_1+\frac{\omega_1}{\omega_3}m,q^{\omega_2\omega_3}\right)\\
&\quad\times B_j^{(N-1)}(\omega_3x_2,q^{\omega_1\omega_2})B_k^{(N-1)}(\omega_2x_3,q^{\omega_1\omega_3})S_{l,q^{\omega_1\omega_3}}(\omega_2-1)\omega_1^{j+k+l}\omega_3^{i+k+l-1}\omega_2^{i+j-1}\\
&=\sum_{m=0}^{\omega_1-1}q^{\omega_2\omega_3 m}\sum_{i+j+k+l=n}\binom{n}{i,j,k,l}B_i^{(N)}\left(\omega_2x_1+\frac{\omega_2}{\omega_1}m,q^{\omega_1\omega_3}\right)\\
&\quad\times B_j^{(N-1)}(\omega_1x_2,q^{\omega_2\omega_3})B_k^{(N-1)}(\omega_3x_3,q^{\omega_1\omega_2})S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\omega_2^{j+k+l}\omega_1^{i+k+l-1}\omega_3^{i+j-1}\\
&=\sum_{m=0}^{\omega_3-1}q^{\omega_1\omega_2 m}\sum_{i+j+k+l=n}\binom{n}{i,j,k,l}B_i^{(N)}\left(\omega_2x_1+\frac{\omega_2}{\omega_3}m,q^{\omega_1\omega_3}\right)\\
&\quad\times B_j^{(N-1)}(\omega_3x_2,q^{\omega_1\omega_2})B_k^{(N-1)}(\omega_1x_3,q^{\omega_2\omega_3})S_{l,q^{\omega_2\omega_3}}(\omega_1-1)\omega_2^{j+k+l}\omega_3^{i+k+l-1}\omega_1^{i+j-1}\\
&=\sum_{m=0}^{\omega_1-1}q^{\omega_2\omega_3 m}\sum_{i+j+k+l=n}\binom{n}{i,j,k,l}B_i^{(N)}\left(\omega_3x_1+\frac{\omega_3}{\omega_1}m,q^{\omega_1\omega_2}\right)\\
&\quad\times B_j^{(N-1)}(\omega_1x_2,q^{\omega_2\omega_3})B_k^{(N-1)}(\omega_2x_3,q^{\omega_1\omega_3})S_{l,q^{\omega_1\omega_3}}(\omega_2-1)\omega_3^{j+k+l}\omega_1^{i+k+l-1}\omega_2^{i+j-1}\\
&=\sum_{m=0}^{\omega_2-1}q^{\omega_1\omega_3 m}\sum_{i+j+k+l=n}\binom{n}{i,j,k,l}B_i^{(N)}\left(\omega_3x_1+\frac{\omega_3}{\omega_2}m,q^{\omega_1\omega_2}\right)\\
&\quad\times B_j^{(N-1)}(\omega_2x_2,q^{\omega_1\omega_3})B_k^{(N-1)}(\omega_1x_3,q^{\omega_2\omega_3})S_{l,q^{\omega_2\omega_3}}(\omega_1-1)\omega_3^{j+k+l}\omega_2^{i+k+l-1}\omega_1^{i+j-1}.
\end{aligned}$$

Letting $x_2 = x_3 = 0$ and $N = 1$ in Theorem 5.4, by (4.3), we get the following corollary. This has also been obtained in [8, Theorem 4.11].

Corollary 5.5. *Let $\omega_1, \omega_2, \omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1, \omega_2, \omega_3$, and we obtain the following six symmetries*

$$\begin{aligned}
&\sum_{m=0}^{\omega_2-1} q^{\omega_1\omega_3 m} \sum_{i+l=n} \binom{n}{i,l} B_i\left(\omega_1 x_1 + \frac{\omega_1}{\omega_2}m, q^{\omega_2\omega_3}\right) S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\omega_1^l \omega_2^{i+l-1}\omega_3^{i-1} \\
&= \sum_{m=0}^{\omega_3-1} q^{\omega_1\omega_2 m} \sum_{i+l=n} \binom{n}{i,l} B_i\left(\omega_1 x_1 + \frac{\omega_1}{\omega_3}m, q^{\omega_2\omega_3}\right) S_{l,q^{\omega_1\omega_3}}(\omega_2-1)\omega_1^l \omega_3^{i+l-1}\omega_2^{i-1} \\
&= \sum_{m=0}^{\omega_1-1} q^{\omega_2\omega_3 m} \sum_{i+l=n} \binom{n}{i,l} B_i\left(\omega_2 x_1 + \frac{\omega_2}{\omega_1}m, q^{\omega_1\omega_3}\right) S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\omega_2^l \omega_1^{i+l-1}\omega_3^{i-1} \\
&= \sum_{m=0}^{\omega_3-1} q^{\omega_1\omega_2 m} \sum_{i+l=n} \binom{n}{i,l} B_i\left(\omega_2 x_1 + \frac{\omega_2}{\omega_3}m, q^{\omega_1\omega_3}\right) S_{l,q^{\omega_2\omega_3}}(\omega_1-1)\omega_2^l \omega_3^{i+l-1}\omega_1^{i-1} \\
&= \sum_{m=0}^{\omega_1-1} q^{\omega_2\omega_3 m} \sum_{i+l=n} \binom{n}{i,l} B_i\left(\omega_3 x_1 + \frac{\omega_3}{\omega_1}m, q^{\omega_1\omega_2}\right) S_{l,q^{\omega_1\omega_3}}(\omega_2-1)\omega_3^l \omega_1^{i+l-1}\omega_2^{i-1} \\
&= \sum_{m=0}^{\omega_2-1} q^{\omega_1\omega_3 m} \sum_{i+l=n} \binom{n}{i,l} B_i\left(\omega_3 x_1 + \frac{\omega_3}{\omega_2}m, q^{\omega_1\omega_2}\right) S_{l,q^{\omega_2\omega_3}}(\omega_1-1)\omega_3^l \omega_2^{i+l-1}\omega_1^{i-1}.
\end{aligned}$$

From (5.5), we get the following theorem.

Theorem 5.6. *Let $\omega_1, \omega_2, \omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1, \omega_2, \omega_3$, and we obtain the following six symmetries*

$$\begin{aligned}
&\sum_{m=0}^{\omega_2-1} q^{\omega_1\omega_3 m} \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3}) \\
&\quad \times B_k^{(N-1)}\left(\omega_3 x_3 + \frac{\omega_3}{\omega_2}m, q^{\omega_1\omega_2}\right) S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\omega_1^{j+k+l}\omega_2^{i+k+l-1}\omega_3^{i+j-1} \\
&= \sum_{m=0}^{\omega_3-1} q^{\omega_1\omega_2 m} \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_1 x_1, q^{\omega_2\omega_3}) B_j^{(N-1)}(\omega_3 x_2, q^{\omega_1\omega_2}) \\
&\quad \times B_k^{(N-1)}\left(\omega_2 x_3 + \frac{\omega_2}{\omega_3}m, q^{\omega_1\omega_3}\right) S_{l,q^{\omega_1\omega_3}}(\omega_2-1)\omega_1^{j+k+l}\omega_3^{i+k+l-1}\omega_2^{i+j-1} \\
&= \sum_{m=0}^{\omega_1-1} q^{\omega_2\omega_3 m} \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_2 x_1, q^{\omega_1\omega_3}) B_j^{(N-1)}(\omega_1 x_2, q^{\omega_2\omega_3}) \\
&\quad \times B_k^{(N-1)}\left(\omega_3 x_3 + \frac{\omega_3}{\omega_1}m, q^{\omega_1\omega_2}\right) S_{l,q^{\omega_1\omega_2}}(\omega_3-1)\omega_2^{j+k+l}\omega_1^{i+k+l-1}\omega_3^{i+j-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\omega_3-1} q^{\omega_1\omega_2 m} \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_2 x_1, q^{\omega_1\omega_3}) B_j^{(N-1)}(\omega_3 x_2, q^{\omega_1\omega_2}) \\
&\quad \times B_k^{(N-1)}\left(\omega_1 x_3 + \frac{\omega_1}{\omega_3} m, q^{\omega_2\omega_3}\right) S_{l,q^{\omega_2\omega_3}}(\omega_1-1)\omega_2^{j+k+l}\omega_3^{i+k+l-1}\omega_1^{i+j-1} \\
&= \sum_{m=0}^{\omega_1-1} q^{\omega_2\omega_3 m} \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_3 x_1, q^{\omega_1\omega_2}) B_j^{(N-1)}(\omega_1 x_2, q^{\omega_2\omega_3}) \\
&\quad \times B_k^{(N-1)}\left(\omega_2 x_3 + \frac{\omega_2}{\omega_1} m, q^{\omega_1\omega_3}\right) S_{l,q^{\omega_1\omega_3}}(\omega_2-1)\omega_3^{j+k+l}\omega_1^{i+k+l-1}\omega_2^{i+j-1} \\
&= \sum_{m=0}^{\omega_2-1} q^{\omega_1\omega_3 m} \sum_{i+j+k+l=n} \binom{n}{i,j,k,l} B_i^{(N)}(\omega_3 x_1, q^{\omega_1\omega_2}) B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3}) \\
&\quad \times B_k^{(N-1)}\left(\omega_1 x_3 + \frac{\omega_1}{\omega_2} m, q^{\omega_2\omega_3}\right) S_{l,q^{\omega_2\omega_3}}(\omega_1-1)\omega_3^{j+k+l}\omega_2^{i+k+l-1}\omega_1^{i+j-1}.
\end{aligned}
$$

Letting $\omega_3 = 1, x_2 = x_3 = 0$ and $N = 1$ in Theorem 5.6, by (4.3), we get the following corollary. This has also been obtained in [8, Corollary 4.12].

Corollary 5.7. *Let ω_1, ω_2 be any positive integers. Then*

$$
\begin{aligned}
&\sum_{i+l=n} \binom{n}{i,l} B_i(\omega_2 x_1, q^{\omega_1}) S_{l,q^{\omega_2}}(\omega_1-1)\omega_2^{l}\omega_1^{i-1} \\
&= \sum_{i+l=n} \binom{n}{i,l} B_i(\omega_1 x_1, q^{\omega_2}) S_{l,q^{\omega_1}}(\omega_2-1)\omega_1^{l}\omega_2^{i-1} \\
&= \sum_{m=0}^{\omega_1-1} q^{\omega_2 m} \sum_{i+k=n} \binom{n}{i,k} B_i(\omega_2 x_1, q^{\omega_1})(\omega_2 m)^k \omega_1^{i-1} \\
&= \sum_{m=0}^{\omega_2-1} q^{\omega_1 m} \sum_{i+k=n} \binom{n}{i,k} B_i(\omega_1 x_1, q^{\omega_2})(\omega_1 m)^k \omega_2^{i-1} \\
&= \sum_{m=0}^{\omega_1-1} q^{\omega_2 m} \sum_{i+k+l=n} \binom{n}{i,k,l} B_i(x_1, q^{\omega_1\omega_2}) m^k S_{l,q^{\omega_1}}(\omega_2-1)\omega_1^{i+l-1}\omega_2^{i+k-1} \\
&= \sum_{m=0}^{\omega_2-1} q^{\omega_1 m} \sum_{i+k+l=n} \binom{n}{i,k,l} B_i(x_1, q^{\omega_1\omega_2}) m^k S_{l,q^{\omega_2}}(\omega_1-1)\omega_2^{i+l-1}\omega_1^{i+k-1}.
\end{aligned}
$$

Letting $\omega_2 = 1$ in Corollary 5.7, we have the following corollary.

Corollary 5.8. *Let ω_1 be any positive integer. Then*

$$
B_n(\omega_1 x_1, q) = \sum_{i+l=n} \binom{n}{i,l} B_i(x_1, q^{\omega_1}) S_{l,q}(\omega_1-1)\omega_1^{i-1}
$$

$$= \sum_{m=0}^{\omega_1-1} q^m \sum_{i+k=n} \binom{n}{i,k} B_i(x_1, q^{\omega_1}) m^k \omega_1^{i-1}.$$

Remark 5.9. Corollary 5.8 is the multiplication formula for the q-Bernoulli polynomials (see [14, Eq. (2.26)]) together with the identity mentioned in Corollary 5.3.

5.3. Symmetric identities from (c-3)

From (2.1), (2.3), and (c-3), we have

$$\begin{aligned} I^{(N)}(\Lambda_{23}^2) = {} & I_q^{(N)}(\omega_2\omega_3, \omega_1 x_1, \omega_2\omega_3 t) \\ & \times \frac{1}{\omega_2} \sum_{l=0}^{\omega_2-1} q^{\omega_1\omega_3 l} e^{\omega_1\omega_3 t l} \\ & \times \frac{1}{\omega_3} \sum_{m=0}^{\omega_3-1} q^{\omega_1\omega_2 m} e^{\omega_1\omega_2 t m} \\ & \times I_q^{(N-1)}(\omega_1\omega_3, \omega_2 x_2, \omega_1\omega_3 t) \\ & \times I_q^{(N-1)}(\omega_1\omega_2, \omega_3 x_3, \omega_1\omega_2 t). \end{aligned} \tag{5.6}$$

From (5.6), we obtain

$$\begin{aligned} I^{(N)}(\Lambda_{23}^2) = {} & \sum_{n=0}^{\infty} \Bigg(\sum_{l=0}^{\omega_2-1} \sum_{m=0}^{\omega_3-1} q^{\omega_1\omega_3 l + \omega_1\omega_2 m} \sum_{i+j+k=n} \binom{n}{i,j,k} \\ & \times B_i^{(N)}\left(\omega_1 x_1 + \frac{\omega_1}{\omega_2} l + \frac{\omega_1}{\omega_3} m, q^{\omega_2\omega_3}\right) \\ & \times B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3}) B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) \omega_1^{j+k} \omega_2^{i+k-1} \omega_3^{i+j-1} \Bigg) \frac{t^n}{n!}, \end{aligned} \tag{5.7}$$

where the inner sum is taken over all nonnegative integers i, j, k, l with $i + j + k + l = n$. Therefore, we get the following theorem.

Theorem 5.10. *Let $\omega_1, \omega_2, \omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1, \omega_2, \omega_3$, and we obtain the following six symmetries*

$$\begin{aligned} & \sum_{l=0}^{\omega_2-1} \sum_{m=0}^{\omega_3-1} q^{\omega_1\omega_3 l + \omega_1\omega_2 m} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}\left(\omega_1 x_1 + \frac{\omega_1}{\omega_2} l + \frac{\omega_1}{\omega_3} m, q^{\omega_2\omega_3}\right) \\ & \quad \times B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3}) B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) \omega_1^{j+k} \omega_2^{i+k-1} \omega_3^{i+j-1} \\ = {} & \sum_{l=0}^{\omega_3-1} \sum_{m=0}^{\omega_2-1} q^{\omega_1\omega_2 l + \omega_1\omega_3 m} \sum_{i+j+k=n} \binom{n}{i,j,k} B_i^{(N)}\left(\omega_1 x_1 + \frac{\omega_1}{\omega_3} l + \frac{\omega_1}{\omega_2} m, q^{\omega_2\omega_3}\right) \\ & \quad \times B_j^{(N-1)}(\omega_3 x_2, q^{\omega_1\omega_2}) B_k^{(N-1)}(\omega_2 x_3, q^{\omega_1\omega_3}) \omega_1^{j+k} \omega_3^{i+k-1} \omega_2^{i+j-1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\omega_1-1}\sum_{m=0}^{\omega_3-1} q^{\omega_2\omega_3 l+\omega_1\omega_2 m} \sum_{i+j+k=n}\binom{n}{i,j,k} B_i^{(N)}\left(\omega_2 x_1+\frac{\omega_2}{\omega_1}l+\frac{\omega_2}{\omega_3}m, q^{\omega_1\omega_3}\right)\\
&\quad\times B_j^{(N-1)}(\omega_1 x_2, q^{\omega_2\omega_3}) B_k^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2})\omega_2^{j+k}\omega_1^{i+k-1}\omega_3^{i+j-1}\\
&= \sum_{l=0}^{\omega_3-1}\sum_{m=0}^{\omega_1-1} q^{\omega_1\omega_2 l+\omega_2\omega_3 m} \sum_{i+j+k=n}\binom{n}{i,j,k} B_i^{(N)}\left(\omega_2 x_1+\frac{\omega_2}{\omega_3}l+\frac{\omega_2}{\omega_1}m, q^{\omega_1\omega_3}\right)\\
&\quad\times B_j^{(N-1)}(\omega_3 x_2, q^{\omega_1\omega_2}) B_k^{(N-1)}(\omega_1 x_3, q^{\omega_2\omega_3})\omega_2^{j+k}\omega_3^{i+k-1}\omega_1^{i+j-1}\\
&= \sum_{l=0}^{\omega_1-1}\sum_{m=0}^{\omega_2-1} q^{\omega_2\omega_3 l+\omega_1\omega_3 m} \sum_{i+j+k=n}\binom{n}{i,j,k} B_i^{(N)}\left(\omega_3 x_1+\frac{\omega_3}{\omega_1}l+\frac{\omega_3}{\omega_2}m, q^{\omega_1\omega_2}\right)\\
&\quad\times B_j^{(N-1)}(\omega_1 x_2, q^{\omega_2\omega_3}) B_k^{(N-1)}(\omega_2 x_3, q^{\omega_1\omega_3})\omega_3^{j+k}\omega_1^{i+k-1}\omega_2^{i+j-1}\\
&= \sum_{l=0}^{\omega_2-1}\sum_{m=0}^{\omega_1-1} q^{\omega_1\omega_3 l+\omega_2\omega_3 m} \sum_{i+j+k=n}\binom{n}{i,j,k} B_i^{(N)}\left(\omega_3 x_1+\frac{\omega_3}{\omega_2}l+\frac{\omega_3}{\omega_1}m, q^{\omega_1\omega_2}\right)\\
&\quad\times B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3}) B_k^{(N-1)}(\omega_1 x_3, q^{\omega_2\omega_3})\omega_3^{j+k}\omega_2^{i+k-1}\omega_1^{i+j-1}.
\end{aligned}
$$

Letting $x_1 = x, x_2 = x_3 = 0$ and $N = 1$ in Theorem 5.10, by (4.3), we get the following corollary. This has also been obtained in [8, Theorem 4.14].

Corollary 5.11. *Let $\omega_1, \omega_2, \omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1, \omega_2, \omega_3$, and we obtain the following three symmetries*

$$
\begin{aligned}
&\omega_2^{n-1}\omega_3^{n-1}\sum_{l=0}^{\omega_2-1}\sum_{m=0}^{\omega_3-1} q^{\omega_1\omega_3 l+\omega_1\omega_2 m} B_n\left(\omega_1 x+\frac{\omega_1}{\omega_2}l+\frac{\omega_1}{\omega_3}m, q^{\omega_2\omega_3}\right)\\
&= \omega_1^{n-1}\omega_3^{n-1}\sum_{l=0}^{\omega_1-1}\sum_{m=0}^{\omega_3-1} q^{\omega_2\omega_3 l+\omega_1\omega_2 m} B_n\left(\omega_2 x+\frac{\omega_2}{\omega_1}l+\frac{\omega_2}{\omega_3}m, q^{\omega_1\omega_3}\right)\\
&= \omega_1^{n-1}\omega_2^{n-1}\sum_{l=0}^{\omega_1-1}\sum_{m=0}^{\omega_2-1} q^{\omega_2\omega_3 l+\omega_1\omega_3 m} B_n\left(\omega_3 x+\frac{\omega_3}{\omega_1}l+\frac{\omega_3}{\omega_2}m, q^{\omega_1\omega_2}\right).
\end{aligned}
$$

Letting $\omega_3 = 1$ in Corollary 5.11, we have the following corollary. This has also been obtained in [8, Corollary 4.15].

Corollary 5.12. *Let ω_1, ω_2 be any positive integers. Then*

$$
\begin{aligned}
&\omega_2^{n-1}\sum_{l=0}^{\omega_2-1} q^{\omega_1 l} B_n\left(\omega_1 x+\frac{\omega_1}{\omega_2}l, q^{\omega_2}\right)\\
&= \omega_1^{n-1}\sum_{l=0}^{\omega_1-1} q^{\omega_2 l} B_n\left(\omega_2 x+\frac{\omega_2}{\omega_1}l, q^{\omega_1}\right)
\end{aligned}
$$

$$= \omega_1^{n-1}\omega_2^{n-1} \sum_{l=0}^{\omega_1-1}\sum_{m=0}^{\omega_2-1} q^{\omega_2 l+\omega_1 m} B_n\left(x+\frac{1}{\omega_1}l+\frac{1}{\omega_2}m, q^{\omega_1\omega_2}\right).$$

Letting $\omega_2 = 1$ in Corollary 5.12, we get the following multiplication theorem for the q-Bernoulli polynomials (cf. Corollary 5.3).

Corollary 5.13. *Let ω_1 be any positive integer. Then we have*

$$B_n(\omega_1 x, q) = \omega_1^{n-1} \sum_{l=0}^{\omega_1-1} q^l B_n\left(x+\frac{1}{\omega_1}l, q^{\omega_1}\right).$$

Remark 5.14. Letting $q \to 1$. From Corollary 5.12, we have

$$\sum_{l=0}^{\omega_2-1} \omega_2^{n-1} B_n\left(\omega_1 x+\frac{\omega_1}{\omega_2}l\right) = \sum_{l=0}^{\omega_1-1} \omega_1^{n-1} B_n\left(\omega_2 x+\frac{\omega_2}{\omega_1}l\right),$$

where $B_n(x)$ is the nth Bernoulli polynomials. When $\omega_2 = 1$, we have the multiplication theorem for the Bernoulli polynomials as follows:

$$B_n(\omega_1 x) = \sum_{l=0}^{\omega_1-1} \omega_1^{n-1} B_n\left(x+\frac{1}{\omega_1}l\right)$$

(see [15, Corollary 4] and [34, Eq. (13)]).

6. Identities of symmetry in $I^{(N)}(\Lambda_{23}^3)$-type

In this section, by using quotient type identities for the N-fold iterated Volkenborn integral given in Subsection 2.4, we prove the corresponding identities of symmetry in three variables related to the q-extension power sums and the higher order q-Bernoulli polynomials.

From (2.3) and (d), we have

$$\begin{aligned}
I^{(N)}(\Lambda_{23}^3) &= \sum_{h=0}^{\infty} B_h^{(N-1)}(\omega_1 x_1, q^{\omega_2\omega_3}) \frac{(\omega_2\omega_3 t)^h}{h!} \\
&\times \frac{1}{\omega_1}\sum_{i=0}^{\infty} S_{i,q^{\omega_2\omega_3}}(\omega_1-1)\frac{(\omega_2\omega_3 t)^i}{i!} \\
&\times \sum_{j=0}^{\infty} B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3}) \frac{(\omega_1\omega_3 t)^j}{j!} \\
&\times \frac{1}{\omega_2}\sum_{k=0}^{\infty} S_{k,q^{\omega_1\omega_3}}(\omega_2-1)\frac{(\omega_1\omega_3 t)^k}{k!} \\
&\times \sum_{l=0}^{\infty} B_l^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2}) \frac{(\omega_1\omega_2 t)^l}{l!} \\
&\times \frac{1}{\omega_3}\sum_{m=0}^{\infty} S_{m,q^{\omega_1\omega_2}}(\omega_3-1)\frac{(\omega_1\omega_2 t)^m}{m!}.
\end{aligned} \tag{6.1}$$

Then we obtain

$$I^{(N)}(\Lambda_{23}^3) = \sum_{n=0}^{\infty}\Bigg(\sum_{h+i+j+k+l+m=n}\binom{n}{h,i,j,k,l,m} \times B_h^{(N-1)}(\omega_1 x_1, q^{\omega_2\omega_3})S_{i,q^{\omega_2\omega_3}}(\omega_1-1) \times B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3})S_{k,q^{\omega_1\omega_3}}(\omega_2-1) \times B_l^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2})S_{m,q^{\omega_1\omega_2}}(\omega_3-1) \times \omega_1^{j+k+l+m-1}\omega_2^{h+i+l+m-1}\omega_3^{h+i+j+k-1}\Bigg)\frac{t^n}{n!}, \tag{6.2}$$

where the inner sum is taken over all nonnegative integers h,i,j,k,l,m with $h+i+j+k+l+m=n$, and

$$\binom{n}{h,i,j,k,l,m} = \frac{n!}{h!i!j!k!l!m!}.$$

Therefore, we get the following theorem.

Theorem 6.1. *Let $\omega_1,\omega_2,\omega_3$ be any positive integers. Then the following expression is invariant under any permutation of $\omega_1,\omega_2,\omega_3$, and we obtain the following six symmetries*

$$\begin{aligned}
&\sum_{h+i+j+k+l+m=n}\binom{n}{h,i,j,k,l,m}B_h^{(N-1)}(\omega_1 x_1, q^{\omega_2\omega_3})S_{i,q^{\omega_2\omega_3}}(\omega_1-1)\\
&\quad\times B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3})S_{k,q^{\omega_1\omega_3}}(\omega_2-1)B_l^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2})S_{m,q^{\omega_1\omega_2}}(\omega_3-1)\\
&\quad\times\omega_1^{j+k+l+m-1}\omega_2^{h+i+l+m-1}\omega_3^{h+i+j+k-1}\\
=&\sum_{h+i+j+k+l+m=n}\binom{n}{h,i,j,k,l,m}B_h^{(N-1)}(\omega_1 x_1, q^{\omega_2\omega_3})S_{i,q^{\omega_2\omega_3}}(\omega_1-1)\\
&\quad\times B_j^{(N-1)}(\omega_3 x_2, q^{\omega_1\omega_2})S_{k,q^{\omega_1\omega_2}}(\omega_3-1)B_l^{(N-1)}(\omega_2 x_3, q^{\omega_1\omega_3})S_{m,q^{\omega_1\omega_3}}(\omega_2-1)\\
&\quad\times\omega_1^{j+k+l+m-1}\omega_3^{h+i+l+m-1}\omega_2^{h+i+j+k-1}\\
=&\sum_{h+i+j+k+l+m=n}\binom{n}{h,i,j,k,l,m}B_h^{(N-1)}(\omega_2 x_1, q^{\omega_1\omega_3})S_{i,q^{\omega_1\omega_3}}(\omega_2-1)\\
&\quad\times B_j^{(N-1)}(\omega_1 x_2, q^{\omega_2\omega_3})S_{k,q^{\omega_2\omega_3}}(\omega_1-1)B_l^{(N-1)}(\omega_3 x_3, q^{\omega_1\omega_2})S_{m,q^{\omega_1\omega_2}}(\omega_3-1)\\
&\quad\times\omega_2^{j+k+l+m-1}\omega_1^{h+i+l+m-1}\omega_3^{h+i+j+k-1}\\
=&\sum_{h+i+j+k+l+m=n}\binom{n}{h,i,j,k,l,m}B_h^{(N-1)}(\omega_2 x_1, q^{\omega_1\omega_3})S_{i,q^{\omega_1\omega_3}}(\omega_2-1)\\
&\quad\times B_j^{(N-1)}(\omega_3 x_2, q^{\omega_1\omega_2})S_{k,q^{\omega_1\omega_2}}(\omega_3-1)B_l^{(N-1)}(\omega_1 x_3, q^{\omega_2\omega_3})S_{m,q^{\omega_2\omega_3}}(\omega_1-1)\\
&\quad\times\omega_2^{j+k+l+m-1}\omega_3^{h+i+l+m-1}\omega_1^{h+i+j+k-1}
\end{aligned}$$

$$= \sum_{h+i+j+k+l+m=n} \binom{n}{h,i,j,k,l,m} B_h^{(N-1)}(\omega_3 x_1, q^{\omega_1\omega_2}) S_{i,q^{\omega_1\omega_2}}(\omega_3 - 1)$$
$$\times B_j^{(N-1)}(\omega_1 x_2, q^{\omega_2\omega_3}) S_{k,q^{\omega_2\omega_3}}(\omega_1 - 1) B_l^{(N-1)}(\omega_2 x_3, q^{\omega_1\omega_3}) S_{m,q^{\omega_1\omega_3}}(\omega_2 - 1)$$
$$\times \omega_3^{j+k+l+m-1} \omega_1^{h+i+l+m-1} \omega_2^{h+i+j+k-1}$$
$$= \sum_{h+i+j+k+l+m=n} \binom{n}{h,i,j,k,l,m} B_h^{(N-1)}(\omega_3 x_1, q^{\omega_1\omega_2}) S_{i,q^{\omega_1\omega_2}}(\omega_3 - 1)$$
$$\times B_j^{(N-1)}(\omega_2 x_2, q^{\omega_1\omega_3}) S_{k,q^{\omega_1\omega_3}}(\omega_2 - 1) B_l^{(N-1)}(\omega_1 x_3, q^{\omega_2\omega_3}) S_{m,q^{\omega_2\omega_3}}(\omega_1 - 1)$$
$$\times \omega_3^{j+k+l+m-1} \omega_2^{h+i+l+m-1} \omega_1^{h+i+j+k-1}.$$

Remark 6.2. Letting $x_1 = x_2 = x_3 = 0$ and $N = 1$ in Theorem 6.1, it will yield no identities of symmetry. However, when $N \geq 2$, we shall obtain new identities of symmetry related to the q-extension power sums and the (higher order) q-Bernoulli polynomials which are generalizations of the corresponding identities of symmetry in three variables.

Now putting $N = 2$ and $\omega_2 = \omega_3 = 1$, by (4.3), we get the following corollary.

Corollary 6.3. *Let ω_1 be any positive integer. Then*

$$\sum_{h+i+j+k=n} \binom{n}{h,i,j,k} B_h(\omega_1 x_1, q) S_{i,q}(\omega_1 - 1) B_j(x_2, q^{\omega_1}) B_k(x_3, q^{\omega_1}) \omega_1^{j+k-1}$$
$$= \sum_{h+i+j+k=n} \binom{n}{h,i,j,k} B_h(x_1, q^{\omega_1}) B_i(\omega_1 x_2, q) S_{j,q}(\omega_1 - 1) B_k(x_3, q^{\omega_1}) \omega_1^{h+k-1}$$
$$= \sum_{h+i+j+k=n} \binom{n}{h,i,j,k} B_h(x_1, q^{\omega_1}) B_i(x_2, q^{\omega_1}) B_i(\omega_1 x_3, q) S_{k,q}(\omega_1 - 1) \omega_1^{h+i-1}.$$

Remark 6.4. Letting $N = 1, 2, \ldots$ in our results, we get a new symmetric identity which is a generalization of Tuenter's [32] and other authors' results which were established in [8, 14, 15, 16, 18, 33, 34].

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References

[1] T. M. Apostol, *On the Lerch zeta function*, Pacific J. Math. **1** (1951), 161–167.

[2] A. Bayad, *Fourier expansions for Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials*, Math. Comp. **80** (2011), no. 276, 2219–2221.

[3] J. G. F. Belinfante, *Problems and Solutions*: *Elementary Problems*: *E3237-E3242*, Amer. Math. Monthly **94** (1987), no. 10, 995–996.

[4] J. G. F. Belinfante and I. Gessel, *Problems and Solutions*: *Solutions of Elementary Problems*: *E3237*, Amer. Math. Monthly **96** (1989), no. 4, 364–365.

[5] H. Cohen, *Number Theory Vol. II*: *Analytic and Modern Tools*, Graduate Texts in Mathematics, 240, Springer, New York, 2007.

[6] E. Y. Deeba and D. M. Rodriguez, *Stirling's series and Bernoulli numbers*, Amer. Math. Monthly **98** (1991), no. 5, 423–426.

[7] F. T. Howard, *Applications of a recurrence for the Bernoulli numbers*, J. Number Theory **52** (1995), no. 1, 157–172.

[8] D. S. Kim, *Identities of symmetry for q-Bernoulli polynomials*, Comput. Math. Appl. **60** (2010), no. 8, 2350–2359.

[9] D. S. Kim, N. Lee, J. Na, and K. H. Park, *Identities of symmetry for higher-order Euler polynomials in three variables* (*II*), J. Math. Anal. Appl. **379** (2011), no. 1, 388–400.

[10] D. S. Kim and K. H. Park, *Identities of symmetry for Euler polynomials arising from quotients of fermionic integrals invariant under* S_3, J. Inequal. Appl. **2010** (2010), Art. ID 851521, 16 pp.

[11] M.-S. Kim, *On Euler numbers, polynomials and related p-adic integrals*, J. Number Theory **129** (2009), no. 9, 2166–2179.

[12] M.-S. Kim and S. Hu, *Sums of products of Apostol-Bernoulli numbers*, Ramanujan J. **28** (2012), no. 1, 113–123.

[13] T. Kim, *On the analogs of Euler numbers and polynomials associated with p-adic q-integral on* $\mathbb{Z}_p$ *at* $q = -1$, J. Math. Anal. Appl. **331** (2007), no. 2, 779–792.

[14] ———, *On the symmetries of the q-Bernoulli polynomials*, Abstr. Appl. Anal. **2008** (2008), Art. ID 914367, 7 pp.

[15] ———, *Symmetry p-adic invariant integral on* $\mathbb{Z}_p$ *for Bernoulli and Euler polynomials*, J. Difference Equ. Appl. **14** (2008), no. 12, 1267–1277.

[16] ———, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys. **16** (2009), no. 1, 93–96.

[17] N. Koblitz, *p-adic Analysis*: *a Short Course on Resent Work*, London Mathematical Society Lecture Note Series, 46, Cambridge University Press, Cambridge-New York, 1980.

[18] H. Liu and W. Wang, *Some identities on the Bernoulli, Euler and Genocchi polynomials via power sums and alternate power sums*, Discrete Math. **309** (2009), no. 10, 3346–3363.

[19] Q.-M. Luo, *Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions*, Taiwanese J. Math. **10** (2006), no. 4, 917–925.

[20] ———, *Fourier expansions and integral representations for the Apostol-Bernoulli and Apostol-Euler polynomials*, Math. Comp. **78** (2009), no. 268, 2193–2208.

[21] Q.-M. Luo and H. M. Srivastava, *Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials*, J. Math. Anal. Appl. **308** (2005), no. 1, 290–302.

[22] V. Namias, *A simple derivation of Stirling's asymptotic series*, Amer. Math. Monthly **93** (1986), no. 1, 25–29.

[23] L. M. Navas, F. J. Ruiz, and J. L. Varona, *Asymptotic estimates for Apostol-Bernoulli and Apostol-Euler polynomials*, Math. Comp. **81** (2012), no. 279, 1707–1722.

[24] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, 1924.

[25] Ju. V. Osipov, *p-adic zeta functions*, (Russian), Uspekhi Mat. Nauk **34** (1979), no. 3, 209–210.

[26] W. H. Schikhof, *Ultrametric Calculus*: *An Introduction to p-Adic Analysis*, Cambridge University Press, 2006.

[27] Y. Simsek, *Twisted* (h, q)*-Bernoulli numbers and polynomials related to twisted* (h, q)*-zeta function and L-function*, J. Math. Anal. Appl. **324** (2006), no. 2, 790–804.

[28] ———, *Complete sum of products of* (h, q)*-extension of Euler polynomials and numbers*, J. Difference Equ. Appl. **16** (2010), no. 11, 1331–1348.

[29] Z. W. Sun, *Introduction to Bernoulli and Euler polynomials*, A Lecture Given in Taiwan on June 6, 2002. http://math.nju.edu.cn/.zwsun/BerE.pdf

[30] B. A. Tangedal and P. T. Young, *On p-adic multiple zeta and log gamma functions*, J. Number Theory **131** (2011), no. 7, 1240–1257.

[31] L. Tao and Z. W. Sun, *A reciprocity law for uniform functions*, Nanjing Univ. J. Math. Biquarterly **21** (2004), no. 2, 201–205.
[32] H. J. H. Tuenter, *A symmetry of power sum polynomials and Bernoulli numbers*, Amer. Math. Monthly **108** (2001), no. 3, 258–261.
[33] W. Wang and W. Wang, *Some results on power sums and Apostol-type polynomials*, Integral Transforms Spec. Funct. **21** (2010), no. 3-4, 307–318.
[34] S.-l. Yang, *An identity of symmetry for the Bernoulli polynomials*, Discrete Math. **308** (2008), no. 4, 550–554.
[35] P.-T. Young, *Degenerate Bernoulli polynomials, generalized factorial sums, and their applications*, J. Number Theory **128** (2008), no. 4, 738–758.

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