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SPECTRAL RADIUS OF BIORTHOGONAL WAVELETS WITH ITS APPLICATION

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Abstract. In this paper, a 2-circular matrix theory is developed, and a concept of spectral radius for biorthogonal wavelet is introduced. We propose a novel design method by minimizing the spectral radius and obtain a wavelet which has better performance than the famous 9-7 wavelet in terms of image compression coding.

1. Introduction

Over the last two decades, wavelets have become a fundamental tool in many areas of applied mathematics and engineering ranging from signal and image processing to numerical analysis. In her celebrated paper [2], Daubechies introduced a general method to construct compactly supported wavelets. It is well known that 2-band orthogonal wavelet suffers from severe constraint conditions. For instance, nontrivial symmetric 2-band orthogonal wavelet does not exist except the Haar wavelet [3]. Biorthogonal wavelet enjoy two important properties of linear phase and higher vanishing moments, was studied [4, 8, 11, 12]. Vetterli and Herley studied theory and design for biorthogonal wavelet [9] and Cohen gave another biorthogonal wavelet family [6]. Guoqiu Wang constructed wavelet filters with free paraments based on 2-circular matrix method [1]. Even though the research on biorthogonal wavelet is relatively mature, further study is still valuable.

Let $(\psi, \widetilde{\psi})$ be a pair of dual wavelet functions,

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^{j/2}x - k),\ j, k \in \mathbb{Z}, \tag{1.1}$$

$$\widetilde{\psi}_{j,k}(x) = 2^{j/2}\widetilde{\psi}(2^{j/2}x - k),\ j, k \in \mathbb{Z}. \tag{1.2}$$

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Here $\{\psi_{j,k}\}$ and $\{\widetilde{\psi}_{j,k}\}$ constitute wavelet frames, respectively, if there exist positive real numbers A, B, $\widetilde{A}$ and $\widetilde{B}$ such that

$$A\|f\|^2 \le \sum_{j,k} |\langle f, \psi_{j,k}(x)\rangle|^2 \le B\|f\|^2, \tag{1.3}$$

$$\widetilde{A}\|f\|^2 \le \sum_{j,k} |\langle f, \widetilde{\psi}_{j,k}(x)\rangle|^2 \le \widetilde{B}\|f\|^2 \tag{1.4}$$

for all $f \in L^2(R)$.

(1.3) and (1.4) show that the energy of biorthogonal wavelet transform is controllable although it is not conservative. However, while $B - A$ or $\widetilde{B} - \widetilde{A}$ are rather large, the biorthogonal wavelet transform may be unstable. i.e., the energy is amplified in some cases and compressed in other cases.

In view of potential applications, estimating the bounds of sub-band operator could be more important. Let's recall the sub-band coding scheme or Mallat algorithm associated to a biorthogonal wavelet. There are four sequences $h = (h_n)_{n\in Z}$, $g = (g_n)_{n\in Z}$, $\widetilde{h} = (\widetilde{h}_n)_{n\in Z}$, $\widetilde{g} = (\widetilde{g}_n)_{n\in Z}$, two of which are used for decomposition $\{h, g\}$ and two for reconstruction $\{\widetilde{h}, \widetilde{g}\}$. Starting from a data sequence $c^0 = (c^0_n)_{n\in Z}$, we convolve with h, g and retain only one sample out of every two for the decomposition:

$$\begin{aligned} c^1_n &= \textstyle\sum_k h_{2n-k} c^0_k, \\ d^1_n &= \textstyle\sum_k g_{2n-k} c^0_k. \end{aligned} \tag{1.5}$$

The reconstruction operation is

$$c^0_k = \sum_n (\widetilde{h}_{2n-k} c^1_n + \widetilde{g}_{2n-k} d^1_n). \tag{1.6}$$

(1.5) and (1.6) can be rewritten as the form of 2-circular matrix [10], which can be defined by the 2-circular operator. Let $v = (v_1, v_2, v_3, \dots, v_n)$. Define

$$\sigma^0(v) = v,\ \sigma(v) = (v_{n-1}, v_n, v_1, v_2, v_3, \dots, v_{n-2}),\ \sigma^k(v) = \sigma^{k-1}(\sigma(v)).$$

If $v = (h_0, h_1, h_2, \dots, h_p, 0, \dots, 0, h_{-p+1}, \dots, h_{-1})$ and $u = (g_0, g_1, g_2, \dots, g_q, 0, \dots, 0, g_{-q+1}, \dots, g_{-1})$ are two $2n$-dimensional row vectors, then

$$M_{2n} = \begin{bmatrix} \sigma^0(v) \\ \sigma^1(v) \\ \vdots \\ \sigma^{n-1}(v) \\ \sigma^0(u) \\ \sigma^1(u) \\ \vdots \\ \sigma^{n-1}(u) \end{bmatrix}$$

is called the 2-circular matrix generated by $\{h, g\}$. An example is as follows:

$$M_8 = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & 0 & 0 & h_{-2} & h_{-1} \\ h_{-2} & h_{-1} & h_0 & h_1 & h_2 & h_3 & 0 & 0 \\ 0 & 0 & h_{-2} & h_{-1} & h_0 & h_1 & h_2 & h_3 \\ h_2 & h_3 & 0 & 0 & h_{-2} & h_{-1} & h_0 & h_1 \\ g_0 & g_1 & g_2 & g_3 & 0 & 0 & g_{-2} & g_{-1} \\ g_{-2} & g_{-1} & g_0 & g_1 & g_2 & g_3 & 0 & 0 \\ 0 & 0 & g_{-2} & g_{-1} & g_0 & g_1 & g_2 & g_3 \\ g_2 & g_3 & 0 & 0 & g_{-2} & g_{-1} & g_0 & g_1 \end{bmatrix}.$$

If c^0 is a periodic signal, we rewrite $c^0 = (c_1^0, c_2^0, \ldots, c_{2n}^0)^T$, which is a whole period.

Let $c^1 = (c_1^1, c_2^1, \ldots, c_n^1, d_1^1, d_2^1, \ldots, d_n^1)^T$. Then there exists a $2n \times 2n$ 2-circular matrix M_{2n} generated by $\{h, g\}$ such that

$$c^1 = M_{2n} c^0. \tag{1.7}$$

It is easy to see that (1.7) is equivalent to (1.5). Let

$$c^0 = \widetilde{M}_{2n}^T c^1, \tag{1.8}$$

where $\widetilde{M}_{2n}$ is a 2-circular $2n \times 2n$ matrix generated by $\{\widetilde{h}, \widetilde{g}\}$.

Clearly, (1.8) is equivalent to (1.6). Then,

$$\|c^1\|^2 = (c^1)^T c^1 = (c^0)^T (M_{2n}^T M_{2n}) c^0. \tag{1.9}$$

Since $M_{2n}^T M_{2n}$ is a positive definite matrix, its eigenvalues λ_i $(i = 1, 2 \ldots, 2n)$ are positive and there exists an orthonormal matrix Q such that

$$M_{2n}^T M_{2n} = Q^T \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n}) Q. \tag{1.10}$$

Let $s = Qc^0 = (s_1, s_2, \ldots, s_{2n})^T$. Then$\|s\|^2 = \|c^0\|^2$. It follows from (1.9) and (1.10) that

$$\|c^1\|^2 = (Qc^0)^T \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n})(Qc^0) = \sum_{i=1}^{2n} \lambda_i s_i^2.$$

Thus,

$$\min\{\lambda_i\}\|c^0\|^2 \le \|c^1\|^2 \le \max\{\lambda_i\}\|c^0\|^2. \tag{1.11}$$

Similarly,

$$\min\{\widetilde{\lambda}_i\}\|c^0\|^2 \le \|c^1\|^2 \le \max\{\widetilde{\lambda}_i\}\|c^0\|^2, \tag{1.12}$$

where $\widetilde{\lambda}_i$ $(i = 1, 2 \ldots, 2n)$ are the eigenvalues of $\widetilde{M}_{2n}^T \widetilde{M}_{2n}$.

Instead of estimating the bounds in (1.3) or (1.4), we try to calculate the eigenvalues of $M_{2n}^T M_{2n}$ or $\widetilde{M}_{2n}^T \widetilde{M}_{2n}$. It has been shown that the eigenvalues of $M_{2n}^T M_{2n}$ appear in pairs of reciprocal, $M_{2n}^T M_{2n}$ and $\widetilde{M}_{2n}^T \widetilde{M}_{2n}$ have the same eigenvalues (cf. [12, Section 2]). It is obvious that $\max\{\lambda_i\} = \max\{\widetilde{\lambda}_i\} = \frac{1}{\min\{\lambda_i\}} = \frac{1}{\min\{\widetilde{\lambda}_i\}}$.

Note that c^0 only has finite length in (1.11). If $c^0 \in l^2$, we shall investigate the limit $\lim_{n\to+\infty} \max\{\lambda_i\}$ to directly evaluate or examine the performances of filters, which is called the spectral radius of the biorthogonal wavelet in this paper.

This paper is organized as follows. In Section 2, we develop some results of the transform matrix, define the concept of spectral radius and prove the existence and uniqueness. An example is provided to illustrate our results in this paper. In Section 3, we propose a novel design method for constructing biorthogonal wavelets and obtain a wavelet with better performance of image compression. Conclusions are summarized in Section 4.

2. Spectral radius of biorthogonal wavelet

Assume that the low pass filters are FIR and symmetric. In addition, we assume

$$\sum_k h_k = \sum_k \widetilde{h}_k = \sqrt{2} \tag{2.1}$$

and let

$$\begin{aligned} g_k &= (-1)^{(1-k)} \widetilde{h}_{1-k}, \\ \widetilde{g}_k &= (-1)^{(1-k)} h_{1-k}. \end{aligned} \tag{2.2}$$

Then

$$\begin{aligned} \textstyle\sum_k h_{2k} &= \textstyle\sum_k h_{2k+1} = \frac{\sqrt{2}}{2}, \\ \textstyle\sum_k \widetilde{h}_{2k} &= \textstyle\sum_k \widetilde{h}_{2k+1} = \frac{\sqrt{2}}{2}. \end{aligned} \tag{2.3}$$

The perfect reconstruction condition is equivalent to the following equation

$$\sum_k h_k \widetilde{h}_{k+2j} = \delta_j, \tag{2.4}$$

where δ_j is the Dirac sequence, i.e., $\delta_j = 1$ for $j = 0$ otherwise $\delta_j = 0$.

Define

$$b_k = \sum_i h_i h_{i+2k}, \quad \widetilde{b}_k = \sum_i \widetilde{h}_i \widetilde{h}_{i+2k}. \tag{2.5}$$

Since h and $\widetilde{h}$ are the FIR filters, let d_1, d_2 be the maximal positive integers such that $b_{d_1} = \sum_i h_i h_{i+2d_1} \neq 0$ and $\widetilde{b}_{d_2} = \sum_i \widetilde{h}_i \widetilde{h}_{i+2d_2} \neq 0$, respectively. Define

$$u_{n,j} = (b_0 + \widetilde{b}_0) + 2 \sum_{i=1}^{d} (b_i + \widetilde{b}_i) \cos\left(\frac{2\pi i j}{n}\right) 1 \leq j \leq \left[\frac{n}{2}\right], \tag{2.6}$$

$$u_0 = \left((b_0 + \widetilde{b}_0) + 2 \sum_{i=1}^{d} (-1)^i (b_i + \widetilde{b}_i) \right), \tag{2.7}$$

where $d = \max\{d_1, d_2\}$ and $[x]$ denotes the largest integer less than or equal to x.

We first present our main theorem of this paper based on a 2-circular matrix theory.

Theorem 2.1. *Assume that* $M_{2n} = [\,{}^{H}_{G}\,]$ *is a 2-circular matrix generated by* $\{h, g\}$, *where* H *and* G *are* $n \times 2n$ *2-circular matrices. Then the characteristic polynomial of* $M_{2n}M_{2n}^T$ *is*

(2.8)
$$p(\lambda) = \begin{cases} (\lambda-1)^2 \prod_{j=1}^{r} \left(\lambda^2 - u_{n,j}\lambda + 1\right)^2 & n = 2r+1; \\ (\lambda-1)^2(\lambda^2 - u_0\lambda + 1) \prod_{j=1}^{r} \left(\lambda^2 - u_{n,j}\lambda + 1\right)^2 & n = 2r+2, \end{cases}$$

where $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$, i *is the imaginary unit.*

To prove Theorem 2.1, we need some results as follows.

Proposition 2.2.

$$\sum_k b_k = 1, \quad \sum_k \widetilde{b}_k = 1, \tag{2.9}$$

where b_k *and* $\widetilde{b}_k$ *are defined in* (2.5).

Proof.

$$\begin{aligned} \sum_k b_k &= \sum_k \sum_i h_i h_{i+2k} = \sum_i h_i^2 + \sum_{k>0} \sum_i h_i h_{i+2k} + \sum_{k<0} \sum_i h_i h_{i+2k} \\ &= \sum_{i \in 2Z+1} h_i^2 + 2 \sum_{k>0} \sum_{i \in 2Z+1} h_i h_{i+2k} + \sum_{i \in 2Z} h_i^2 + 2 \sum_{k>0} \sum_{i \in 2Z} h_i h_{i+2k} \\ &= \left(\sum_{i \in 2Z+1} h_i\right)^2 + \left(\sum_{i \in 2Z} h_i\right)^2. \end{aligned}$$

It follows from (2.3) that $\sum_k b_k = 1$. Similarly, $\sum_k \widetilde{b}_k = 1$. □

Proposition 2.3. $M_{2n}^T M_{2n}, M_{2n}M_{2n}^T, \widetilde{M}_{2n}^T \widetilde{M}_{2n}, \widetilde{M}_{2n}\widetilde{M}_{2n}^T$ *have the same eigenvalues.*

Proof. Note that $M_{2n}^T M_{2n}$ and $\widetilde{M}_{2n}^T \widetilde{M}_{2n}$ have the same eigenvalues (cf. [12, Section 2]). The proof of the others is trivial. □

Proposition 2.4. *Assume that both* $M_{2n} = [\,{}^{H}_{G}\,]$ *and* $\widetilde{M}_{2n} = \left[\,{}^{\widetilde{H}}_{\widetilde{G}}\,\right]$ *are 2-circular matrices generated by* $\{h, g\}$ *and* $\{\widetilde{h}, \widetilde{g}\}$, *respectively, where* $H, G, \widetilde{H}, \widetilde{G}$ *are all* $n \times 2n$ *2-circular matrices and* n *is large enough. Then*

(1) HH^T, HG^T, GH^T, GG^T, $\widetilde{H}\widetilde{H}^T$, $\widetilde{H}\widetilde{G}^T$, $\widetilde{G}\widetilde{H}^T$, $\widetilde{G}\widetilde{G}^T$ *are all 1-circular matrices.*

(2) $HH^T GG^T - HG^T GH^T = I_n$.

Proof. (1) The element at the jth row and the kth column in HH^T can be written as

$$\sum_i h_{i+2j} h_{i+2k} = \sum_i h_{i+2(j+1)} h_{i+2(k+1)}.$$

Since $\sum_i h_{i+2(j+1)}h_{i+2(k+1)}$ is the element at the $(j+1)$th row and the $(k+1)$th column in HH^T, it is easy to see that HH^T is a 1-circular matrix. The proof of the others is similar.

(2) Note that $(M_{2n}^T M_{2n})(\widetilde{M}_{2n}^T \widetilde{M}_{2n}) = I_n$. It implies

$$(HH^T)(\widetilde{H}\widetilde{H}^T) + (HG^T)(\widetilde{G}\widetilde{H}^T) = I_n \tag{2.10}$$

and

$$(HH^T)(\widetilde{H}\widetilde{G}^T) + (HG^T)(\widetilde{G}\widetilde{G}^T) = O_n, \tag{2.11}$$

where O_n denotes an $n \times n$ zero matrix.

Firstly, we verify that

$$GG^T = \widetilde{H}\widetilde{H}^T,\ HH^T = \widetilde{G}\widetilde{G}^T,\ HG^T = -\widetilde{G}\widetilde{H}^T,\ \widetilde{H}\widetilde{G}^T = -GH^T. \tag{2.12}$$

In fact, the element in GG^T can be written as

$$\begin{aligned}\sum_i g_{i+2j}g_{i+2k} &= \sum_i (-1)^{1-i-2j}\widetilde{h}_{1-i-2j}(-1)^{1-i-2k}\widetilde{h}_{1-i-2k} \\ &= \sum_i \widetilde{h}_{i-2j}\widetilde{h}_{i-2k} = \sum_i \widetilde{h}_{i+2j}\widetilde{h}_{i+2k}.\end{aligned}$$

It is just the element in $\widetilde{H}\widetilde{H}^T$ at the same position. Thus, $GG^T = \widetilde{H}\widetilde{H}^T$ holds.

Similarly, the others in (2.8) can be verified.

Note that if A and B are 1-circular matrices, then AB and $A \pm B$ are 1-circular matrices, moreover, $AB = BA$ (cf. [5, Section 2]).

It follows from (2.11) and (2.12) that

$$\begin{aligned}O_n &= (HH^T)(\widetilde{H}\widetilde{G}^T) + (HG^T)(\widetilde{G}\widetilde{G}^T) = (HH^T)(\widetilde{H}\widetilde{G}^T) + (\widetilde{G}\widetilde{G}^T)(HG^T) \\ &= (\widetilde{G}\widetilde{G}^T)(\widetilde{H}\widetilde{G}^T) + (\widetilde{G}\widetilde{G}^T)(-\widetilde{G}\widetilde{H}^T) = (\widetilde{G}\widetilde{G}^T)[(\widetilde{H}\widetilde{G}^T) - (\widetilde{G}\widetilde{H}^T)].\end{aligned}$$

Since $M_{2n}M_{2n}^T$ is a positive definite matrix, and its principal minor determinants are all positive, we have $\det(HH^T) = \det(\widetilde{G}\widetilde{G}^T) > 0$, thus, $\widetilde{G}\widetilde{H}^T = \widetilde{H}\widetilde{G}^T$.

Finally, by (2.10) and (2.12), we have

$$\begin{aligned}(HH^T)(GG^T) - (HG^T)(GH^T) &= (HH^T)(\widetilde{H}\widetilde{H}^T) + (HG^T)(\widetilde{H}\widetilde{G}^T) \\ &= (HH^T)(\widetilde{H}\widetilde{H}^T) + (HG^T)(\widetilde{G}\widetilde{H}^T) = I_n.\end{aligned}$$

The proof is completed. □

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. The characteristic polynomial of $M_{2n}M_{2n}^T$ is

$$p(\lambda) = \det\left(M_{2n}M_{2n}^T - \lambda I_{2n}\right) = \det\begin{pmatrix} HH^T - \lambda I_n & HG^T \\ GH^T & GG^T - \lambda I_n \end{pmatrix}.$$

Note that if $AC = CA$, then

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB),$$

which is a simple result in linear algebra.

Thus,

$$\begin{aligned} p(\lambda) &= \det\left(\left(HH^T - \lambda I_n\right)\left(GG^T - \lambda I_n\right) - GH^T HG^T\right) \\ &= \det\left(\lambda^2 I_n - \lambda(HH^T + GG^T) + HH^T GG^T - GH^T HG^T\right). \end{aligned}$$

By Proposition 2.4, we have

$$p(\lambda) = \det\left(\lambda^2 I_n - \lambda(HH^T + GG^T) + I_n\right). \tag{2.13}$$

It is simple to verify that HH^T, $\widetilde{H}\widetilde{H}^T$ are $n \times n$ 1-circular matrices generated by $(b_0, b_1, \ldots, b_{d_1}, 0, \ldots, 0, b_{d_1}, \ldots, b_1)$ and $(\widetilde{b}_0, \widetilde{b}_1, \ldots, \widetilde{b}_{d_2}, 0, \ldots, 0, \widetilde{b}_{d_2}, \ldots, \widetilde{b}_1)$, respectively, where b_k and $\widetilde{b}_k$ are defined in (2.5).

Note that $\lambda^2 I_n - \lambda(HH^T + GG^T) + I_n$ is also a 1-circular matrix. We define

$$f(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_{d+1} x^d + a_{d+1} x^{n-d} + \cdots + a_2 x^{n-1}, \tag{2.14}$$

where $a_1 = \lambda^2 - (b_0 + \widetilde{b}_0)\lambda + 1$, $a_{i+1} = -(b_i + \widetilde{b}_i)\lambda$ $(1 \le i \le d)$.

By the property of 1-circular matrix determinant in [5], we have

$$p(\lambda) = \begin{cases} f(1)\prod_{j=1}^{r}\left|f(\omega^j)\right|^2 & n = 2r+1; \\ f(1)f(-1)\prod_{j=1}^{r}\left|f(\omega^j)\right|^2 & n = 2r+2, \end{cases}$$

where $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$.

Since

$$f(1) = \lambda^2 - \lambda\left((b_0 + \widetilde{b}_0) + 2\sum_{i=1}^{d}(b_i + \widetilde{b}_i)\right) + 1 = \lambda^2 - \lambda\left(\sum_i (b_i + \widetilde{b}_i)\right) + 1,$$

by Proposition 2.2, we have

$$f(1) = \lambda^2 - 2\lambda + 1. \tag{2.15}$$

Using the formulas

$$\cos(\frac{2\pi(n-k)j}{n}) = \cos(2\pi j - \frac{2\pi kj}{n}) = \cos(\frac{2\pi kj}{n})$$

and

$$\sin(\frac{2\pi(n-k)j}{n}) = \sin(2\pi j - \frac{2\pi kj}{n}) = -\sin(\frac{2\pi kj}{n}),$$

we have

$$\begin{aligned} &\left|f(\omega^j)\right|^2 \\ = &\left|a_1 + a_2\omega^j + a_3\omega^{2j} + \cdots + a_{d+1}\omega^{dj} + a_{d+1}\omega^{(n-d)j} + \cdots + a_2\omega^{(n-1)j}\right|^2 \\ = &\left(a_1 + a_2\cos(\frac{2\pi j}{n}) + \cdots + a_{d+1}\cos(\frac{2\pi dj}{n}) + a_{d+1}\cos(\frac{2\pi(n-d)j}{n}) + \cdots \right. \\ &\left. + a_2\cos(\frac{2\pi(n-1)j}{n})\right)^2 \end{aligned}$$

$$+\left(a_2\sin(\frac{2\pi j}{n})+\cdots+a_{d+1}\sin(\frac{2\pi dj}{n})+a_{d+1}\sin(\frac{2\pi(n-d)j}{n})+\cdots\right.$$
$$\left.+a_2\sin(\frac{2\pi(n-1)j}{n})\right)^2$$
$$=\left(a_1+2a_2\cos(\frac{2\pi j}{n})+\cdots+2a_{d+1}\cos(\frac{2\pi dj}{n})\right)^2$$
$$=\left(\lambda^2-\lambda\left((b_0+\widetilde{b}_0)+2\sum_{i=1}^{d}(b_i+\widetilde{b}_i)\cos\left(\frac{2\pi ij}{n}\right)\right)+1\right)^2,$$

i.e.,

$$\left|f(\omega^j)\right|^2=\left(\lambda^2-\lambda\left((b_0+\widetilde{b}_0)+2\sum_{i=1}^{d}(b_i+\widetilde{b}_i)\cos(\frac{2\pi ij}{n})\right)+1\right)^2, \tag{2.16}$$

where $u_{n,j}$ is defined by (2.6).

If $n=2r+2$, let $x=-1$ in (2.14), then

$$f(-1)=\lambda^2-\lambda\left((b_0+\widetilde{b}_0)+2\sum_{i=1}^{d}(-1)^i(b_i+\widetilde{b}_i)\right)+1=\lambda^2-u_0\lambda+1, \tag{2.17}$$

where u_0 is defined by (2.7). The proof is completed. □

Definition 2.5. Let M be an $n\times n$ matrix with real elements, and $\lambda(M)$ be all eigenvalues of MM^T. Then $\lambda(M)$ is called the spectrum of matrix M, and $\rho(M)=\max_i|\lambda_i|$ is called the spectral radius of M.

Theorem 2.6. *$\rho(M_{2n})\to\beta(n\to+\infty)$, where β is a finite positive real number.*

Proof. Define

$$g(x)=(b_0+\widetilde{b}_0)+2(b_1+\widetilde{b}_1)\cos x+\cdots+2(b_d+\widetilde{b}_d)\cos(dx),\ x\in[0,\pi].$$

By (2.6), $g(x)$ is a continuous function in $[0,\pi]$. Therefore, there exists an $x_0\in[0,\pi]$ such that

$$g(x_0)=\max_{x\in[0,\pi]}\{g(x)\}.$$

Define

$$A_n=\left\{\frac{2\pi j}{n},\ j=1,2,\ldots,[n/2]\right\}.$$

For $x_0\in[0,\pi]$, there exists an $x_n=\frac{2\pi j_n}{n}\in A_n$ such that when $n\to+\infty$, $x_n\to x_0$. Thus,

$$g(x_n)\to g(x_0)\ (n\to+\infty).$$

Note that $u_{n,j_n} = g\left(\frac{2\pi j_n}{n}\right)$, $\frac{\sqrt{x^2-4}}{2}$ $(x > 2)$ is a monotonously increasing function, and

$$\frac{u_{n,j_n} + \sqrt{u_{n,j_n}^2 - 4}}{2} \le \rho(M_{2n}) \le \frac{g(x_0) + \sqrt{g^2(x_0) - 4}}{2} = \beta.$$

This leads to $\rho(M_{2n}) \to \beta(n \to +\infty)$. The proof of this theorem is completed. □

Definition 2.7. $\beta = \lim_{n\to+\infty} \rho(M_{2n})$ is called the spectral radius of a biorthogonal wavelet.

Now we go back to (1.5) and (1.12). By Theorem 2.6, we can obtain the following theorem directly.

Theorem 2.8. *Let* $c^0 = (c_n^0)_{n\in Z} \in l^2$, $c^1 = (\ldots, c_1^1, c_2^1, \ldots, c_n^1, \ldots, d_1^1, d_2^1, \ldots, d_n^1, \ldots)$. *Define a operator* $T : Tc^0 = c^1$, *where* c^0 *and* c^1 *are defined by* (1.5). *Then*

$$\frac{1}{\sqrt{\beta}}\|c^0\| \le \|Tc^0\| \le \sqrt{\beta}\|c^0\|. \tag{2.18}$$

In the rest of this section, we will illustrate our results by an example. The well-known 9-7 wavelet, which was studied thoroughly by Cohen et al. [1], is denoted as CDF 9-7. In this example, $d = 4$, numerical computation gives

$$\begin{aligned}\lambda(M_{18}) = \{&0.7720, 0.7720, 0.8561, 0.8561, 0.8980, 0.8980, 0.9545, 0.9545, 1, 1,\\ &1.0477, 1.0477, 1.1136, 1.1136, 1.1681, 1.1681, 1.2953, 1.2953\},\\ \lambda(M_{20}) = \{&0.7567, 0.8025, 0.8025, 0.8751, 0.8751, 0.9053, 0.9053, 0.9617,\\ &0.9617, 1, 1, 1.0399, 1.0399, 1.1045, 1.1045, 1.1427, 1.1427, 1.2460,\\ &1.2460, 1.3216\}.\end{aligned}$$

We can also calculate that

$$b_0 + \widetilde{b}_0 \approx 2.0234,\ b_1 + \widetilde{b}_1 \approx -0.0159,\ b_2 + \widetilde{b}_2 \approx 0.0064,\ b_3 + \widetilde{b}_3 \approx -0.0036,$$

$$b_4 + \widetilde{b}_4 \approx 0.0014,\ \text{and } u_{n,j} \to u\ (u \approx 2.078)\ (j = [n/2],\ n \to +\infty).$$

Thus, $\beta \approx 1.3216$.

3. A design method based on minimizing spectral radius

Define

$$\begin{aligned} m_0(\xi) &= 2^{-1/2} \textstyle\sum_n h_n e^{-in\xi},\\ \widetilde{m}_0(\xi) &= 2^{-1/2} \textstyle\sum_n \widetilde{h}_n e^{-in\xi}.\end{aligned} \tag{3.1}$$

Assume that (3.1) can be factored as

$$\begin{aligned} m_0(\xi) &= \left(\tfrac{1+e^{-i\xi}}{2}\right)^L F(\xi),\\ \widetilde{m}_0(\xi) &= \left(\tfrac{1+e^{-i\xi}}{2}\right)^{\widetilde{L}} \widetilde{F}(\xi).\end{aligned} \tag{3.2}$$

The idea of classical design method for biorthogonal wavelet is to choose as high $(L, \widetilde{L})$ as possible in (3.2)(CDF9-7 was successfully designed in [1]). Applying this method to 8-8-tap wavelets, we obtain an 8-8-tap wavelet denoted as OR (8-8):

$$h = \sqrt{2}(0.0534975, -0.0872258, -0.0692208, 0.602949),$$
$$\widetilde{h} = \sqrt{2}(-0.0228179, -0.0372038, 0.133432, 0.42659),$$

(whereas the other half is symmetric and so skipped), and the highest $(L, \widetilde{L}) = (5, 3)$. However, this wavelet is not as good as CDF9-7 in terms of image compression. Computation shows that its spectral radius is slightly large, see Table 1.

Table 1. Spectral radius

Wavelets	spectrum radius β	$\beta^{\frac{1}{2}}$
OR8-8	2.6432	1.6258
OP8-8	1.7612	1.3271
OP12-8	1.4714	1.2130
OP16-8	1.3824	1.1758
CDF9-7	1.3216	1.1496

From (2.18), it is obvious that T is a unitary operator if and only if $\sqrt{\beta} = 1$. If T is derived from a biorthogonal wavelet, then it is not a unitary operator. In order to keep the stability of transform, we hope the operator is as near a unitary one as possible, i.e., $\sqrt{\beta}$ is as near 1 as possible. Based on this idea, we can minimize the spectral radius to obtain some biorthogonal wavelets.

Now we relax the condition of the highest $\widetilde{L}$, for example, let $\widetilde{L}$ decrease to 1 from 3 in the case of 8-8-tap. Then we can use an extra degree of freedom to minimize the spectral radius.

Assume that a wavelet system contains a vector of s variables, which is represented by $X = (x_1, x_2, \ldots, x_s)$. Then h, $\widetilde{h}$, b_k and $\widetilde{b}_k$ in (2.5), $u_{n,j}$ in (2.6) are functions of X. To formulate the optimization problem, define

$$u_{n,0} = \begin{cases} 0, & n = 2r + 1; \\ u_0, & n = 2r + 2, \end{cases}$$

where u_0 is defined by (2.7). Consider the optimization problem for a large enough n as follows:

$$u_n = \min_X \{ \max_{0 \le j \le r} \{u_{n,j}\}\}. \tag{3.3}$$

Let $\beta_n = \frac{u_n + \sqrt{u_n^2 - 4}}{2}$. Then the limit of β_n is the spectral radius. In fact, the optimization process stops whenever $|\beta_{n+1} - \beta_n| < \varepsilon$, where $\varepsilon > 0$ is given. We take $\varepsilon = 10^{-5}$ for numerical computations in this paper. Generally speaking, n has to be larger than 400. Obviously, the optimization process is very complex and only approximate solutions can be obtained.

Applying the same process to 8-8-tap, 12-8-tap and 16-8-tap wavelets, we obtained three optimal wavelets with other half symmetric part skipped as before.

OP8-8:

$h \approx \sqrt{2}(0.10588478, -0.21250827, 0.13072889, 0.47589460),$

$\widetilde{h} \approx \sqrt{2}(-0.03146955, -0.06315864, 0.12478045, 0.46984774),$

with $(L, \widetilde{L}) = (5, 1)$.

OP12-8:

$h \approx \sqrt{2}(0.01438339, -0.03075211, 0.10103289, -0.12189856, 0.05633416, 0.48090023),$

$\widetilde{h} \approx \sqrt{2}(-0.03625410, -0.07751231, 0.11999590, 0.49377051),$

with $(L, \widetilde{L}) = (5, 1)$.

OP16-8:

$h \approx \sqrt{2}(0.00720413, -0.0156142, -0.00506077, 0.0575831, 0.00975006, -0.0917248, 0.0684645, 0.469398),$

$\widetilde{h} \approx \sqrt{2}(-0.037533, -0.0813489, 0.118717, 0.500165),$

with $(L, \widetilde{L}) = (5, 3)$. See Fig. 1 for the graphs of the scaling functions and wavelets in this example.

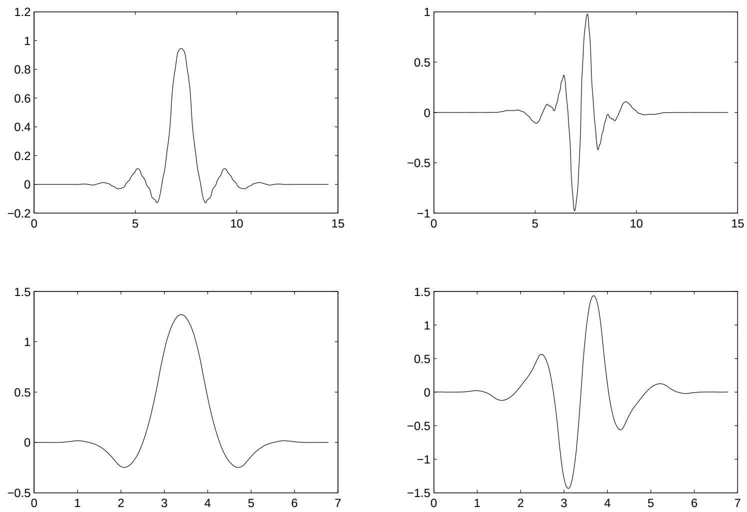

Fig. 1. OP16-8: $\phi, \widetilde{\phi}$ a dual scale functions; $\psi, \widetilde{\psi}$ a dual wavelet functions.

In order to test the performance of these wavelets, we apply the SPIHT algorithms [7] to the experiments of image compression coding. In these experiments, we use the typical and standard test images such as Lena and Barbara, which are decomposed 5 levels, with the Huffman entropy coding technologies. The PSNR (dB) is used to evaluate the quality of decoded images. From the test results in Table 2 and Table 3, we can see that the OP16-8 has significant advantage in image compression coding. In fact, OP16-8 is superior for almost all test images, with further details of test skipped due to space limit.

Note that, in general, smaller norm does not necessarily make the performance better. A counter example is Haar wavelet with obviously poor performance. As the optimization procedure is applied to the wavelet systems with even lengths filters, the smallest spectral radius must be 1 if no other constraints imposed on the filters. Usually, the high vanishing moments are necessary.

Table 2. Test results for Lena

	1:4	1:8	1:16	1:32
OR8-8	43.51	39.24	36.06	32.79
OP8-8	43.97	39.53	36.33	33.10
OP12-8	44.33	39.82	36.53	33.35
OP16-8	44.73	40.11	36.81	33.58
CDF9-7	44.52	40.03	36.78	33.54

Table 3. Test results for Barbara

	1:4	1:8	1:16	1:32
OR8-8	41.35	34.90	29.64	25.69
OP8-8	41.61	35.51	30.24	26.51
OP12-8	41.78	35.76	30.60	26.87
OP16-8	42.44	36.45	31.38	27.45
CDF9-7	42.22	36.01	30.99	27.20

4. Conclusions

Algebraic methods are often effective in studying the wavelet problems. In this paper, an important index of spectral radius for biorthogonal wavelet is proposed based on the eigenvalue-method. It provides us another viewpoint to examine and evaluate the wavelets. The circular matrix theory is no doubt an effective tool in wavelet analysis. The concepts, results and method proposed in this paper could be helpful for the estimate of bounds of wavelet frames.

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