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DISJOINT CYCLES WITH PRESCRIBED LENGTHS AND INDEPENDENT EDGES IN GRAPHS

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Abstract. We conjecture that if $k \geq 2$ is an integer and G is a graph of order n with minimum degree at least $(n+2k)/2$, then for any k independent edges $e_1, \ldots, e_k$ in G and for any integer partition $n = n_1 + \cdots + n_k$ with $n_i \geq 4$ $(1 \leq i \leq k)$, G has k disjoint cycles $C_1, \ldots, C_k$ of orders $n_1, \ldots, n_k$, respectively, such that C_i passes through e_i for all $1 \leq i \leq k$. We show that this conjecture is true for the case $k = 2$. The minimum degree condition is sharp in general.

1. Introduction

It is well known [9] that if a graph G of order n with minimum degree at least $(n+2)/2$, then for each edge e, G has a cycle of order l passing through e for each $3 \leq l \leq n$. A set of graphs is said to be disjoint if no two of them have any vertex in common. We ask this question: Given a graph G of order $n = n_1 + \cdots + n_k$ with $n_i \geq 3$ $(1 \leq i \leq k)$ and k independent edges $e_1, \ldots, e_k$ in G, when does G have k disjoint cycles of orders $n_1, \ldots, n_k$, respectively, such that C_i passes through e_i for each $1 \leq i \leq k$? If the orders of the k cycles are not restricted, a similar problem was proposed in [8]. It was conjectured that for each integer $k \geq 2$, there exists $n_0(k)$ such that if G is a graph of order $n \geq n_0(k)$ and $d(x) + d(y) \geq n + 2k - 2$, then for any k independent edges $e_1, \ldots, e_k$ of G, G has k disjoint cycles $C_1, \ldots, C_k$ covering all the vertices of G such that C_i passes through e_i for all $1 \leq i \leq k$. This conjecture was confirmed and completely solved by Egawa, Faudree, Györi, Ishigami, Schelp and Wang in [4]. Here we propose the following conjecture:

Conjecture A. *Let $k \geq 2$ be an integer and let G be a graph of order n with minimum degree at least $(n+2k)/2$. Then for any k independent edges $e_1, \ldots, e_k$ in G and for any integer partition $n = n_1 + \cdots + n_k$ with $n_i \geq 4$ $(1 \leq i \leq k)$, G has k disjoint cycles $C_1, \ldots, C_k$ of orders $n_1, \ldots, n_k$, respectively, such that C_i contains e_i for all $1 \leq i \leq k$.*

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To see the sharpness in general, we observe $K_{(n-2(k-1))/2,(n-2(k-1))/2} + K_{2(k-1)}$. This graph has minimum degree $(n+2k)/2-1$. Let $e_1,\ldots,e_k$ be k independent edges such that $e_1,\ldots,e_{k-1}$ are taken from the clique $K_{2(k-1)}$. Let $n = n_1+\cdots+n_k$ be such that n_k is odd. Then the graph does not contain k required cycles.

In Conjecture A, the condition $n_i \geq 4$ $(1 \leq i \leq k)$ is necessary in general. This can be demonstrated in the following example with $n_i = 3$ $(1 \leq i \leq k)$. Choose positive integers a, b and k such that $a \geq k/2+1$, $b \geq 2$, $k > a+b$ and $k-b$ is even. Let K be the complete graph on $V = \{x_1, y_1, \ldots, x_k, y_k, z_1, \ldots, z_k\}$. Let (V, E) be a graph of order $3k$ with $V = \{x_1, y_1, \ldots, x_k, y_k, z_1, \ldots, z_k\}$ such that $E = E(K) - \{y_i z_j \mid a+1 \leq i \leq k,\ 1 \leq j \leq (k-b)/2\} - \{x_i z_j \mid a+1 \leq i \leq k,\ (k-b)/2+1 \leq j \leq k-b\}$. This graph does not contain k disjoint triangles containing k independent edges $x_i y_i$ $(1 \leq i \leq k)$ since $k-b > a$ and a triangle containing a vertex of $\{z_1, \ldots, z_{k-b}\}$ and an edge of $\{x_i y_i \mid 1 \leq i \leq k\}$ must contain an edge of $\{x_i y_i \mid 1 \leq i \leq a\}$. Its minimum degree is $\min\{2k-1+(k+b)/2, 2k-1+a\} \geq 5k/2$.

Magnant and Ozeki [7] discussed similar questions about disjoint cycles with approximately prescribed lengths and fixed edges where the condition on $\sigma_2(G)$ is used.

If the k disjoint cycles are not required to pass through given edges, we have El-Zahar's conjecture [5]. The conjecture says that if G is a graph of order $n = n_1 + \cdots + n_k$ with $n_i \geq 3$ $(1 \leq i \leq k)$ and minimum degree at least $\lceil n_1/2 \rceil + \cdots + \lceil n_k/2 \rceil$, then G contains k disjoint cycles of order $n_1, \ldots, n_k$, respectively. It was confirmed for the case $k = 2$ in [5]. Abbasi [1] announced a solution of this conjecture for large n using regularity lemma.

In this paper, we prove Conjecture A for the case $k = 2$:

Theorem B. *Let G be a graph of order n with minimum degree at least $(n+4)/2$. Then for any two independent edges e_1 and e_2 in G and for any integer partition $n = n_1+n_2$ with $n_1 \geq 3$ and $n_2 \geq 3$, G has two disjoint cycles C_1 and C_2 of orders n_1 and n_2, respectively, such that $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$.*

We shall use terminology and notation from [2] except as indicated. Let $G = (V, E)$ be a graph. Let $x \in V(G)$. Let H be a subset of $V(G)$ or a subgraph of G. We define $N(x, H) = \{u \in N(x) \mid u \text{ belongs to } H\}$. Let $d(x, H) = |N(x, H)|$. If X is a subset of $V(G)$ or a subgraph of G, define $N(X, H) = \cup_x N(x, H)$ and $d(X, H) = \sum_x d(x, H)$, where x runs over X. Clearly, if X and H do not have any common vertex, then $d(X, H)$ is the number of edges of G between X and H. We also use $[H]$ to denote the induced subgraph of G by the vertices in H. For $x, y \in V(G)$, define $I(xy, H) = N(x, H) \cap N(y, H)$ and let $i(xy, H) = |I(xy, H)|$. We use $e(G)$ to denote $|E(G)|$. The order of G is denoted by $|G|$.

A path from u to v is called a u-v path. If P is a path of G and v is an endvertex of P, we use $\alpha(P, v)$ to denote the order of the longest u-v subpath of P with $uv \in E(G)$. Clearly, if $\alpha(P, v) \geq 3$, then $P + uv$ has a cycle of order

$\alpha(P, v)$. Let $w \in V(G)$ ($e \in E(G)$, respectively). Let $P = w_1 w_2 \cdots w_t$ be a longest path starting at $w = w_1$ ($e = w_1 w_2$, respectively). We say that P is an optimal path at w (e, respectively) in G if $\alpha(P', x_t) \leq \alpha(P, w_t)$ for any longest path $P' = x_1 x_2 \cdots x_t$ starting at $w = x_1$ ($e = x_1 x_2$, respectively) in G. If $e \in E(P)$, we define $\sigma(P, e) = \min\{|E(P_1)|, |E(P_2)|\}$, where P_1 and P_2 are the two components of $P - e$. Thus if $\sigma(P, e) = 0$, then e is an end edge of P. For an edge $e \in E(G)$, an e-path or e-cycle is a path or a cycle that passes through e. If P is a u-v path, we define $d^*(P, H) = d(uv, H)$.

A cycle C of G is called an *end-cycle* at $u \in V(C)$ if $N(x, G) \subseteq V(C)$ and $[C]$ has a u-x hamiltonian path for each $x \in V(C - u)$.

If $C = x_1 \cdots x_t x_1$ is a cycle of G, we assume an orientation of C is given by default such that x_2 is the successor of x_1. Then $C[x_i, x_j]$ is the x_i-x_j path on C along the orientation of C and $C^-[x_i, x_j]$ is the x_i-x_j path on C in the direction against the orientation of C. Define $C[x_i, x_j) = C[x_i, x_j] - x_j$ and $C(x_i, x_j] = C[x_i, x_j] - x_i$. The predecessor and successor of x_i on C are denoted by x_i^- and x_i^+. We will use similar definitions for a path.

Let $P = x_1 \cdots x_t$ be a path of G. If $\{x_1 x_{i+1}, x_t x_i\} \subseteq E$ with $1 \leq i \leq t - 1$, we say that $x_i x_{i+1}$ is an *accessible edge* of P. Let $C = u_1 u_2 \cdots u_m u_1$ be a cycle of G. Let u_i and u_j be two distinct vertices of C. For each $e \in E(C)$, if e is an *accessible edge* of either $C[u_i, u_j]$ or $C[u_j, u_i]$, then we say that e is an *accessible edge* of C with respect to $\{u_i, u_j\}$.

2. Proof of Theorem B

In this section, we list Lemmas 2.1-2.7 and use them to prove the theorem. The proofs of these lemmas are in Section 4. Let $G = (V, E)$ be a graph order n with $\delta(G) \geq (n + 4)/2$. Suppose, for a contradiction, that theorem fails for G. Let G be a counter example with n minimal. Let $n = n_1 + n_2$ be an integer partition with $n_1 \geq 3$ and $n_2 \geq 3$ and let e_1 and e_2 be two independent edges such that G does not contain two disjoint cycles of orders n_1 and n_2 passing through e_1 and e_2, respectively. For each $X \subseteq V$ with $|X| \leq 3$, $\delta(G - X) \geq (n + 4)/2 - |X| \geq ((n - |X|) + 1)/2$ and by Lemma 3.4, $G - X$ is hamiltonian connected. If $n_1 = 3$ or $n_2 = 3$, say $n_1 = 3$ and $e_1 = xy$, then x and y have a common neighbor z that is not an endverex of e_2 because $\delta(G) \geq (n + 4)/2$. Since $G - \{x, y, z\}$ is hamiltonian connected, it has a hamiltonian cycle passing through e_2. Thus the theorem holds if $n_1 = 3$ or $n_2 = 3$. Therefore $n_1 \geq 4$, $n_2 \geq 4$ and so $n \geq 8$.

For the sake of convenience, for each $i \in \{1, 2\}$, let $\mathcal{P}_i$ be the set of all the subgraphs of G which have e_i-hamiltonian paths and $\mathcal{H}_i$ the set of all the subgraphs of G which have e_i-hamiltonian cycles. Furthermore, for each $i \in \{1, 2\}$ and $J \in \mathcal{P}_i$, let $\mathcal{P}_i(J)$ denote the set of all the e_i-hamiltonian paths of J and let $\mathcal{P}_i^*(J)$ denote the subset of $\mathcal{P}_i(J)$ such that a path $P \in \mathcal{P}_i(J)$ belongs to $\mathcal{P}_i^*(J)$ if and only if $\sigma(P, e_i) \geq 1$.

For each $i \in \{1,2\}$ and $J \in \mathcal{P}_i$, let $S_i(J)$ be the set of all the vertices x of $J - V(e_i)$ such that x is an end vertex of some $P \in \mathcal{P}_i(J)$ and let $\delta_i(J) = \min\{d(x,J) \mid x \in S_i(J)\}$.

As $\delta(G) \geq (n+4)/2$, G has a hamiltonian cycle containing both e_1 and e_2. Thus G has two disjoint subgraphs G_1 and G_2 such that for each $i \in \{1,2\}$, $|G_i| = n_i$ and $G_i \in \mathcal{P}_i$. We choose G_1 and G_2 such that

$$e(G_1) + e(G_2) \text{ is maximum.} \tag{1}$$

Let $P_1 = x_1 \cdots x_{n_1}$ and $P_2 = y_1 \cdots y_{n_2}$ be two paths such that, $P_1 \in \mathcal{P}_1(G_1)$, $P_2 \in \mathcal{P}_2(G_2)$, $x_1 \in S_1(G_1)$, $y_1 \in S_2(G_2)$, $d(x_1,G_1) = \delta_1(G_1)$ and $d(y_1,G_2) = \delta_2(G_2)$. For any $x \in V(G_1)$ and $y \in V(G_2)$, we use $\xi(x,y)$ to denote $d(x,G_2) - d(x,G_1) + d(y,G_1) - d(y,G_2) - 2d(x,y)$. Thus $e(G_1 - x + y) + e(G_2 - y + x) = e(G_1) + e(G_2) + \xi(x,y)$. By (1), we readily obtain the following Property A and Property B. The first one is evident.

Property A. Let $x \in V(G_1)$ and $y \in V(G_2)$. If $G_1 - x + y \in \mathcal{P}_1$ and $G_2 - y + x \in \mathcal{P}_2$, then $\xi(x,y) \leq 0$.

Property B. Either $\mathcal{P}_1^*(G_1) \neq \emptyset$ or $\mathcal{P}_2^*(G_2) \neq \emptyset$.

Proof. Say $\mathcal{P}_1^*(G_1) = \emptyset$ and $\mathcal{P}_2^*(G_2) = \emptyset$. Then $e_1 = x_{n_1-1}x_{n_1}$ and $N(x_{n_1}, G_1) \subseteq \{x_{n_1-1}, x_{n_1-2}\}$. Thus $n_2 \geq d(x_{n_1}, G_2) \geq (n_1 + n_2 + 4)/2 - 2 = (n_1 + n_2)/2$ and so $n_2 \geq n_1$. Similarly, $n_1 \geq (n_1+n_2)/2$. It follows that $n_1 = n_2$, $N(x_{n_1}, G_1) = \{x_{n_1-2}, x_{n_1-1}\}$ and $N(y_{n_2}, G_2) = \{y_{n_2-2}, y_{n_2-1}\}$. Thus $N(e_1, G_1) = \{x_{n_1-2}, x_{n_1-1}, x_{n_1}\}$ and $N(e_2, G_2) = \{y_{n_2-2}, y_{n_2-1}, y_{n_2}\}$. Consequently, $G_1 - V(e_1) + V(e_2) \in \mathcal{P}_2$, $G_2 - V(e_2) + V(e_1) \in \mathcal{P}_1$, $e(G_1 - V(e_1) + V(e_2)) + e(G_2 - V(e_2) + V(e_1)) > e(G_1) + e(G_2)$. This contradicts (1). □

To reach a contradiction, we will investigate the structure of G_1 and G_2 which lead us to construct a sequence $(G_1, G_2), (G_3, G_4), \ldots, (G_{2k-1}, G_{2k})$ of pairs of disjoint subgraphs of G. This will be accomplished by seven lemmas. Lemmas 2.1-2.6 are the steps to Lemma 2.7 and we use Lemma 2.7 to show that the sequence yields a contradiction.

Lemma 2.1. *Either* $d(x_1, G_1) \leq (n_1+1)/2$ *or* $d(y_1, G_2) \leq (n_2+1)/2$.

Lemma 2.2. *Either* $d(x_1, G_1) \geq (n_1+2)/2$ *or* $d(y_1, G_2) \geq (n_2+2)/2$.

By Lemma 2.1 and Lemma 2.2, we may assume without loss of generality that $d(x_1, G_1) \leq (n_1+1)/2$ and $d(y_1, G_2) \geq (n_2+2)/2$, i.e., $\delta_1(G_1) \leq (n_1+1)/2$ and $\delta_2(G_2) \geq (n_2+2)/2$. Clearly, $d(x_1, G_2) \geq (n_2+3)/2$.

Lemma 2.3. $G_2 \notin \mathcal{H}_2$.

By Lemma 2.3, $G_2 \notin \mathcal{H}_2$. As $\delta_2(G_2) \geq (n_2+2)/2$ and by Lemma 3.3, $\mathcal{P}_2^*(G_2) = \emptyset$. Let $P = v_{n_2} v_{n_2-1} \cdots v_1$ be an optimal path of G_2 at $e_2 = v_{n_2} v_{n_2-1}$. Say $\alpha(P, v_1) = r$. As $G_2 \notin \mathcal{H}_2$, $r \leq n_2 - 1$. As $\delta_2(G_2) \geq (n_2+2)/2$ and by Lemma 3.9, $J = v_1 v_2 \cdots v_r v_1$ is an end-cycle at v_r in G_2 such that $d(v_i, J) \geq (n_2+2)/2$ for all $i \in \{1, \ldots, r-1\}$. Let $J^* = \{v_2, v_3, \ldots, v_{r-2}\}$. Clearly, $r \geq (n_2+2)/2 + 1 = (n_2+4)/2$.

Lemma 2.4. *There exists no* $u \in V(G_1) - V(e_1)$ *such that* $G_1 - u \in \mathcal{P}_1$, $G_2 + u \in \mathcal{H}_2$ *and* $d(u, J^*) > 0$.

Lemma 2.5. $\delta_1(G_1) \leq (n_1 - 1)/2$.

Let $w_1 \in S_1(G_1)$ with $d(w_1, G_1) = \delta_1(G_1)$. Then $d(w_1, G_2) \geq (n_1 + n_2 + 4)/2 - (n_1 - 1)/2 = (n_2 + 5)/2$. Clearly, $d(w_1, J) \geq (n_2 + 5)/2 - (n_2 - r) \geq 9/2$. Thus $d(w_1, J^*) > 0$. By Lemma 2.4, $G_2 + w_1 \notin \mathcal{H}_2$. This implies that $w_1 v_{n_2} \notin E$ and if $v_{n_2} v_{n_2 - 2} \in E$, then $w_1 v_{n_2 - 1} \notin E$. Hence $\mathcal{P}_2^*(G_2 + w_1) = \emptyset$. For each $v \in S_2(G_2 + w_1)$, if $d(v, G_2 + w_1) \leq (n_2 + 4)/2$, then $d(v, G_1 - w_1) \geq n_1/2$ and so $G_1 - w_1 + v \in \mathcal{P}_1$ by Lemma 3.2(a). But $e(G_1 - w_1 + v) + e(G_2 + w_1 - v) > e(G_1) + e(G_2)$, contradicting (1). Hence $\delta_2(G_2 + w_1) \geq (n_1 + 5)/2$. In the meantime, we see that $n_2 - 1 \geq \lceil (n_2 + 5)/2 \rceil$. Thus $n_2 \geq 7$. With $G_1 - w_1$ and $G_2 + w_1$, this argument also implies the existence of the following two subgraphs G_3 and G_4.

Let G_3 and G_4 be two disjoint subgraphs of G with $e(G_3) + e(G_4)$ maximal such that $|G_3| = n_1 - 1$, $|G_4| = n_2 + 1$, $G_3 \in \mathcal{P}_1$, $G_4 \in \mathcal{P}_2$ and $\mathcal{P}_2^*(G_4) = \emptyset$. By the above argument, $e(G_3) + e(G_4) \geq e(G_1) + e(G_2) - (n_1 - 1)/2 + (n_2 + 5)/2$. If $d(v, G_4) \leq (|G_4| + 3)/2$ for some $v \in S_2(G_4)$, then $d(v, G_3) \geq (|G_3| + 1)/2$ and $e(G_3 + v) + e(G_4 - v) > e(G_1) + e(G_2)$. This contradicts (1) since $G_3 + v \in \mathcal{P}_1$ by Lemma 3.2(a). Thus $\delta_2(G_4) \geq (n_2 + 5)/2 = (|G_4| + 4)/2$. This argument is the key for a generalization leading to the following definition and the proofs of Lemma 2.6 and Lemma 2.7.

Let $k \geq 2$ be the largest integer such that there exist a sequence (G_1, G_2), $(G_3, G_4), \ldots, (G_{2k-1}, G_{2k})$ of disjoint pairs of subgraphs of G such that for each $i \in \{1, \ldots, k-1\}$, $G_{2i-1} \in \mathcal{P}_1$, $G_{2i} \in \mathcal{P}_2$, $\mathcal{P}_2^*(G_{2i}) = \emptyset$ and there exists $w_i \in S_1(G_{2i-1})$ such that $\delta_1(G_{2i-1}) = d(w_i, G_{2i-1}) \leq (|G_{2i-1}| - 1)/2$, $d(w_i, G_{2i}) \geq (|G_{2i}| + 5)/2$ and $G_{2i} + w_i \notin \mathcal{H}_2$. Moreover, for each $i \in \{1, \ldots, k-1\}$, $e(G_{2i+1}) + e(G_{2i+2})$ is maximal such that $|G_{2i+1}| = |G_{2i-1}| - 1$, $|G_{2i+2}| = |G_{2i}| + 1$, $G_{2i+1} \in \mathcal{P}_1$, $G_{2i+2} \in \mathcal{P}_2$ and $\mathcal{P}_2^*(G_{2i+2}) = \emptyset$. By the above argument, k is well defined.

Lemma 2.6. *The following two statements hold*:

(a) *For each* $i \in \{1, \ldots, k\}$, $|G_{2i-1}| = n_1 - i + 1$ *and* $|G_{2i}| = n_2 + i - 1$.

(b) *For each* $i \in \{1, \ldots, k\}$, $\delta_2(G_{2i}) \geq (|G_{2i}| + 4)/2$.

Say $s = |G_{2k-1}|$ and $|G_{2k}| = t$. As $n_2 \geq 7$, $t \geq 8$. By Lemma 2.6, $\delta_2(G_{2k}) \geq (t + 4)/2$. Let $L = y_t y_{t-1} \cdots y_1$ be an optimal path at $e_2 = y_t y_{t-1}$ in G_{2k}. Say $r = \alpha(L, y_1)$. Then $r \geq \delta_2(G_2) + 1 \geq \lceil (t + 4)/2 + 1 \rceil = \lceil (t + 6)/2 \rceil \geq 7$. As $\mathcal{P}_2^*(G_{2k}) = \emptyset$, $r \leq t - 1$. Let $R = [y_1, y_2, \ldots, y_r]$ and $R' = R - y_r$. By Lemma 2.6 and Lemma 3.9, $y_1 y_2 \cdots y_r y_1$ is an end-cycle at y_r in G_{2k} and so $\delta(R') \geq (t + 4)/2 - 1 \geq (|R'| + 4)/2$. By the minimality of $|G|$, Theorem B holds for R'. Note that $R' - \{x, y\}$ is hamiltonian connected for all $\{x, y\} \subseteq V(R')$ by Lemma 3.4. Clearly, $s \geq d(y_t, G_{2k-1}) \geq (s + t + 4)/2 - 2 = (s + t)/2$. This implies that $s \geq t$ and if equality holds, then $N(y_t, G_{2k}) = \{y_{t-1}, y_{t-2}\}$ and $r \leq t - 2$.

Lemma 2.7. *For no* $x \in V(G_{2k-1})$, $G_{2k-1} - x \in \mathcal{P}_1$, $G_{2k} + x \in \mathcal{H}_2$ *and* $d(x, R' - \{y_1, y_{r-1}\}) > 0$.

To prove Theorem B, let $y_c \in V(R' - \{y_1, y_{r-1}\})$. Then $d(y_c y_t, G_{2k-1}) \geq s+t+4-(t-1) = s+5$ and so $i(y_c y_t, G_{2k-1}) \geq 5$. By Lemma 2.7, $G_{2k-1} - x \notin \mathcal{P}_1$ for all $x \in I(y_c y_t, G_{2k-1})$ and so $G_{2k-1} \notin \mathcal{H}_1$. If $\delta_1(G_{2k-1}) \leq (s-1)/2$, let $w_k \in S_1(G_{2k-1})$ with $d(w_k, G_{2k-1}) = \delta_1(G_{2k-1})$. As $d(w_k, G_{2k}) \geq (t+5)/2$, $d(w_k, R' - \{y_1, y_{r-1}\}) \geq 1$. By Lemma 2.7, $G_{2k} + w_k \notin \mathcal{H}_2$. Thus $w_k y_t \notin E$ and if $y_t y_{t-2} \in E$, then $w_k y_{t-1} \notin E$. Therefore $\mathcal{P}_2^*(G_{2k} + w_k) = \emptyset$. This allows us to define (G_{2k+1}, G_{2k+2}) to lengthen the sequence $(G_1, G_2), \ldots, (G_{2k-1}, G_{2k})$. This contradicts the maximality of k. Therefore $\delta_1(G_{2k-1}) \geq s/2$. Recall that $d(y_t, G_{2k-1}) \geq (s+t)/2$. If $\mathcal{P}_1^*(G_{2k-1}) \neq \emptyset$, then by Lemma 3.5(c), we see that G_{2k-1} has a u-v e_1-hamiltonian path such that $v \notin V(e_1)$, $d(v, G_{2k-1}) = s/2$ and $v y_t \in E$. As $d(v, G_{2k}) \geq (t+4)/2$, $d(v, R' - \{y_1, y_{r-1}\}) > 0$ and so $G_{2k} + v \in \mathcal{H}_2$, contradicting Lemma 2.7. Therefore $\mathcal{P}_1^*(G_{2k-1}) = \emptyset$. Let $P = z_s z_{s-1} \cdots z_1$ be an optimal path at $e_1 = z_s z_{s-1}$ in G_{2k-1}. Say $\alpha(P, z_1) = q$. As $d(z_s, G_{2k-1}) \leq 2$, $t \geq d(z_s, G_{2k}) \geq (s+t+4)/2 - 2$ and so $t \geq s$. Since $s \geq t$, it follows that $s = t$ and $d(z_s, G_{2k}) = t = d(y_t, G_{2k-1})$. By Lemma 2.7, we see that $d(z_i, R' - \{y_1, y_{r-1}\}) = 0$ for all $i \in \{1, \ldots, q-1\}$. Then $t + 2 \leq d(y_c, G) \leq r - 1 + d(y_c, G_{2k-1}) \leq r - 1 + t - q + 1 = t + r - q$. Thus $r - q \geq 2$. Then $t + 2 \leq d(z_1, G) \leq q - 1 + d(z_1, G_{2k}) \leq q - 1 + t - r + 3 \leq t$, a contradiction. This proves the theorem.

3. Auxiliary lemmas

In the following, $G = (V, E)$ is a graph. We will use the following lemmas. Lemma 3.1 is an easy observation.

Lemma 3.1. *Let* $P = x_1 \cdots x_r$ *be a path of order* r *in* G. *Let* u *and* v *be two vertices of* $G - V(P)$. *Suppose that* $d(uv, P) \geq r+1$ *and* $\{ux_{i+1}, vx_i\} \not\subseteq E$ *for all* $i \in \{1, \ldots, r-1\}$. *Then* $d(uv, P) = r+1$ *and* $\{ux_1, vx_r\} \subseteq E$. *Moreover, either* $N(u, P) = \{x_1, \ldots, x_a\}$ *and* $N(v, P) = \{x_a, \ldots, x_r\}$ *for some* $a \in \{1, \ldots, r\}$, *or* $d(x_i, uv) = 0$ *for some* $1 < i < r$. □

Lemma 3.2. *Let* P *be a* u-v *path of order* r *in* G, $e \in E(P)$ *and* $x \in V(G) - V(P)$. *The following five statements hold*:

(a) *If* $d(x, P) > r/2$, *then* $P + x$ *has an* e*-hamiltonian path.*
(b) *If* $d(x, P) > (r+1)/2$, $P + x$ *has an* e*-hamiltonian path ending at* v.
(c) *If* $d(x, P) > (r+2)/2$, *then* $P + x$ *has a* u-v e*-hamiltonian path.*
(d) *If* $d(xv, P) \geq r+2$, *then* $[P + x]$ *has a* u-x e*-hamiltonian path.*
(e) *If* $d(xv, P) \geq r+1$, *then* $[P + x]$ *has an* e*-hamiltonian path.*
(f) *If* $d(x, P) > (r+1)/2$ *and* $uv \in E$, *then* $P + uv + x$ *has an* e*-hamiltonian cycle.*

Proof. Let P_1 and P_2 be the two components of $P - e$ with v in P_2. If $d(x, f) = 2$ for some $f \in E(P - e)$, then (a), (b) and (c) hold. So if one of (a), (b) and (c) fails, then $d(x, f) \le 1$ for all $f \in E(P) - \{e\}$. This implies that $d(x, P_i) \le (|P_i| + 1)/2$ for $i \in \{1, 2\}$ and so $d(x, P) \le (r + 2)/2$. Furthermore, for each $i \in \{1, 2\}$, if $d(x, P_i) = (|P_i| + 1)/2$, then $|P_i|$ is odd and x is adjacent to the two endvertices of P_i and so the first three statements follow.

If one of (d) and (e) fails, then $\{vz, xz^+\} \not\subseteq E$ for each $z \in V(P)$ with $zz^+ \ne e$. This implies that $d(xv, P) \le r + 1$. So (d) holds. Obviously, (e) would hold if $uv \in E$ or $d(x, uv) > 0$. To see (e), say $uv \notin E$ and $d(x, uv) = 0$. Then apply (d) to $P - u$ and x.

To obtain (f), we see that there exists an edge e' on $P + uv$ with $e' \ne e$ such that $d(x, e') = 2$. □

Lemma 3.3. *Let P be a u-v path of order $r \ge 3$ in G. Let $e \in E(P)$. Suppose that $d(uv, P) \ge r + \epsilon$ where $\epsilon = 0$ if $\sigma(P, e) = 0$ and $\epsilon = 1$ if $\sigma(P, e) > 0$. Then $[P]$ has an e-hamiltonian cycle.*

Proof. If $uv \in E$, nothing to prove. So assume $uv \notin E$. Then the condition implies that some edge $f \in E(P) - \{e\}$ is an accessible edge and this yields a required cycle. □

Lemma 3.4 ([3]). *If H is a graph of order $r \ge 3$ and $d(xy, H) \ge r+1$ for each pair x and y of nonadjacent vertices of H, then H is hamiltonian connected and so for each $e \in E(H)$, H has an e-hamiltonian cycle.*

Lemma 3.5. *Let $P = x_1 \cdots x_r$ be a path of order $r \ge 3$ in G. Let $e \in E(P)$. Suppose that $[P]$ does not have an e-hamiltonian cycle and $d(x_1x_r, P) \ge r$. Let $R = \{x_i \mid d(x_i, x_1x_r) = 0, 1 < i < r\}$ and $\mathcal{P}$ be the set of all the components of $P - R \cup \{x_1, x_r\} - e$. Then $\sigma(e, P) > 0$, $d(x_1x_r, P) = r$ and the following three statements hold*:

(a) *$R \cup \{x_1, x_r\}$ is an independent set*;
(b) *$d(x_l, P') \le 1$ for all $x_l \in R$ and $P' \in \mathcal{P}$*;
(c) *If $d^*(L, P) \ge r$ for every e-hamiltonian path L of $[P]$ with $\sigma(L, e) > 0$, then either $V(P)$ has a partition $X \cup Y$ such that $|X| = r/2$, $V(e) \subseteq X$, $Y = R \cup \{x_1, x_r\}$ and $N(y, P) = X$ for all $y \in Y$, or $[P] - V(e)$ has two complete components H_1 and H_2 such that $|H_1| + |H_2| = r - 2$ and $V(H_1 \cup H_2) \subseteq N(x)$ for each $x \in V(e)$.*

Proof. By Lemma 3.3, $\sigma(e, P) > 0$ and $d(x_1x_r, P) = r$. Clearly, $|\mathcal{P}| \le |R| + 2$ and $|\mathcal{P}| + |R| \le \sum_{P' \in \mathcal{P}} |P'| + |R| \le r - 2$. Say $e = x_a x_{a+1}$. Since $[P]$ does not have an e-hamiltonian cycle, each $x_i x_{i+1}$ with $i \ne a$ is not an accessible edge of P. By Lemma 3.1, $d(x_1x_r, P') \le |P'| + 1$ for each $P' \in \mathcal{P}$. Thus $d(x_1x_r, P) \le (r-2) - |R| + |\mathcal{P}| \le r$. It follows that $|\mathcal{P}| = |R| + 2$ and $d(x_1x_r, P') = |P'| + 1$ for each $P' \in \mathcal{P}$. Consequently, $\{x_2, x_a, x_{a+1}, x_{r-1}\} \cap R = \emptyset$, R does not contain two consecutive vertices of P, and for each $P' = P[x_i, x_j] \in \mathcal{P}$ there exists $i \le k \le j$ such that $N(x_1, P') = \{x_i, \ldots, x_k\}$ and $N(x_r, P') = \{x_k, \ldots, x_j\}$. In particular, $\{x_1x_{a+1}, x_r x_a\} \subseteq E$. It is easy to see that R is an independent set for otherwise $[P]$ has an e-hamiltonian cycle. So (a) holds.

To see (b), say $d(x_l, P') \geq 2$ for some $x_l \in R$ and $P' = P[x_i, x_j] \in \mathcal{P}$. Let $x_k \in V(P')$ be such that $N(x_1, P') = \{x_i, \ldots, x_k\}$ and $N(x_r, P') = \{x_k, \ldots, x_j\}$. Say without loss of generality that $l < i$. Let $x_p \in V(P')$ be such that $x_l x_p \in E$ and $p \neq i$. If $p \leq k$, then

$$x_1 P[x_1, x_{l-1}]P^-[x_r, x_p]x_l P[x_{l+1}, x_{p-1}]x_1$$

is an e-hamiltonian cycle of $[P]$ and if $p > k$, then

$$x_1 P[x_1, x_l]P[x_p, x_r]P^-[x_{p-1}, x_{l+1}]x_1$$

is an e-hamiltonian cycle of $[P]$, a contradiction. Hence (b) holds.

To see (c), it is easy to observe that for each $x_l \in R$, $[P]$ has an x_1-x_l e-hamiltonian path and an x_r-x_l e-hamiltonian path. If $R \neq \emptyset$, then $d(x_l x_1, P) \geq r$, $d(x_l x_r, P) \geq r$ and so $d(x_l, P) \geq r/2$ for each $x_l \in R$. Since $|\mathcal{P}| = |R| + 2$ and $|\mathcal{P}| + |R| \leq r - 2$, it follows that $|\mathcal{P}| = r/2$ and $|P'| = 1$ for all $P' \in \mathcal{P}$. Thus $X \cup Y$ with $Y = R \cup \{x_1, x_r\}$ and $X = V(P) - Y$ is a partition of $V(P)$ satisfying (c). Next, assume that $R = \emptyset$. Let $2 \leq b \leq a$ and $a + 1 \leq c \leq r - 1$ be such that $N(x_1, P) = \{x_2, \ldots, x_b\} \cup \{x_{a+1}, \ldots, x_c\}$ and $N(x_r, P) = \{x_b, \ldots, x_a\} \cup \{x_c, \ldots, x_{r-1}\}$. Then we readily see that for each $x_i \in N(x_1, P) - \{x_b, x_a, x_{a+1}, x_c\}$ and $x_j \in N(x_r, P) - \{x_b, x_a, x_{a+1}, x_c\}$, $[P]$ has an x_i-x_r e-hamiltonian path, an x_1-x_j e-hamiltonian path, an x_i-x_j e-hamiltonian path and so $x_i x_j \notin E$. It follows that $N(x_i, P) \cup \{x_i\} = N(x_1, P) \cup \{x_1\}$ and $N(x_j, P) \cup \{x_j\} = N(x_r, P) \cup \{x_r\}$ for all $x_i \in N(x_1, P) - \{x_b, x_a, x_{a+1}, x_c\}$ and $x_j \in N(x_r, P) - \{x_b, x_a, x_{a+1}, x_c\}$. Thus if $b < a$, then $x_1 P^-[x_c, x_{b+1}]P^-[x_r, x_{c+1}]P^-[x_b, x_1]$ is an e-hamiltonian cycle of $[P]$, a contradiction. Hence $b = a$. Similarly, $c = a + 1$. This proves (c). □

Lemma 3.6. *Let C be a cycle of order r in G. Let u and v be two distinct vertices on C and e an edge of C with $e \notin \{uu^+, vv^+\} = \emptyset$. Set $R = \{x \mid d(x, uv) = 0, x \in V(C) - \{u, v\}\}$. Let $\mathcal{P}$ be the set of all the components of $C - (R \cup \{u, v\}) - e$. Suppose that $d(uv, C) \geq r + 1$ and $[C]$ does not have a u^+-v^+ e-hamiltonian path. Then $d(uv, C) = r + 1$ and the following four statements hold*:

- (a) *Each edge of $C - e$ is inaccessible on C with respect to $\{u, v\}$;*
- (b) *$V(e) \cap (R \cup \{u, v\}) = \emptyset$, $d(uv, P) = |P| + 1$ for all $P \in \mathcal{P}$ and $|\mathcal{P}| = |R| + 3$.*
- (c) *R is an independent set and $d(x, P) \leq 1$ for all $x \in R$ and $P \in \mathcal{P}$.*
- (d) *If $d(z, C) \geq (r+1)/2$ for all $z \in V(C) - V(e)$, then r is odd. Moreover, either $[C]$ has a vertex-cut X with $V(e) \subseteq X$ and $|X| = 3$ such that $[C]$ has exactly two components isomorphic to $K_{(r-3)/2}$ and $X \subseteq N(y)$ for all $y \in V(C) - X$, or $V(C)$ has a partition $X \cup Y$ such that $|X| = (r+1)/2$, $|Y| = (r-1)/2$, $Y = R \cup \{u, v\}$, $V(e) \subseteq X$ and $N(y, C) = X$ for all $y \in Y$.*

Proof. It is easy to check that (a) holds since $[C]$ does not have a u^+-v^+ e-hamiltonian path. In particular, $uv \notin E$. Clearly, $|\mathcal{P}| \leq |R| + 3$ and $|\mathcal{P}| + |R| \leq$

$\sum_{P\in\mathcal{P}} |P| + |R| = r - 2$. By (a) and Lemma 3.1, $d(uv, P) \le |P| + 1$ for each $P \in \mathcal{P}$ and so $d(uv, C) \le r + 1$. Since $d(uv, C) \ge r + 1$, it follows that $d(uv, C) = r + 1$, $|\mathcal{P}| = |R| + 3$, $V(e) \cap (R \cup \{u, v\}) = \emptyset$, and $d(uv, P) = |P| + 1$ for all $P \in \mathcal{P}$. So (b) holds.

As $|\mathcal{P}| = |R|+3$, R does not contain two consecutive vertices of C. To proves (c), Let $C = x_1 \cdots x_r x_1$ be such that $x_1 = u$, $x_2 = u^+$, $x_p = v$ and $x_{p+1} = v^+$. Without loss of generality, say $e = x_q x_{q+1}$ for some $q \in \{p + 1, \ldots, r - 1\}$. We first check that R is an independent set. Let $L_1 = C(x_1, x_p)$, $L_2 = C(x_p, x_q]$ and $L_3 = C[x_{q+1}, x_r]$. Let $R_i = R \cap V(L_i)$ for $i \in \{1, 2, 3\}$. Say $x_i x_j \in E$ for some $\{x_i, x_j\} \subseteq R$ with $i < j$. We shall obtain a contradiction by showing that $[C]$ has an x_2-x_{p+1} e-hamiltonian path. According to the locations of x_i and x_j in $R = R_1 \cup R_2 \cup R_3$, there are six cases to check, which are very similar in the verification. So we just show one example with $x_i \in R_1$ and $x_j \in R_3$. In this case, $\{x_1 x_{i+1}, x_p x_{i-1}\} \subseteq E$ and $\{x_1 x_{j-1}, x_p x_{j+1}\} \subseteq E$ by (a), (b) and Lemma 3.1. Then

$$x_2 C[x_2, x_{i-1}] x_p C^-[x_p, x_i] x_i x_j C[x_j, x_1] x_{j-1} C^-[x_{j-1}, x_{p+1}]$$

is an x_2-x_{p+1} e-hamiltonian path of $[C]$, a contradiction.

Next, we show that $d(x, P) \le 1$ for all $x \in R$ and $P \in \mathcal{P}$. On the contrary, say $d(x, P) \ge 2$ for some $x \in R$ and $P \in \mathcal{P}$. We shall obtain a contradiction by showing that $[C]$ has an x_2-x_{p+1} e-hamiltonian path. According to the locations of x in $R_1 \cup R_2 \cup R_3$ and P on $L_1 \cup L_2 \cup L_3$, there are nine cases to check, which are also very similar in the verification. So we just show one example with $x \in R_1$ and P on L_3. Say $P = C[x_i, x_j]$. By (a), (b) and Lemma 3.1, $N(x_1, P) = \{x_a, \ldots, x_j\}$ and $N(x_p, P) = \{x_i, \ldots, x_a\}$ for some $i \le a \le j$. Since $d(x, P) \ge 2$, $x x_t \in E$ for some $x_t \in V(P)$ with $t \ne x_i$. If $t > a$, then

$$x_2 C[x_2, x^-] x_p C^-[x_p, x] x x_t C[x_t, x_1] x_{t-1} C^-[x_{t-1}, x_{p+1}]$$

is an x_2-x_{p+1} e-hamiltonian path of $[C]$, a contradiction. Thus $t \le a$. Then

$$x_2 C[x_2, x] x x_t C[x_t, x_1] x^+ C[x^+, x_p] x_{t-1} C^-[x_{t-1}, x_{p+1}]$$

is an x_2-x_{p+1} e-hamiltonian path of $[C]$, a contradiction.

To prove (d), we have $d(x, C) \le |\mathcal{P}|$ for all $x \in R$ by (c). Since $|\mathcal{P}| \le r - |R| - 2$ and $|\mathcal{P}| = |R| + 3$, we obtain $d(x, C) \le (r + 1)/2$ for all $x \in R$. It follows that if $R \ne \emptyset$, then r is odd and $|P| = 1$ for all $P \in \mathcal{P}$. Consequently, if $Y = R \cup \{u, v\}$ and $X = V(C) - Y$, then $N(y, C) = X$ for all $y \in Y$ and so (d) holds. So assume that $R = \emptyset$. By (a), (b) and Lemma 3.1, there exists $x_{a_i} \in V(L_i)$ for $i \in \{1, 2, 3\}$ such that

$$N(x_1, C) = V(L_1[x_2, x_{a_1}]) \cup V(L_2[x_{a_2}, x_q]) \cup V(L_3[x_{a_3}, x_r]),$$
$$N(x_p, C) = V(L_1[x_{a_1}, x_{p-1}]) \cup V(L_2[x_{p+1}, x_{a_2}]) \cup V(L_3[x_{q+1}, x_{a_3}]).$$

We claim that for each vertex x of $L_1[x_2, x_{a_1}) \cup L_2(x_{a_2}, x_q) \cup L_3(x_{a_3}, x_r]$, $N(x, C) \subseteq N(x_1, C) \cup \{x_1\}$. If this is false, say $xy \in E(G)$ for some vertex x of $L_1[x_2, x_{a_1}) \cup L_2(x_{a_2}, x_q) \cup L_3(x_{a_3}, x_r]$ and $y \in V(C) - N(x_1, C) - \{x_1\}$. We shall obtain a contradiction by showing that $[C]$ has an x_2-x_{p+1} e-hamiltonian path. According to the locations of x in $L_1[x_2, x_{a_1}) \cup L_2(x_{a_2}, x_q) \cup L_3(x_{a_3}, x_r]$ and y on $L_1 \cup L_2 \cup L_3$, there are nine cases to check, which are very similar in the verification. So we just show one example with x in $L_3(x_{a_3}, x_r]$ and y on $L_1(x_{a_1}, x_{p-1}]$. In this case,

$$x_2C[x_2, y^-]x_pC^-[x_p, y]xC[x, x_1]x^-C^-[x^-, x_{p+1}]$$

is an x_2-x_{p+1} e-hamiltonian path of $[C]$, a contradiction.

Similarly, $N(y, C) \subseteq N(x_p, C) \cup \{x_p\}$ for each vertex y of $L_1(x_{a_1}, x_{p-1}] \cup L_2[x_{p+1}, x_{a_2}) \cup L_3(x_{q+1}, x_{a_3})$. As $d(x, C) \geq (r+1)/2$ for all $x \in V(C) - V(e)$, we see that r is odd and $d(x_1, C) = d(x_p, C) = (r+1)/2$. Furthermore, if $\{x_{a_2}, x_{a_3}\} = \{x_q, x_{q+1}\}$, then $\{x_{a_1}, x_q, x_{q+1}\}$ is a vertex-cut of $[C]$ and each component of $[C] - \{x_{a_1}, x_q, x_{q+1}\}$ is isomorphic to $K_{(r-3)/2}$. Consequently, (d) holds. So assume that $\{x_{a_2}, x_{a_3}\} \neq \{x_q, x_{q+1}\}$. We shall obtain a contradiction by showing that $[C]$ has an x_2-x_{p+1} e-hamiltonian path. If $x_{q+1} \neq x_{a_3}$, then

$$x_2C[x_2, x_{a_1}]x_1C^-[x_1, x_{q+2}]x_{a_1+1}C[x_{a_1+1}, x_p]x_{q+1}C^-[x_{q+1}, x_{p+1}]$$

is an x_2-x_{p+1} e-hamiltonian path of $[C]$, a contradiction. Therefore $x_{q+1} = x_{a_3}$ and $x_q \neq x_{a_2}$. Then

$$x_2C[x_2, x_p]x_{q+1}x_qx_1C^-[x_1, x_{q+1}^+]x_{q-1}C^-[x_{q-1}, x_{p+1}]$$

is an x_2-x_{p+1} e-hamiltonian path of $[C]$, a contradiction. This proves the lemma. □

Lemma 3.7. *Let C be a cycle of order r in G. Let λ be a positive integer. Let $e \in E(C)$. Suppose that $d^*(P, C) \geq r + \lambda$ for every e-hamiltonian path P of $[C]$. Then $d(xy, C) \geq r + \lambda$ for every pair x and y of distinct vertices of C with $V(e) \neq \{x, y\}$.*

Proof. On the contrary, say that there are two distinct vertices x and y on C with $V(e) \neq \{x, y\}$ such that $d(xy, C) \leq r + \lambda - 1$. Clearly, either $e \notin \{xx^-, yy^-\}$ or $e \notin \{xx^+, yy^+\}$. Say without loss of generality the former holds. Then $d(xx^-, C) \geq r + \lambda$ and $d(yy^-, C) \geq r + \lambda$. Thus $d(x^-y^-, C) \geq 2(r+\lambda) - (r + \lambda - 1) \geq r + 2$. By Lemma 3.6, $[C]$ has an x-y e-hamiltonian path and therefore $d(xy, C) \geq r + \lambda$, a contradiction. □

Lemma 3.8. *Let $C = x_1 \cdots x_r x_1$ be a cycle in G. Let $e = x_1x_2$. Suppose that $d^*(P, C) \geq r+1$ for each e-hamiltonian path P of $[C]$ with $\sigma(P, e) > 0$. If there exists $x_j \in V(C) - V(e)$ such that $d(x_j, C) \leq r/2$, then one of the following two statement holds*:

(a) *If $4 \leq j \leq r-1$, then $d(x_i, C) \geq (r+2)/2$ for all $3 \leq i \leq r$ with $i \neq j$;*
(b) *If $j \in \{3, r\}$, then $d(x_i, C) \geq (r+2)/2$ for all $4 \leq i \leq r-1$.*

Proof. To prove (a), say $4 \leq j \leq r-1$. Then $d(x_{j-1}, C) \geq r + 1 - d(x_j, C) \geq (r+2)/2$. Similarly, $d(x_{j+1}, C) \geq (r+2)/2$. If $d(x_i, C) \leq (r+1)/2$ for some $3 \leq i \leq r$ with $i \neq j$, let x_i be the one closest to x_j on $C - e$. Say without loss of generality $i > j$. Then $d(x_{i-1}, C) \geq (r+2)/2$. Thus $d(x_{j-1}x_{i-1}, C) \geq r+2$. By Lemma 3.6, $[C]$ has an x_j-x_i e-hamiltonian path and so $d(x_ix_j, C) \geq r+1$. Thus $d(x_i, C) \geq r+1-r/2 = (r+2)/2$, a contradiction.

To prove (b), say without loss of generality that $d(x_3, C) \leq r/2$, i.e., $d(x_3, C) \leq \lfloor r/2 \rfloor$. If $r \leq 4$, nothing to prove. So assume $r \geq 5$. Then $d(x_4, C) \geq r+1-\lfloor r/2 \rfloor = \lceil (r+2)/2 \rceil$. Similarly, if $d(x_r, C) \leq r/2$, then $d(x_{r-1}, C) \geq \lceil (r+2)/2 \rceil$ and so $d(x_4x_{r-1}, C) \geq r+2$. If $d(x_r, C) \not\leq r/2$, i.e., $d(x_r, C) \geq \lceil (r+1)/2 \rceil$, then $d(x_4x_r, C) \geq \lceil (r+2)/2 \rceil + \lceil (r+1)/2 \rceil = r+2$. Let $s \in \{r-1, r\}$ be maximal such that $d(x_4x_s, C) \geq r+2$. If $d(x_i, C) \leq (r+1)/2$ for some $i \in \{5, \ldots, r-1\}$, let x_i be the one closest to x_s on $C-e$. Then $d(x_4x_{i+1}, C) \geq r+2$. By Lemma 3.6, $[C]$ has an x_3-x_i e-hamiltonian path and so $d(x_i, C) \geq r+1-r/2 = (r+2)/2$, a contradiction. □

Lemma 3.9 ([6])**.** *Let $P = x_tx_{t-1}\cdots x_1$ be an optimal path at x_t in G. Let $r = \alpha(P, x_1)$ and $c > r/2$. Suppose that for each $v \in V(G)$, if there exists a longest path starting at x_t in G such that the path ends at v, then $d(v) \geq c$. Then $N(x_i) \subseteq \{x_1, x_2, \ldots, x_r\}$, $[P]$ has an x_t-x_i hamiltonian path and $d(x_i) \geq c$ for all $i \in \{1, 2, \ldots, r-1\}$. Moreover, if $t > r$, then x_r is a cut-vertex of G.*

Lemma 3.10. *Let $P = x_tx_{t-1}\cdots x_1$ be an optimal path at x_t in G. Let $r = \alpha(P, x_1)$. Suppose that $r \geq 3$ and for each $v \in V(G)$, if there exists a longest path starting at x_t in G such that the path ends at v, then $d(v) \geq (r+2)/2$. Then for each pair x_i and x_j of distinct vertices in $\{x_1, x_2, \ldots, x_{r-1}\}$, the following three statements hold*:

(a) *If $d(x_r, \{x_1, x_2, \ldots, x_{r-1}\}) \geq 3$, then $[P]-x_i$ has an x_t-x_j hamiltonian path*;

(b) *If $N(x_r, \{x_1, x_2, \ldots, x_{r-1}\}) = \{x_1, x_{r-1}\}$ but $i \notin \{1, r-1\}$, then $[P]-x_i$ has an x_t-x_j hamiltonian path*;

(c) *If $N(x_r, \{x_1, x_2, \ldots, x_{r-1}\}) = \{x_1, x_{r-1}\}$ and $i \in \{1, r-1\}$ but $j \notin \{1, r-1\}$, then $[P]-x_i$ has an x_t-x_j hamiltonian path.*

Proof. Obviously, the lemma is true if $r \leq 4$. So assume $r \geq 5$. Let $H = [\{x_1, \ldots, x_r\} - \{x_i\}]$. By Lemma 3.9, for each $x_l \in \{x_1, \ldots, x_{r-1}\}$, $[P]$ has an x_t-x_l hamiltonian path, $N(x_l, G) \subseteq V(H) \cup \{x_i\}$ and $d(x_l, H + x_i) \geq (r+2)/2$. Moreover, x_r is a cut-vertex of $[P]$ if $t > r$, and consequently, $H + x_i$ has an x_r-x_i hamiltonian path and so H has a hamiltonian path starting at x_r. Obviously, for each $v \in V(H - x_r)$, $d(v, H) \geq (r+2)/2 - 1 = ((r-1)+1)/2$. Let L be an optimal path at x_r in H. Say L is an x_r-y path. Then $\alpha(L, y) \leq r-1$. As $\delta(H - x_r) \geq (r+2)/2 - 2 = (r-2)/2$, $H - x_r$ is hamiltonian. If $d(x_r, H) \geq 2$, then H is 2-connected and by applying Lemma 3.9 to L in H, we see that $\alpha(L, y) = r-1$. Consequently, H has an x_r-x_j hamiltonian path and so $[P - x_i]$ has an x_t-x_j hamiltonian path. Therefore (a) and (b) hold. If $d(x_r, H) = 1$, then $x_i \in \{x_1, x_{r-1}\}$ and so $\alpha(L, y) = r-2$. Moreover, the vertex z with $\{x_i, z\} = \{x_1, x_{r-1}\}$ is a cut-vertex of H. To see (c), we have $x_j \notin \{x_1, x_{r-1}\}$ and H has an x_r-x_j hamiltonian path. □

4. Proof of Lemmas 2.1-2.7

Proof of Lemma 2.1. On the contrary, say $d(x_1, G_1) \geq (n_1+2)/2$ and $d(y_1, G_2) \geq (n_2+2)/2$, i.e., $\delta_1(G_1) \geq (n_1+2)/2$ and $\delta_2(G_2) \geq (n_2+2)/2$. Say without loss of generality $G_1 \notin \mathcal{H}_1$. By Lemma 3.3, we see that $\mathcal{P}_1^*(G_1) = \emptyset$. Let $P = u_{n_1}u_{n_1-1}\cdots u_1$ be an optimal path at $e_1 = u_{n_1}u_{n_1-1}$ in G_1. Then $N(u_{n_1}, G_1) \subseteq \{u_{n_1-1}, u_{n_1-2}\}$. Say $\alpha(P, u_1) = r$. As $\delta_1(G_1) \geq (n_1+2)/2$ and by Lemma 3.9, $u_1 \cdots u_r u_1$ is an end-cycle at u_r in G_1 and for each $j \in \{1, \ldots, r-1\}$, G_1 has a u_{n_1}-u_j e_1-hamiltonian path and $d(u_j, G_1) \geq (n_1+2)/2$. Since $n_2 \geq d(u_{n_1}, G_2) \geq (n+4)/2 - d(u_{n_1}, G_1) \geq n/2$, we obtain $n_2 \geq n_1$. Note that $r - 1 \geq (n_1+2)/2$ and so $n_1 \geq 6$.

By Property B, $\mathcal{P}_2^*(G_2) \neq \emptyset$. As $\delta_2(G_2) \geq (n_2+2)/2$ and by Lemma 3.3, $G_2 \in \mathcal{H}_2$. Thus $d(y, G_2) \geq (n_2+2)/2$ for all $y \in V(G_2) - V(e_2)$. Let $v_1 \cdots v_{n_2} v_1$ be a hamiltonian cycle of G_2 with $e_2 = v_1v_2$. Let $i, j \in \{1, \ldots, r-1\}$ with $i \notin \{1, r-1\}$. By Lemma 3.10, $G_1 - u_i$ has an u_{n_1}-u_j e_1-hamiltonian path. Clearly, $d(u_{n_1}u_j, G_2) \geq n + 4 - (n_1 - 1) = n_2 + 5$. Thus for some $s \in \{4, \ldots, n_2-1\}$, $d(v_s, u_{n_1}u_j) = 2$ and so $G_1 - u_i + v_s \in \mathcal{H}_1$. Thus $G_2 - v_s + u_i \notin \mathcal{H}_2$. As $d(v_{s-1}v_{s+1}, G_2 - v_s) \geq n_2 + 2 - 2 = n_2$ and by Lemma 3.3, $G_2 - v_s \in \mathcal{H}_2$. Let $C = w_1 \cdots w_t w_1$ be an e_2-hamiltonian cycle of $G_2 - v_s$ with $t = n_2 - 1$. As $d(u_i, G_1) \leq n_1 - 2$, $d(u_i, C) \geq (n+4)/2 - (n_1 - 2) - 1 \geq 3$. As $C + u_i \notin \mathcal{H}_2$, we see that there are two distinct vertices u and v in C such that $\{u, v\} \cap V(e_2) = \emptyset$ and either $\{u^+, v^+\} \subseteq N(u_i)$ or $\{u^-, v^-\} \subseteq N(u_i)$. Say without loss of generality $\{u^+, v^+\} \subseteq N(u_i)$. As $G_2 - v_s + u_i \notin \mathcal{H}_2$, $[C]$ does not have a u^+-v^+ e_2-hamiltonian path. Clearly, $d(x, C) \geq (n_2+2)/2 - 1 = (t+1)/2$ for all $x \in V(C) - V(e_2)$. Thus we may apply Lemma 3.6(d) to $[C]$. First, assume that $[C]$ has a vertex-cut X with $|X| = 3$ and $V(e_2) \subseteq X$ such that each of the two components $[C] - X$ is isomorphic to $K_{(t-3)/2}$. As $G_2 - v_s + u_i \notin \mathcal{H}_2$, we see that $N(u_i, C) = X$. Thus $v_s x \in E$ for all $x \in V(C) - X$ as $\delta_2(G_2) \geq (n_2+2)/2$. Let $v' \in I(u_{n_1}u_j, C - X)$. Then $G_1 - u_i + v' \in \mathcal{H}_1$ and $G_2 - v' + u_i \in \mathcal{H}_2$ by Lemma 3.10, a contradiction. Therefore $V(C)$ has a partition $X \cup Y$ such that $|X| = (t+1)/2$, $V(e_2) \subseteq X$, $|Y| = (t-1)/2$, $\{u, v\} \subseteq Y$ and $N(y, C) = X$ for all $y \in Y$. As $\delta_2(G_2) \geq (n_2+2)/2$, we obtain $Y \subseteq N(v_s)$. As $G_2 - v_s + u_i \notin \mathcal{H}_2$, we see that $N(u_i, C) \subseteq X$. As $d(u_{n_1}, G_1) \leq 2$, we readily see that $d(u_{n_1}, Y) > 0$. Let $v' \in N(u_{n_1}, Y)$. Clearly, $d(v', G_1) \geq (n+4)/2 - (n_2+2)/2 = (n_1+2)/2$. Thus $v'u_p \in E$ for some $p \in \{1, \ldots, r-1\}$ with $p \neq i$. By Lemma 3.10, $G_1 - u_i$ has a u_{n_1}-u_p e_1-hamiltonian path. With v' and u_p in place of v_s and u_j in the above argument, we see that $V(G_2 - v')$ has a partition $X' \cup Y'$ such that $|X'| = (t+1)/2$, $V(e_2) \subseteq X'$, $|Y'| = (t-1)/2$, $N(y, G_2 - v') = X'$ for all $y \in Y'$, $Y' \subseteq N(v')$ and $N(u_i, G_2 - v') \subseteq X'$. Since $Y' \neq Y$ and Y is an independent set, we see that $Y \subseteq X' \cup \{v'\}$. Thus $N(u_i, Y) \neq \emptyset$, a contradiction. □

Proof of Lemma 2.2. On the contrary, say $d(x_1, G_1) \leq (n_1+1)/2$ and $d(y_1, G_2) \leq (n_2+1)/2$. Then $d(x_1, G_2) \geq (n_2+3)/2$ and $d(y_1, G_1) \geq (n_1+3)/2$. By Lemma 3.2(a), $G_1 - x_1 + y_1 \in \mathcal{P}_1$ and $G_2 - y_1 + x_1 \in \mathcal{P}_2$. By Property A, $\xi(x_1, y_1) \leq 0$. This implies that $d(x_1, G_1) = (n_1+1)/2$, $d(x_1, G_2) = (n_2+3)/2$,

$d(y_1, G_2) = (n_2 + 1)/2$, $d(y_1, G_1) = (n_1 + 3)/2$ and $x_1y_1 \in E$. Since either $G_1 \notin \mathcal{H}_1$ or $G_2 \notin \mathcal{H}_2$, say without loss of generality $G_1 \notin \mathcal{H}_1$. As $\delta_1(G_1) = (n_1 + 1)/2$ and by Lemma 3.3, $\mathcal{P}_1^*(G_1) = \emptyset$. Therefore $e_1 = x_{n_1}x_{n_1-1}$ and $N(x_{n_1}, G_1) \subseteq \{x_{n_1-1}, x_{n_1-2}\}$. Thus $n_2 \geq d(x_{n_1}, G_2) \geq (n + 4)/2 - 2$. This implies $n_2 \geq n_1$.

By Property B, $\mathcal{P}_2^*(G_2) \neq \emptyset$. As $\delta_2(G_2) = (n_2 + 1)/2$ and by Lemma 3.3, $G_2 \in \mathcal{H}_2$. Then $d(y, G_2) \geq (n_2 + 1)/2$ for all $y \in V(G_2) - V(e_2)$. Let $H_1 = G_1 - x_1$ and $H_2 = G_2 + x_1$. By Property A and Lemma 3.2(a) as above, we readily see that $H_2 - y \in \mathcal{P}_2$, if $d(y, G_2) = (n_2 + 1)/2$, then $yx_1 \in E$, and so $d(y, H_2) \geq (n_2 + 3)/2$ for all $y \in V(H_2) - V(e_2)$. Let $C = v_1v_2 \cdots v_tv_1$ be a hamiltonian cycle of H_2 with $t = n_2+1$ and $e_2 = v_1v_2$. Let Y be the set of those vertices $y \in V(H_2)-V(e_2)$ such that $H_2-y \in \mathcal{H}_2$. Then $H_1+y \notin \mathcal{H}_1$ for all $y \in Y$. For each $v_s \in V(C)-\{v_1, v_2, v_3, v_t\}$, $d(v_{s-1}v_{s+1}, C-v_s) \geq n_2+3-2 = n_2+1$ and so $H_2 - v_s \in \mathcal{H}_2$ by Lemma 3.3. Thus $V(C) - \{v_1, v_2, v_3, v_t\} \subseteq Y$. Since $\mathcal{P}_1^*(G_1) = \emptyset$ and $N(x_{n_1}, G_1) \subseteq \{x_{n_1-1}, x_{n_1-2}\}$, we see that $d(x_2x_{n_1}, H_1) \leq n_1 - 2$. It follows that $d(x_2x_{n_1}, H_2) \geq n + 4 - (n_1 - 2) = t + 5$. Consequently, $v_s \in I(x_2x_{n_1}, H_2)$ for some $v_s \in V(C) - \{v_1, v_2, v_3, v_t\}$ and so $H_1 + v_s \in \mathcal{H}_1$, a contradiction. □

Proof of Lemma 2.3. On the contrary, say that $G_2 \in \mathcal{H}_2$. Then $y \in S_2(G_2)$ and so $d(y, G_2) \geq (n_2 + 2)/2$ for all $y \in V(G_2) - V(e_2)$ and $G_1 \notin \mathcal{H}_1$. As $d(x_1, G_2) \geq (n_2 + 3)/2$, $G_2 + x_1 \in \mathcal{H}_2$ by Lemma 3.2(f) and so $S_2(G_2 + x_1) = V(G_2 + x_1) - V(e_2)$. By Property A and Lemma 3.2(a), we readily see that $d(y, G_2 + x_1) \geq (n_2 + 3)/2$ for all $y \in V(G_2) - V(e_1)$. Set $H_1 = G - x_1$ and $H_2 = G_2 + x_1$. Let $A = \{v \in V(H_2) - V(e_2) \mid H_2 - v \in \mathcal{H}_2\}$. Then $H_1 + v \notin \mathcal{H}_1$ for each $v \in A$. Let $C = v_1v_2 \cdots v_{n_2}v_1$ be a hamiltonian cycle of G_2 with $e_1 = v_1v_2$. Say $X_0 = \{v_{n_2}, v_1, v_2, v_3\}$. We claim:

Claim 1. The following two statements hold:

(a) $V(H_2) - X_0 \subseteq A$;

(b) If $d(v_1, H_2 - X_0) \geq 1$, then $v_{n_2} \in A$ and if $d(v_2, H_2 - X_0) \geq 1$, then $v_3 \in A$.

Proof. Clearly, $x_1 \in A$. Let $v_i \in V(G_2) - X_0$. Then $d(v_{i-1}v_{i+1}, G_2 - v_i) \geq (n_2+2)-2 = (n_2-1)+1$ and by Lemma 3.3, $G_2-v_i \in \mathcal{H}_2$. Since $d(x_1, G_2-v_i) \geq (n_2 + 3)/2 - 1 = ((n_2 - 1) + 2)/2$, $H_2 - v_i \in \mathcal{H}_2$. Hence (a) holds.

To see (b), we just need show the first assertion by the symmetry. If $x_1v_1 \in E$, then $x_1v_1 \cdots v_{n_2-1} \in \mathcal{P}_2(H_2 - v_{n_2})$ and $d(x_1v_{n_2-1}, H_2 - v_{n_2}) \geq n_2 + 3 - 2 = n_2 + 1$. By Lemma 3.3, $H_2 - v_{n_2} \in \mathcal{H}_2$. If $v_1v_i \in E$ for some $v_i \in V(G_2) - X_0$, then $v_{i-1}v_{i-2} \cdots v_2v_1v_iv_{i+1} \cdots v_{n_2-1} \in \mathcal{P}_2(G_2 - v_{n_2})$ and $d(v_{i-1}v_{n_2-1}, G_2 - v_{n_2}) \geq n_2$. As above, we see $H_2 - v_{n_2} \in \mathcal{H}_2$. Hence (b) holds. □

We now divide the proof of the lemma into the following two cases. Say $l = n_1 - 1$.

Case 1. $H_1 \notin \mathcal{H}_1$.

Let $P = z_1 \cdots z_l$ be an arbitrary path in $\mathcal{P}_1(H_1)$. Then $I(z_1z_l, A) = \emptyset$. Thus $d(z_1z_l, H_2) \leq n_2+5$ and so $d(z_1z_l, H_1) \geq l$. By Lemma 3.3, $d(z_1z_l, H_1) = l$ and $\sigma(P, e_1) > 0$. Thus $d(z_1z_l, H_2) = n_2 + 5$, $X_0 = I(z_1z_l, H_2)$, $A = V(H_2) - X_0$ and $d(x, z_1z_l) = 1$ for all $x \in A$. By Claim 1, $N(v_1v_2, H_2) \subseteq X_0$. Then $n_1 - 1 = l \geq d(v_1, H_1) \geq (n_1 + n_2 + 4)/2 - d(v_1, G_2) \geq (n_1 + n_2 + 4)/2 - 3$ and $d(x_1, G_2) \leq (n_2 - 2)$. As $d(x_1, G_2) \geq (n_2 + 3)/2$, we see that $n_2 \geq 7$. As $n_2 - 3 \geq d(v_5, G_2) \geq (n_2+2)/2$, it follows that $n_1 \geq n_2 \geq 8$ and $d(x_1, H_1) \geq 4$.

We apply Lemma 3.5 to H_1. First, assume that $V(H_1)$ has a partition $X \cup Y$ such that $|X| = l/2$, $V(e_1) \subseteq X$ and $N(y, H_1) = X$ for all $y \in Y$. Then every two distinct vertices in Y can play the role of z_1 and z_l. Hence $d(x_1, Y) \geq l/2 - 1 \geq 2$ and so $G_1 \in \mathcal{H}_1$, a contradiction. Therefore $H_1 - V(e_1)$ has two components J_1 and J_2 such that $H_1 - V(e_1) = J_1 \cup J_2$, each of J_1 and J_2 is complete and $d(x, H_1) = l - 1$ for each $x \in V(e_1)$. Say without loss of generality $z_1 \in V(J_1)$ and $d(z_1, H_1) \leq d(z_l, H_1)$. Then $d(z_1, G_1) \leq (n_1 + 1)/2$ and so $d(z_1, G_2) \geq (n_2 + 3)/2$. Clearly, $G_1 - z_1 \in \mathcal{P}_1$ and $G_1 - z_1$ has an x_1-z_l hamiltonian e_1-path. Switching the roles of z_1 and x_1 in the above argument, we also obtain $X_0 = I(x_1u, G_2 + z_1)$. By Claim 1, $\{v_3, v_{n_2}\} \subseteq A$, a contradiction.

Case 2. $H_1 \in \mathcal{H}_1$.

Let $L = u_1u_2 \cdots u_lu_1$ be a hamiltonian cycle of H_1 with $e_1 = u_1u_2$, $B = V(L - u_1)$ and $a = n_2 + 1 - |A|$. If $a \geq 3$, then $N(v_1, H_2) \subseteq X_0$ or $N(v_2, H_2) \subseteq X_0$ by Claim 1. As $\delta_2(G_2) \geq (n_2 + 2)/2$, it follows that $n_2 \geq 6$ if $a \geq 3$. We divide this case into the following three subcases.

Subcase 2.1. $d^*(P, H_1) \geq l + 2$ for all $P \in \mathcal{P}_1(H_1)$.

By Lemma 3.7, $d(xy, H_1) \geq l + 2$ for all $x, y \in V(H_1)$ with $x \neq y$ and $xy \neq e_1$. By Lemma 3.6, for all $x, y \in V(H_1)$ with $x \neq y$ and $xy \neq e_1$, H_1 has an x-y e_1-hamiltonian path. Since $H_1 + v_i \notin \mathcal{H}_1$ for all $v_i \in A$, we see that the following Claim 2 holds:

Claim 2. For each $v_i \in A$, if $d(v_i, H_1) \geq 2$, then $N(v_i, H_1) = V(e_1)$.

By Claim 2, $n_2 \geq (n_1 + n_2 + 4)/2 - d(v_i, H_1) \geq (n_1 + n_2 + 4)/2 - 2$ for all $v_i \in A$. Thus $n_2 \geq n_1$. By Claim 2, $d(v_i, B) \leq 1$ for all $v_i \in A$ and so $d(A, B) \leq |A| = n_2 + 1 - a$. On the other hand, $d(A, B) \geq \sum_{u \in B} d(u, A) \geq \sum_{u \in B}((n_1+n_2+4)/2 - d(u, H_1) - a) \geq (n_1 - 2)((n_1+n_2+4)/2 - (n_1 - 2) - a)$. Therefore $(n_1 - 2)((n_1 + n_2 + 4)/2 - (n_1 - 2) - a) - (n_2 + 1 - a) \leq 0$. Denote the left side of this inequality by $f(n_1)/2$ with $n_2 = n - n_1$. Then $f(n_1) = -2n_1^2 + (n+14-2a)n_1 + (-4n-18+6a) \leq 0$ for $4 \leq n_1 \leq n/2$. As $f''(n_1) < 0$, $f(n_1) \geq \min\{f(4), f(n/2)\} = \min\{6 - 2a, 3n - an - 18 + 6a\}$. Thus $a \geq 3$ for otherwise $f(n_1) > 0$. Thus $N(v_1, H_2) \subseteq X_0$ or $N(v_2, H_2) \subseteq X_0$. Say without loss of generality $N(v_1, H_2) \subseteq X_0$. Then $n_1 - 1 \geq d(v_1, H_1) \geq (n_1+n_2+4)/2 - 3$ which implies that $n_1 \geq n_2$. Let $v_i \in A - X_0$. Then $n_2 - 1 \geq d(v_i, H_2) \geq (n_1 + n_2 + 4)/2 - 2$ which implies that $n_2 \geq n_1 + 2$, a contradiction.

Subcase 2.2. $d^*(P, H_1) \geq l + 1$ for all $P \in \mathcal{P}_1(H_1)$.

By the above subcase, $d^*(P, H_1) = l + 1$ for some $P \in \mathcal{P}_1(H_1)$. Thus $d^*(P, H_2) \geq n_1 + n_2 + 4 - l - 1 = n_2 + 4$. As $d^*(P, v_i) \leq 1$ for all $v_i \in A$. Thus $d^*(P, v') = 2$ and so $v' \notin A$ for some $v' \in \{v_3, v_{v_2}\}$. It follows that $a \geq 3$ and so $n_2 \geq 6$. By Claim 1, $N(v_1, H_2) \subseteq X_0$ and we may assume that $v_{n_2} \notin A$. As in the above paragraph, this implies that $n_1 \geq n_2$. Let z be an arbitrary vertex in $A - X_0$. Then $n_1 - 1 \geq d(z, H_1) \geq (n_1 + n_2 + 4)/2 - (n_2 - 1) \geq 3$. It is easy to see that there exist two distinct vertices u and w on L such that either $\{u^-, w^-\} \subseteq N(z)$ and $e_1 \notin \{uu^-, ww^-\}$ or $\{u^+, w^+\} \subseteq N(z)$ and $e_1 \notin \{uu^+, ww^+\}$. Say without loss of generality $\{u^+, w^+\} \subseteq N(z)$ and $e_1 \notin \{uu^+, ww^+\}$. By Lemma 3.7, $d(xy, H_1) \geq l + 1$ for all $\{x, y\} \subseteq V(H_1)$ with $x \neq y$ and $xy \neq e_1$. We claim that $d(x, H_1) \geq (l + 1)/2$ for all $x \in V(H_1)$. If this is false, say $d(x_0, H_1) \leq l/2$ for some $x_0 \in V(H_1)$. Then $d(x, H_1) \geq (l + 2)/2$ for all $x \in V(H_1 - x_0)$ with $x_0 x \neq e_1$ and $d(x_0, H_2) \geq (n_1 + n_2 + 4)/2 - l/2 \geq (n_2 + 5)/2 \geq 5$. Thus $d(x_0, A - X_0) > 0$. It is easy to see that in the choices of the vertices u, w and z in the above, we can choose u, w and z such that $x_0 \notin \{u, w\}$. Thus $d(uw, H_1) \geq l + 2$ and by Lemma 3.6, H_1 has a u^+-w^+ e_1-hamiltonian path and so $H_1 + z \in \mathcal{H}_1$, a contradiction. Hence $d(x, H_1) \geq (l + 1)/2$ for all $x \in V(H_1)$.

We now apply Lemma 3.6(d) to H_1 since H_1 does not have a u^+-w^+ e_1-hamiltonian path. First, assume that H_1 has a vertex-cut X with $|X| = 3$ and $V(e_1) \subseteq X$ such that $H_1 - X = H_1' \cup H_1''$, where H_1' and H_1'' are isomorphic to $K_{(l-3)/2}$. Then $N(z, H_1) = X$ as $H_1 + z \notin \mathcal{H}_1$. As z is arbitrary in $A - X_0$, $N(A - X_0, H_1) = X$. It follows that $d(x, G) \leq (l + 1)/2 + 4 < (n_1 + n_2 + 4)/2$ for $x \in V(H_1 - X)$, a contradiction. Therefore $V(H_1)$ has a partition $X \cup Y$ such that $|X| = (l + 1)/2$, $V(e_1) \subseteq X$, $\{u, w\} \subseteq Y$, and $N(y, H_1) = X$ for all $y \in Y$. Clearly, $\{u^+, w^+\} \subseteq X$. Thus $N(z, H_1) \subseteq X$ as $H_1 + z \notin \mathcal{H}_1$. Let $y \in Y$. As $d(y, A - X_0) \geq (n_1 + n_2 + 4)/2 - (l + 1)/2 - 4 > 0$, let $z' \in N(y, A - X_0)$. With z' in place of z in this argument, we see that $V(H_1)$ has a partition $X' \cup Y'$ such that $|X'| = (l + 1)/2$, $V(e_1) \subseteq X'$, $N(y', H_1) = X'$ for all $y' \in Y'$ and $N(z', H_1) \subseteq X'$. It follows that $Y' \cap X \neq \emptyset$ and so $Y' \subseteq X$. Thus $|X| \geq (l + 1)/2 + 1 = (l + 3)/2$, a contradiction.

Subcase 2.3. For some $P \in \mathcal{P}_1(H_1)$, $d^*(P, H_1) \leq l$.

For each $P \in \mathcal{P}_1(H_1)$, as $d^*(P, A) \leq |A|$, $d^*(P, H_1) \geq n_1 + n_2 + 4 - (n_2 + 1 + a) = l + 4 - a \geq l$. Thus $a = 4$ and by Claim 1, $N(v, H_2) \subseteq X_0$ for $v \in \{v_1, v_2\}$. As before, it follows that $n_1 \geq n_2 \geq 6$. Let z be an arbitrary vertex in A. Then $d(z, H_1) \geq (n_1 + n_2 + 4)/2 - (n_2 - 2) \geq 4$.

First, assume that there exists $P \in \mathcal{P}_1^*(H_1)$ such that $d^*(P, H_1) = l$. As $H_1 + v \notin \mathcal{H}_1$ for all $v \in A$, it follows that $d^*(P, v) = 1$ for all $v \in A$ and $d^*(P, X_0) = 8$. Say $P = z_1 z_2 \cdots z_l$ with $d(z_1, P) \leq d(z_l, P)$. Then $d(z_1, P) \leq l/2$, $d(z_1, H_2) \geq \lceil (n_1 + n_2 + 4)/2 \rceil - \lfloor l/2 \rfloor \geq 5$. Let $z_c \in \{z_1, z_l\}$ and $v_b \in A$ be such that $v_b z_c \in E$. We claim that $G_2 + z_c - v_j \in \mathcal{H}_2$ for all $v_j \in V(G_2) - V(e_2)$. To see this, say $G_2 + z_c - v_j \notin \mathcal{H}_2$ for some $v_j \in V(G_2) - V(e_2)$. Clearly, $v_{j-1} v_{j+1} \notin E$ otherwise $G_2 + z_l - v_j \in \mathcal{H}_2$. First assume that $v_j \notin \{v_3, v_{n_2}\}$. Then $d(v_{j-1} v_{j+1}, G_2 - v_j) \geq n_2 + 2 - 2 = (n_2 - 1) + 1$. This implies that $C - v_j$ has an accessible edge e' with $e' \neq e_2$. Since $N(v_1v_2, G_2) \subseteq X_0$ and $d(z_c, X_0) = 4$, it follows that $G_2 - v_j + z_c \in \mathcal{H}_2$, a contradiction. Hence $v_j \in \{v_3, v_{n_2}\}$. Say without loss of generality $v_j = v_3$. Then $P' = v_4 \cdots v_b z_c v_2 v_1 v_{n_2} v_{n_2-1} \cdots v_{b+1}$ is an e_2-hamiltonian path of $G_2 - v_3 + z_c$ with $d(v_4 v_{b+1}, G_2 - v_3 + z_c) \geq n_2$. As $d(v_4, e_2) = 0$, this implies that P' has an accessible edge e'' with $e'' \neq e_2$ and so $G_2 - v_j + z_c \in \mathcal{H}_2$, a contradiction. Hence this claim holds. Let $H_1' = G_1 - z_c$ and $H_2' = G_2 + z_c$. We claim that $H_1' \notin \mathcal{P}_1$. To see this, say $H_1' \in \mathcal{P}_1$. Then for any $Q \in \mathcal{P}_1(H_1')$ and $v \in V(H_2') - V(e_2)$, $H_1' + v \notin \mathcal{H}_1$ and so $d^*(Q, v) \leq 1$. Thus for any $Q \in \mathcal{P}_1(H_1')$, $d^*(Q, H_2') \leq n_2 + 3$ and so $d^*(Q, H_1') \geq l + 2$. Let $v_j \in A - \{x_1\}$. Then $d(v_j, H_1') \geq (n_1 + n_2 + 4)/2 - d(v_j, G_2) - d(v_j, z_c) \geq (n_1 + n_2 + 4)/2 - (n_2 - 3) - 1 \geq 4$. By Lemma 3.6 and Lemma 3.7, we see that $H_1' + v_j \in \mathcal{H}_1$, a contradiction.

Therefore $H_1' \notin \mathcal{P}_1$. As $d(z_1, H_1) \leq \lfloor l/2 \rfloor$, $d(z_1, G_2) \geq \lceil (n_1 + n_2 + 4)/2 \rceil - \lfloor l/2 \rfloor - 1 \geq 5$. The above argument implies that $H_1 - z_1 + x_1 \notin \mathcal{P}_1$ and so $x_1 z_l \notin E$. Thus $z_1 x_1 \in E$ and so $H_1 - z_l + x_1 \in \mathcal{P}_1$. Consequently, the above argument implies that $d(z_l, G_2) = d(z_l, X_0) = 4$. Thus $d(x_1 z_l, H_1 - z_1) \geq n_1 + n_2 + 4 - (n_2 - 2) - 4 - 2 = (l - 1) + 2$. By Lemma 3.2, $H_1 - z_1 + x_1 \in \mathcal{P}_1$, a contradiction.

Therefore for each $P \in \mathcal{P}_1^*(H_1)$, $d^*(P, H_1) \geq l + 1$. Recall that $L = u_1 u_2 \cdots u_l u_1$ is a hamiltonian cycle of H_1 with $e_1 = u_1 u_2$. To apply Lemma 3.8, let us first assume that $d(u_t, H_1) \leq l/2$ for some $u_t \in V(L) - V(e_1)$. If $4 \leq t \leq l - 1$, then $d(u_j, H_1) \geq (l + 2)/2$ for all $3 \leq j \leq l$ with $j \neq t$. As $d(z, H_1) \geq 4$, it is easy to see that there exist two distinct vertices u and w on L with $u_t \notin \{u, w\}$ such that either $\{u^-, w^-\} \subseteq N(z)$ and $e_1 \notin \{u^- u, w^- w\}$ or $\{u^+, w^+\} \subseteq N(z)$ and $e_1 \notin \{u^+ u, w^+ w\}$. By Lemma 3.6, we see that $H_1 + z \in \mathcal{H}_1$, a contradiction. Hence $u_t \in \{u_3, u_l\}$. By Lemma 3.8, $d(u_j, H_1) \geq (l + 2)/2$ for all $5 \leq j \leq l - 1$. To avoid the existence of u and w as above such that $H_1 + z \in \mathcal{H}_1$, we see that $N(z, H_1) = \{u_1, u_2, u_4, u_{l-1}\}$. As z is an arbitrary vertex in A, we see that $d(u_t, A) = 0$ and so $d(u_t, H_2) \leq 5$. Thus $d(u_t, H_1) \geq (n_1 + n_2 + 4)/2 - 5 \geq (l + 1)/2$, a contradiction.

Therefore $d(u_j, H_1) \geq (l + 1)/2$ for all $u_j \in V(H_1) - V(e_1)$. As $d(z, H_1) \geq 4$, there exist two distinct vertices u and w on C such that either $\{u^-, w^-\} \subseteq N(z)$ and $e_1 \notin \{uu^-, ww^-\}$ or $\{u^+, w^+\} \subseteq N(z)$ and $e_1 \notin \{uu^+, ww^+\}$. Say without loss of generality $\{u^+, w^+\} \subseteq N(z)$ and $e_1 \notin \{uu^+, ww^+\}$. We now apply word by word the argument in the last paragraph of Subcase 2.2 to H_1 and H_2 and a contradiction follows. □

Proof of Lemma 2.4. As $\mathcal{P}_2^*(G_2) = \emptyset$, $N(v_{n_2}, G_2) \subseteq \{v_{n_2-1}, v_{n_2-2}\}$ and so $n_1 \geq d(v_{n_2}, G_1) \geq (n_1 + n_2 + 4)/2 - d(v_{n_2}, G_2)$. Thus $n_1 \geq n_2$ and if $n_1 = n_2$, then $N(v_{n_2}, G_2) = \{v_{n_2-2}, v_{n_2-1}\}$ and so $r \leq n_2 - 2$. Since $n_2 - 2 \geq r - 1 \geq (n_2 + 2)/2$, we see that $n_2 \geq 6$ and if $r \leq n_2 - 2$, then $n_2 \geq 8$.

On the contrary, say that the lemma fails. Let $u_0 \in V(G_1) - V(e_1)$ with $d(u_0, G_1)$ minimal be such that $G_1 - u_0 \in \mathcal{P}_1$, $G_2 + u_0 \in \mathcal{H}_2$ and $d(u_0, J^*) > 0$. Let $v_c \in J^*$ with $u_0 v_c \in E$. As $G_2 + u_0 \in \mathcal{H}_2$, we see that $u_0 v_{n_2} \in E$ if

$v_{n_2}v_{n_2-2} \notin E$ and $d(u_0, v_{n_2}v_{n_2-1}) \geq 1$ if $v_{n_2}v_{n_2-2} \in E$. Thus we may assume without loss of generality that $u_0v_{n_2} \in E$. Let B be the set of all the vertices v_i in G_2 such that $G_2 - v_i + u_0 \in \mathcal{H}_2$. By Lemma 3.10, $V(J) - \{v_c, v_r\} \subseteq B$, and if $d(u_0, J - v_r) \geq 2$, then $v_c \in B$. Set $H = G_1 - u_0$ and $l = |H| = n_1 - 1$. We claim the following:

Claim A. If $d(u_0, G_1) \leq (n_1 + 1)/2$, then $r \in \{n_2 - 2, n_2 - 1\}$ and $B = \{v_1, \ldots, v_{r-1}\}$. Moreover, for each $P \in \mathcal{P}_1(H)$ we have that $d^*(P, v_i) \leq 1$ for all $1 \leq i \leq r - 1$, $d^*(P, H) \geq l$ and if $d^*(P, H) = l$, then $r = n_2 - 2$, $d^*(P, v_{n_2-2}v_{n_2-1}v_{n_2}) = 6$, $d^*(P, u_0) = 2$ and $d^*(P, v_i) = 1$ for all $1 \leq i \leq r - 1$.

Proof. Say $d(u_0, G_1) \leq (n_1 + 1)/2$. Then $d(u_0, G_2) \geq (n_2 + 3)/2$. As $G_2 + u_0 \in \mathcal{H}_2$, for each $y \in V(G_2) - V(e_2)$, $G_2 - y + u_0 \in \mathcal{P}_2$ and so if $G_1 - u_0 + y \in \mathcal{P}_1$, then $\xi(u_0, y) \leq 0$ by Property A. Let y be an arbitrary vertex of $G_2 - V(e_2)$. If $d(y, G_2) \leq (n_2 + 1)/2$, then $d(y, G_1) \geq (n_1 + 3)/2$ and so $G_1 - u_0 + y \in \mathcal{P}_1$ by Lemma 3.2(a). Consequently, $\xi(u_0, y) \leq 0$. This implies that $d(y, G_2) = (n_2+1)/2$ and $u_0y \in E$. Therefore $d(y, G_2) \geq (n_2+1)/2$ for all $y \in V(G_2)-V(e_2)$. Consequently, $r \in \{n_2-2, n_2-1\}$. As $d(u_0, G_2) \geq (n_2+3)/2$ and $r-1 \geq \lceil (n_2+2)/2 \rceil$, we see that $d(u_0, J-v_r) \geq \lceil (n_2+3)/2 \rceil - (n_2-r) - 1 \geq 3$. By Lemma 3.10, $B = \{v_1, \ldots, v_{r-1}\}$. Let P be an arbitrary path in $\mathcal{P}_1(H)$. Say u and w are the two endvertices of P. Then $I(uw, G_2) \cap B = \emptyset$, i.e., $d^*(P, v_i) \leq 1$ for all $i \in \{1, \ldots, r-1\}$. It follows that $d(uw, G_2) \leq n_2 + 3$ and if equality holds, then $r = n_2 - 2$ and $\{v_{n_2-2}, v_{n_2-1}, v_{n_2}\} = I(uw, G_2)$. Clearly, $d(uw, G_1) \geq n_1 + n_2 + 4 - (n_2 + 3) = l + 2$ and so $d(uw, H) \geq l$. Claim A follows.

We now break into two cases here.

Case 1. $H \notin \mathcal{H}_1$.

Then $d^*(P, H) \leq l$ by Lemma 3.3 and so $d^*(P, G_2) \geq n_1 + n_2 + 4 - l - d^*(P, u_0) \geq n_2+3$ for all $P \in \mathcal{P}_1(H)$. First, assume that $d(u_0, H) \leq (n_1+1)/2$. By Claim A and Lemma 3.3, $r = n_2 - 2$ and for each $P \in \mathcal{P}_1(H)$, $d^*(P, H) = l$, $\sigma(e_1, P) \neq 0$, $d^*(P, \{u_0, v_{n_2-2}, v_{n_2-1}, v_{n_2}\}) = 8$, and $d^*(P, v_i) = 1$ for all $1 \leq i \leq r - 1$. We apply Lemma 3.5(c) to H. First, assume that $V(H)$ has a partition $X \cup Y$ such that $|X| = l/2$, $V(e_1) \subseteq X$ and $N(y, H) = X$ for all $y \in Y$. Then any two distinct vertices in Y can play the role of the two endvertices of P. Hence $d(v_1, Y) \geq l/2 - 1 \geq 2$ and so $H + v_1 \in \mathcal{H}_1$, a contradiction. Therefore $H - V(e_1)$ has exactly two components H_1 and H_2 such that both H_1 and H_2 are complete and $d(x, H_1 \cup H_2) = l - 2$ for each $x \in V(e_1)$. It follows that $V(H_1 \cup H_2) \subseteq N(u_0)$. Thus $n_1 - 3 \leq d(u_0, G_1) \leq (n_1 + 1)/2$. This implies that $n_1 \leq 7$. As mentioned in the beginning paragraph, we have $r = n_2 - 2$ and $n_1 \geq n_2 \geq 8$, a contradiction.

Therefore $d(u_0, H) \geq (n_1 + 2)/2$. Let $P = z_1 \cdots z_l$ be arbitrary in $\mathcal{P}_1(H)$ with $z_1 \in S_1(H)$. We claim $d(z_1, G_1) \geq (n_1 + 2)/2$. If this is not true, say $d(z_1, G_1) \leq (n_1 + 1)/2$. Then $d(z_1, G_2) \geq (n_2 + 3)/2$. Clearly, $d(z_1, J) \geq \lceil (n_2 + 3)/2 - (n_2 - r) \rceil \geq 4$ as $r - 1 \geq (n_2 + 2)/2$ and so $d(z_1, J^*) > 0$.

By Lemma 3.2(a), $G_1 - z_1 = H - z_1 + u_0 \in \mathcal{P}_1$. As $d(z_1, G_1) < d(u_0, G_1)$ and by the minimality of $d(u_0, G_1)$, $G_2 + z_1 \notin \mathcal{H}_2$, i.e., $z_1 v_{n_2} \notin E$ and if $v_{n_2} v_{n_2-2} \in E$, then $z_1 v_{n_2-1} \notin E$. By Lemma 3.2(b), $G_2 + z_1 \in \mathcal{P}_2$. If $v \in S_2(G_2 + z_1)$, then $d(v, G_2 + z_1) \geq (n_2 + 3)/2$ for otherwise $\xi(z_1, v) > 0$, $d(v, G_1) \geq (n_1 + 2)/2$ and $G_1 - z_1 + v \in \mathcal{P}_1$ by Lemma 3.2(a), contradicting (1). Let s be the maximal index such that $z_1 v_s \in E$. Set $r' = \max\{r, s\}$. By Lemma 3.9, for all $v \in \{z_1, v_1, \dots v_{r'-1}\}$, $d(v, G_2 + z_1) \geq (n_2 + 3)/2$, $N(v, G_2 + z_1) \subseteq \{z_1, v_1, \dots, v_{r'}\}$ and $G_2 + z_1$ has a v_{n_2}-v e_2-hamiltonian path. Therefore $d(v, G_2) \geq (n_2 + 1)/2$ for all $v \in \{v_1, \dots, v_{r'-1}\}$. It follows that $r' = r$ or $r' = r + 1$. As $d(z_1 z_l, G_2) \geq n_2 + 3$, $i(z_1 z_l, J + v_{r'}) \geq 3$. As $I(z_1 z_l, B) = \emptyset$, we see that $I(z_1 z_l, G_2) = \{v_c, v_r, v_{r+1}\}$. It follows that $d(z_1 z_l, H) = l$, $N(z_1 z_l, G_2) = V(G_2)$, $d(u_0, z_1 z_l) = 2$, $B = V(J) - \{v_c, v_r\}$ and $d(v_i, z_1 z_l) = 1$ for all $v_i \in B$. This argument implies that for any u-v path in $\mathcal{P}_1^*(H)$, $d(uv, H) = l$ because $\min\{d(u, H), d(v, H)\} \leq l/2$ and so $\min\{d(u, G_1), d(v, G_1)\} \leq (n_1 + 1)/2$.

We now apply Lemma 3.5(c) to H. First, assume that $V(H)$ has a partition $X \cup Y$ such that $|X| = l/2$, $V(e_1) \subseteq X$ and $N(y, H) = X$ for all $y \in Y$. Then any two distinct vertices in Y can play the role of the two endvertices of P. Hence $d(v_i, Y) \geq l/2 - 1 \geq 2$ and so $H + v_i \in \mathcal{H}_1$ for each $v_i \in B$, a contradiction. Therefore $H - V(e_1)$ has exactly two components H_1 and H_2. Say $z_1 \in V(H_1)$ and $z_l \in V(H_2)$. Then z_1 can be any vertex in H_1 and z_l can be any vertex in H_2 for the above argument. Consequently, $V(H_2) \subseteq N(v_{n_2})$ and $V(H_1 \cup H_2) \subseteq N(v_c) \cap N(u_0)$. Clearly, $G_1 - x + v_c \in \mathcal{H}_1$ for any $x \in V(H_2)$. Let $v_d \in B - \{v_c\}$. If $x v_d \in E$ for some $x \in V(H_2)$, then $G_2 - v_c + x \in \mathcal{H}_2$, a contradiction. Therefore $d(v_d, H_2) = 0$ and so $N(v_d, H_1) = V(H_1)$. As $d(v_d, G_1) \geq (n_1 + n_2 + 4)/2 - (n_2 - 2) \geq 4$, $|H_1| \geq 2$. As $V(H_1 \cup H_2) \subseteq N(u_0)$, we see $G_1 - z_l + v_d \in \mathcal{H}_1$. As $G_2 - v_d$ has a v_{n_2}-v_c e_2-hamiltonian path, $G_2 - v_d + z_l \in \mathcal{H}_2$, a contradiction.

Therefore $d(z_1, G_1) \geq (n_1 + 2)/2$ and so $d(z_1, H) \geq (l + 1)/2$. Thus $\delta_1(H) \geq (l + 1)/2$. As $H \notin \mathcal{H}_1$ and by Lemma 3.3, $\mathcal{P}_1^*(H) = \emptyset$ and so $e_1 = z_l z_{l-1}$. As $d(u_0, H) \geq (n_1 + 2)/2$ and by Lemma 3.2(a), $H - z_1 + u_0 \in \mathcal{P}_1$. As $H \notin \mathcal{H}_1$, $d(z_1 z_l, H) \leq l - 1$ by Lemma 3.3. Choose P to be an optimal path at e_1 in H. Say $t = \alpha(P, z_1)$. By Lemma 3.9, $C = z_1 z_2 \cdots z_t z_1$ is an end-cycle at z_t in H such that $d(z_i, C) \geq (l + 1)/2$ for all $i \in \{1, 2, \dots, t - 1\}$. Thus for all $i \in \{1, 2, \dots, t-1\}$, each z_i can play the role of z_1 in the above and so $d(z_i, G_1) \geq (n_1 + 2)/2$. Clearly, $d(u_0, C - z_t) > 0$. Say without loss of generality $u_0 z_1 \in E$. As $\mathcal{P}_1^*(H) = 0$, $N(z_l, G_1) \subseteq \{z_{l-1}, z_{l-2}, u_0\}$. Clearly, $d(z_l, J - v_r) \geq (n_1 + n_2 + 4)/2 - 3 - (n_2 - r + 1) > 0$. Recall that $u_0 v_{n_2} \in E$. Therefore if we set $G' = G_1 - V(e_1) + V(e_2)$ and $G'' = G_2 - V(e_2) + V(e_1)$, then $G' \in \mathcal{H}_2$ and $G'' \in \mathcal{H}_1$. Recall that if $z_l z_{l-2} \in E$, then $N(z_{l-1}, G_1) \subseteq \{z_l, z_{l-2}, u_0\}$ as $\mathcal{P}_1^*(H) = \emptyset$. We readily see that $d(e_1, G_1 - V(e_1)) \leq l$, $d(e_1, G_2) \geq (n_1 + n_2 + 4) - l - 2 = n_2 + 3$, $d(e_2, G_2 - V(e_2)) \leq n_2 - 2$ and $d(e_2, G_1) \geq n_1 + n_2 + 4 - (n_2 - 2) - 2 = n_1 + 4$. Thus

$$(2) \qquad e(G') + e(G'') = e(G_1) - d(e_1, G_1 - V(e_1)) + d(e_1, G_2) + e(G_2)$$

$$- d(e_2, G_2 - V(e_1)) + d(e_2, G_1) - 2d(e_1, e_2)$$
$$\geq e(G_1) + e(G_2) + 10 - 2d(e_1, e_2).$$

As $10 - 2d(e_1, e_2) \geq 2$ and by (1), we see that $n_1 = |G'| \neq n_2$. As $n_1 \geq n_2$, $n_1 > n_2$. As $N(z_l, G_1) \subseteq \{z_{l-2}, z_{l-1}, u_0\}$, we obtain that $n_2 \geq d(z_l, G_2) \geq \lceil (n_1 + n_2 + 4)/2 - d(z_l, G_1) \rceil = n_2$. It follows that $N(z_l, G_1) = \{z_{l-2}, z_{l-1}, u_0\}$ and $d(z_l, G_2) = n_2$. As $H + v_j \notin \mathcal{H}_1$ for all $v_j \in B$, it follows that $z_i v_j \notin E$ for all $i \in \{1, \ldots, t-1\}$ and $v_j \in J - \{v_c, v_r\} \subseteq B$. Thus $d(z_1, G_1) + d(z_1, G_2) \leq t + n_2 - r + 2$. Let $v \in J - \{v_c, v_r\}$. Then $d(v, G_1) + d(v, G_2) \leq l - t + 2 + r - 1$. Consequently, $d(z_1) + d(v) \leq n_1 + n_2 + 2$. But $d(z_1) + d(z_2) \geq n_1 + n_2 + 4$ as $\delta(G) \geq (n_1 + n_2 + 4)/2$, a contradiction.

Case 2. $H \in \mathcal{H}_1$.

Let $C = z_1 z_2 \cdots z_l z_1$ be an e_1-hamiltonian cycle of H with $e_1 = z_1 z_2$. Let $v_i \in B$. With the details stated in the beginning paragraph, we see that $d(v_i, H) \geq \lceil (n_1 + n_2 + 4)/2 \rceil - (r-1) - d(v_i, u_0) \geq 4$ and if equality holds, then $v_i u_0 \in E$, $r \in \{n_2 - 2, n_2 - 1\}$ and $d(v_i, G_2) = r - 1$. We divide this case into the following two subcases.

Subcase 2.1. For each path $P \in \mathcal{P}_1^*(H)$, $d^*(P, H) \geq l + 1$.

First, assume that $d(w, C) \leq l/2$ for some $w \in V(C) - V(e_1)$. If $w \notin \{z_3, z_l\}$, then $d(x, C) \geq (l+2)/2$ for all $x \in V(C - w) - V(e_1)$ by Lemma 3.8. As $d(v_i, C) \geq 4$ and $H + v_i \notin \mathcal{H}_1$, we readily see that there exist two distinct vertices z_j and z_h in $N(v_i, C)$ such that either $\{z_j^-, z_h^-\} \subseteq V(C) - \{z_1, z_2, w\}$ or $\{z_j^+, z_h^+\} \subseteq V(C) - \{z_1, z_2, w\}$. Consequently, by Lemma 3.6, H has a z_j-z_h e_1-hamiltonian path and so $H + v_i$ is hamiltonian, a contradiction. Therefore $d(z_j, C) \geq (l+2)/2$ for all $4 \leq j \leq l-1$. As above, $N(v_i, C)$ does not contain two distinct vertices z_j and z_h such that either $\{z_j^-, z_h^-\} \subseteq V(C) - \{z_1, z_2, z_3, z_l\}$ or $\{z_j^+, z_h^+\} \subseteq V(C) - \{z_1, z_2, z_3, z_l\}$. It follows that $N(v_i, C) = \{z_1, z_2, z_4, z_{l-1}\}$. The above argument allows us to conclude that $r \in \{n_2 - 2, n_2 - 1\}$, and for all $v \in B$, $N(v, H) = \{z_1, z_2, z_4, z_{l-1}\}$. As $V(J) - \{v_c, v_r\} \subseteq B$, $d(w, G_2) \leq 4$ and so $d(w, C) \geq (n_1 + n_2 + 4)/2 - 5 \geq (l+1)/2$, a contradiction.

Therefore $d(z_i, H) \geq (l+1)/2$ for all $i \in \{3, \ldots, l\}$. As $d(v_i, H) \geq 4$ and $H + v_i \notin \mathcal{H}_1$, there exist two distinct vertices u and v in $C - V(e_1)$ such that either $\{u^+, v^+\} \subseteq N(v_i)$ or $\{u^-, v^-\} \subseteq N(v_i)$. Say without loss of generality $\{u^+, v^+\} \subseteq N(v_i)$. Then H does not have a u^+-v^+ e_1-hamiltonian path. We apply Lemma 3.6(d) to H. First, assume that H has a vertex-cut X with $V(e_1) \subseteq X$ and $|X| = 3$ such that $H - X$ has exactly two components isomorphic to $K_{(l-3)/2}$ and $X \subseteq N(y)$ for all $y \in V(C) - X$. Obviously, $H + v_i \in \mathcal{H}_1$, a contradiction. Thus $V(H)$ has a partition $X \cup Y$ such that $|X| = (l+1)/2$, $|Y| = (l-1)/2$, $\{u^+, v^+\} \cup V(e_1) \subseteq X$ and $N(y, H) = X$ for all $y \in Y$. As $H + v_i \notin \mathcal{H}_1$, it follows that $N(v_i, H) \subseteq X$. Let $y \in Y$. Then $d(y, G_2) \geq (n_1 + n_2 + 4)/2 - (l+1)/2 - 1 = (n_2 + 2)/2$. Thus $d(y, B) > 0$. Let $v_j \in N(y, B)$. With v_j in place of v_i in the above argument, we see that $V(H)$ has a partition X' and Y' such that $|X'| = (l+1)/2$, $V(e_1) \subseteq X'$,

$N(v_j, H) \subseteq X'$ and $X' = N(y', H)$ for all $y' \in Y'$. As Y is an independent set, it follows that $Y \subseteq X'$ and so $|X'| \geq (l-1)/2 + 2 = (l+3)/2$, a contradiction.

Subcase 2.2. There exists $P = z_1 z_2 \cdots z_l \in \mathcal{P}_1^*(H)$ such that $d(z_1 z_l, H) \leq l$.

Then $d(z_1 z_l, G_2) \geq n_1 + n_2 + 4 - l - d(u_0, z_1 z_l) \geq n_2 + 3$ and so $i(z_1 z_l, G_2) \geq 3$. Say $d(z_1, G_1) \leq d(z_l, G_1)$. Then $d(z_1, G_1) \leq l/2 + 1 = (n_1+1)/2$. Thus $d(z_1, G_2) \geq (n_2+3)/2$. As $r \geq \delta_2(G_2) + 1 \geq (n_2+2)/2 + 1$, $d(z_1, J - v_r) \geq \lceil (n_2+3)/2 - (n_2 - r) - 1 \rceil \geq 3$. Therefore $G_2 + z_1$ has a hamiltonian path from e_1 to z_1. We claim that $G_1 - z_1 \in \mathcal{P}_1$. If this is not true, then $d(u_0, P - z_1) \leq (l-1)/2$ by Lemma 3.2(a) and so $d(u_0, G_1) \leq (l+1)/2$. By Claim A, it follows that

$$(3) \qquad \begin{aligned} &d(z_1 z_l, H) = l, \ r = n_2 - 2, \\ &I(z_1 z_l, G_2) = \{v_{n_2-2}, v_{n_2-1}, v_{n_2}\} \text{ and } d(u_0, z_1 z_l) = 2. \end{aligned}$$

Therefore $G_1 - z_1 \in \mathcal{P}_1$. By the minimality of u_0, $d(u_0, G_1) \leq d(z_1, G_1) \leq (n_1+1)/2$. Therefore (3) still holds and $d(u_0, G_1) \leq (n_1+1)/2$ in any case. Moreover, $d(u_0, J) \geq \lceil (n_2+3)/2 \rceil - (n_2 - r) \geq 4$ and so $B = V(J) - \{v_r\}$ as mentioned in the paragraph above Claim A. As $r - 1 \geq (n_2+2)/2$, $n_2 \geq 8$.

We claim that for each $\{u, v\} \subseteq V(J) - \{v_r\}$ with $u \neq v$, $G_2 - \{u, v\} + \{u_0, z_l\} \in \mathcal{H}_2$. To see this, we note that $u_0 v_{n_2} v_{n_2-1} v_{n_2-2} z_l u_0$ is a cycle in G. Moreover, we have that for all $x \in V(J - \{u, v, v_r\})$, $d(x, J - \{u, v, v_r\}) \geq (n_2+2)/2 - 3 = ((n_2 - 5) + 1)/2$ and so $J - \{u, v, v_r\}$ is hamiltonian connected. Clearly, for each $y \in \{u_0, z_l\}$ $d(y, J - \{u, v, v_r\}) \geq \lceil (n_2+3)/2 \rceil - 5 \geq 1$ as $n_2 \geq 8$. Thus if $G_2 - \{u, v\} + \{u_0, z_l\} \notin \mathcal{H}_2$, then $d(y, J - \{u, v, v_r\})) = 1$ for each $y \in \{u_0, z_l\}$. Consequently, $n_2 \leq 9$. As $\delta_2(G_2) \geq \lceil (n_2+2)/2 \rceil$, it follows that J is complete and obviously $G_2 - \{u, v\} + \{u_0, z_l\} \in \mathcal{H}_2$, a contradiction. Hence the claim holds. □

Therefore $H - z_l + u + v \notin \mathcal{H}_1$ for all $u, v \in V(J - v_r)$ with $u \neq v$. For each vertex $v \in V(J - v_r)$, it is easy to see that $uv \in E$ for some $u \in N(z_1, J - v_r)$ since $d(z_1, G_2) \geq (n_2+3)/2$ and $d(v, J) \geq (n_2+2)/2$. Therefore $d(z_{l-1}, J - v_r) = 0$ for otherwise $H - z_l + u + v \in \mathcal{H}_1$ for some $v \in N(z_{l-1}, J - v_r)$ and $u \in N(z_1, J - v_r)$ with $uv \in E$. Thus $d(z_{l-1}, H - z_l) \geq (n_1 + n_2 + 4)/2 - 5 = (n_1 + n_2)/2 - 3$. Let $uv \in E(J - v_r)$ with $uz_1 \in E$. Clearly, $d(v, H - z_l) \geq (n_1 + n_2 + 4)/2 - (r-1) - 2 = (n_1 - n_2)/2 + 3$. Thus $d(vz_{l-1}, H - z_l) \geq (l-1) + 2$. By Lemma 3.2(d), $H - z_l + v$ has an e_1-hamiltonian path from z_1 to v and so $H - z_l + u + v \in \mathcal{H}_1$, a contradiction. This proves the lemma. □

Proof of Lemma 2.5. Choose $v' \in J^*$. Then $d(v' v_{n_2}, G_1) \geq n_1 + n_2 + 4 - (n_2 - 1) = n_1 + 5$. Thus $i(v' v_{n_2}, G_1) \geq 5$. By Lemma 2.4, $G_1 - u \notin \mathcal{P}_1$ for all $u \in I(v' v_{n_2}, G_1) - V(e_1)$. Therefore $G_1 \notin \mathcal{H}_1$. By Property B, $\mathcal{P}_1^*(G_1) \neq \emptyset$. We claim $\delta_1(G_1) \leq (n_1 - 1)/2$. To see this, say $\delta_1(G_1) \geq n_1/2$. Choose any path from $\mathcal{P}_1^*(G_1)$ and then apply Lemma 3.5(c) with this path in G_1. As $d(v_{n_2}, G_1) \geq (n_1 + n_2 + 4)/4 - 2 = (n_1 + n_2)/2$, we see that G_1 has an x-y e_1-hamiltonian path such that $y \notin V(e_1)$, $d(y, G_1) = n_1/2$ and $yv_{n_2} \in E$. As

$d(y, G_2) \geq (n_2+4)/2$, $d(y, J^*) > 0$ and so $G_2 + y \in \mathcal{H}_2$, contradicting Lemma 2.4. □

Proof of Lemma 2.6. The statement (a) is evident by the definition of $(G_{2i-1}, G_{2i})(1 \leq i \leq k)$. We show (b) by contradiction. Say on the contrary that $d(v, G_{2i}) \leq (|G_{2i}|+3)/2$ for some $v \in S_2(G_{2i})$ and $i \in \{1, \ldots, k\}$. Let i be minimal. Then $d(v, G_{2i-1}) \geq (|G_{2i-1}|+1)/2$ and so $G_{2i-1} + v \in \mathcal{P}_1$ by Lemma 3.2(a). As $\mathcal{P}_2^*(G_{2i}) = \emptyset$, $\mathcal{P}_2^*(G_{2i} - v) = \emptyset$. By the maximality of $e(G_{2(i-1)-1}) + e(G_{2(i-1)})$, we shall have

$$\begin{aligned} & e(G_{2(i-1)-1}) + e(G_{2(i-1)}) \\ (4) \qquad & \geq e(G_{2i-1}+v) + e(G_{2i}-v) \\ & \geq e(G_{2i-1}) + e(G_{2i}) - (|G_{2i}|+3)/2 + (|G_{2i-1}|+1)/2. \end{aligned}$$

Let $P = v_q v_{q-1} \cdots v_1$ be an optimal path at $e_2 = v_q v_{q-1}$ in $G_{2(i-1)}$, where $q = |G_{2(i-1)}|$. Say $\alpha(P, v_1) = r$. As $\delta_2(G_{2(i-1)}) \geq (|G_{2(i-1)}| + 4)$ and $\mathcal{P}_2^*(G_{2(i-1)}) = \emptyset$, we see that $v_1 v_2 \cdots v_r v_1$ is an end-cycle at v_r in $G_{2(i-1)}$. As $d(w_{i-1}, G_{2(i-1)}) \geq (|G_{2(i-1)}|+5)/2$ and $G_{2(i-1)} + w_{i-1} \notin \mathcal{H}_2$, we see that $\mathcal{P}_2^*(G_{2(i-1)} + w_{i-1}) = \emptyset$. By the maximality of $e(G_{2i-1}) + e(G_{2i})$, we shall have

$$\begin{aligned} & e(G_{2i-1}) + e(G_{2i}) \\ (5) \qquad & \geq e(G_{2(i-1)-1} - w_{i-1}) + e(G_{2(i-1)} + w_{i-1}) \\ & \geq e(G_{2(i-1)-1}) + e(G_{2(i-1)}) - (|G_{2(i-1)-1}|-1)/2 + (|G_{2(i-1)}+5)/2. \end{aligned}$$

By (4) and (5), we see that

$$e(G_{2(i-1)-1}) + e(G_{2(i-1)}) > e(G_{2(i-1)-1}) + e(G_{2(i-1)}),$$

a contradiction. □

Proof of Lemma 2.7. On the contrary, say the claim fails. Let $x_0 \in V(G_{2k-1})$ such that $G_{2k-1} - x_0 \in \mathcal{P}_1$, $G_{2k} + x_0 \in \mathcal{H}_2$ and $d(x_0, R' - \{y_1, y_{r-1}\}) > 0$. Let $y_c \in V(R') - \{y_1, y_{r-1}\}$ with $x_0 y_c \in E$. Since $G_{2k} + x_0 \in \mathcal{H}_2$ and $\mathcal{P}_2^*(G_{2k}) = \emptyset$, either $x_0 y_t \in E$ or $x_0 y_{t-1} \in E$ with $y_t y_{t-2} \in E$. Say without loss of generality $x_0 y_t \in E$.

Set $H = G_{2k-1} - x_0$ and $p = |H| = s - 1$. As $s \geq t$ and $t - 1 \geq r$, for each $y \in V(R')$, $d(y, H) \geq \lceil (s+t+4)/2 - (r-1) - d(y, x_0) \rceil \geq 3$.

Assume for the moment that for every $P \in \mathcal{P}_1(H)$, $d^*(P, H) \geq p+2$ for each $P \in \mathcal{P}_1(H)$. By Lemma 3.3, $H \in \mathcal{H}_1$. By Lemma 3.7, $d(uv, H) \geq p+2$ for all $u, v \in V(H)$ with $u \neq v$ and $\{u, v\} \neq V(e_1)$. Let y_i and y_j be two distinct vertices of $R' - y_c$ such that $\{y_i, y_j\} \neq \{y_1, y_{r-1}\}$ and $y_i y_j \in E$. Let C be an e_1-hamiltonian cycle of H. Then there is an orientation of C such that for some $u, v \in V(C)$ with $u \neq v$ and $V(e_1) \neq \{u, v\}$, we have $e_1 \notin \{uu^+, vv^+\}$ and $\{y_i u^+, y_j v^+\} \subseteq E$. Let $y' \in N(y_r, R' - y_c)$ be such that $y' \notin \{y_i, y_j\}$. By Lemma 3.6, H has a u^+-v^+ e_1-hamiltonian path. Since Theorem B holds for R', R' has two disjoint paths P'' and P' such that $|P''| = n_1 - p$, $|P'| =$

$r-1-|P''|$, P'' is from y_i to y_j and P' is from y' to y_c. Thus $[H, P''] \in \mathcal{H}_1$ and $G_{2k} - V(P'') + x_0 \in \mathcal{H}_2$, i.e., G contains two required cycles, a contradiction.

Therefore $d^*(P, H) \leq p+1$ for some $P \in \mathcal{P}_1(H)$. Say $P = z_1 \cdots z_p$. First, assume that $d(y_i, z_1 z_p) > 0$ for some $y_i \in V(R') - \{y_1, y_{r-1}, y_c\}$. Say without loss of generality $z_1 y_i \in E$. Then $y_i z_p \notin E$. If there exists $z_p y_j \in E$ for some $y_j \in N(y_i, R') - \{y_c\}$, then we obtain the two required cycles as above. Therefore $z_p y_j \notin E$ for all $y_j \in N(y_i, R') - \{y_c\}$. Thus $d(z_p, R) \leq r - (d(y_i, R) - 2)$ and so $d(z_p, G_{2k}) \leq t - r + r - (d(y_i, R) - 2) = t - d(y_i, R) + 2$. As $d(y_i, R) \geq (t+4)/2$, $d(z_p, G_{2k}) \leq t/2$. Therefore $d(z_p, H) \geq (s+t+4)/2 - t/2 - d(z_p, x_0) \geq (s+2)/2$. Similarly, if $z_p y_1 \in E$, then $z_1 y_a \notin E$ for each $y_a \in N(y_1, R' - \{y_{r-1}, y_c\})$. Consequently, $d(z_1, R) \leq r - (d(y_1, R) - 3)$ and $d(z_1, G_{2k}) \leq t - d(y_1, R) + 3 \leq (t+2)/2$. It follows that $d(z_1, H) \geq s/2$ and so $d(z_1 z_p, H) \geq s + 1 = p + 2$, a contradiction. Therefore $z_p y_1 \notin E$. Similarly, $z_p y_{r-1} \notin E$. Thus $N(z_p, R) \subseteq \{y_r, y_c\}$ and so $d(z_p, G_{2k}) \leq t - r + 2$. Let $y_j \in N(y_i, R') - \{y_c\}$. Then $d(z_p y_j, G_{2k}) \leq t - r + 2 + r - 1 = t + 1$. Thus $d(z_p y_j, H) \geq s + t + 4 - (t+1) - d(x_0, z_p y_j) \geq p + 2$. By Lemma 3.2(d), $H + y_j$ has a z_1-y_j e_1-hamiltonian path and so $H + y_i + y_j$ has a y_i-y_j e_1-hamiltonian path. As above, we see that G contains two required cycles, a contradiction.

Therefore $N(z_1, R) \cup N(z_p, R) \subseteq \{y_1, y_{r-1}, y_r, y_c\}$ and so $d(z_1 z_p, G_{2k}) \leq 2(t-r) + 8$. As $r \geq \delta_2(G_{2k}) + 1 \geq (t+6)/2$, we get $d(z_1 z_p, G_{2k}) \leq t + 2$. Therefore $p + 1 \geq d(z_1 z_p, H) \geq s + t + 4 - (t+2) - d(x_0, z_1 z_p) \geq p + 1$. This implies that $N(z_1, R) = N(z_p, R) = \{y_1, y_{r-1}, y_r, y_c\}$, $r = (t+6)/2$ and $R \cong K_{(t+6)/2}$. It follows that G contains two required cycles as above. □

References

[1] S. Abbasi, *Spanning cycles in dense graphs*, Ph.D. thesis, Rutgers University, 1999.

[2] B. Bollobás, *Extremal Graph Theory*, Academic Press, London, 1978.

[3] J. Bondy and V. Chvátal, *A method in graph theory*, Discrete Math. **15** (1976), no. 2, 111–135.

[4] Y. Egawa, R. Faudree, E. Gyori, Y. Ishigami, R. Schelp, and H. Wang, *Vertex-disjoint cycles containing specified edges*, Graphs Combin. **16** (2000), no. 1, 81–92.

[5] M. H. El-Zahar, *On circuits in graphs*, Discrete Math. **50** (1984), no. 2-3, 227–230.

[6] P. Erdős and T. Gallai, *On maximal paths and circuits of graphs*, Acta Math. Acad. Sci. Hungar **10** (1959), 337–356.

[7] C. Magnant and K. Ozeki, *Partitioning graphs into paths and cycles*, J. Comb. **3** (2012), no. 2, 135–161.

[8] H. Wang, *Covering a graph with cycles passing through given edges*, J. Graph Theory **26** (1997), no. 2, 105–109.

[9] J. E. Williamson, *Panconnected graph II*, Period. Math. Hungar. **8** (1977), no. 2, 105–116.

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