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REMARKS ON LEVI HARMONICITY OF CONTACT SEMI-RIEMANNIAN MANIFOLDS

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Abstract. In a recent paper [10] we introduced the notion of *Levi harmonic* map f from an almost contact semi-Riemannian manifold $(M, \varphi, \xi, \eta, g)$ into a semi-Riemannian manifold M'. In particular, we computed the tension field $\tau_{\mathcal{H}}(f)$ for a CR map f between two almost contact semi-Riemannian manifolds satisfying the so-called *φ-condition*, where $\mathcal{H} = \mathrm{Ker}(\eta)$ is the Levi distribution. In the present paper we show that the condition (A) of Rawnsley [17] is related to the φ-condition. Then, we compute the tension field $\tau_{\mathcal{H}}(f)$ for a CR map between two arbitrary almost contact semi-Riemannian manifolds, and we study the concept of Levi pluriharmonicity. Moreover, we study the harmonicity on quasi-cosymplectic manifolds.

1. Introduction

As a natural continuation of the ideas in [2], and following the ideas of B. Fuglede (who started the study of the semi-Riemannian case within harmonic map theory, cf. [11] and [1], pp. 427–455), in the recent paper [10] S. Dragomir and the present author introduced the concept of *Levi harmonic* map f from an almost contact semi-Riemannian manifold $(M, \varphi, \xi, \eta, g)$ into a semi-Riemannian manifold (M', g'), i.e., C^∞ solutions of $\tau_{\mathcal{H}}(f) \equiv \mathrm{trace}_g(\Pi_{\mathcal{H}}\beta_f) = 0$, where β_f is the second fundamental form of f, and $\Pi_{\mathcal{H}}\beta_f$ is the restriction of β_f to the Levi distribution $\mathcal{H} = \mathrm{Ker}(\eta)$. Thus, we studied the Levi harmonicity for CR maps between two almost contact semi-Riemannian manifolds. This is perhaps the most general geometric setting (metrics are but semi-Riemannian and in general the contact condition (2.2) is not satisfied and the underlying almost CR structures are not integrable).

In the study of [10] an important role is played by the notion of *φ-condition* (cf. (3.1) in §3), in particular we computed the tension field $\tau_{\mathcal{H}}(f)$ for a CR map

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f between two almost contact semi-Riemannian manifolds satisfying the the φ-*condition*. Moreover, we emphasized that the class of almost contact semi-Riemannian manifolds obeying to (3.1) is quite large. For instance, contact semi-Riemannian manifolds, quasi-cosimplectic manifolds and orientable real hypersurfaces in an indefinite Kaehler manifold (with the induced almost contact semi-Riemannian structure) satisfy the φ-condition. On the other hand, Rawnsley [17] introduced the so-called condition (A) in order to study the harmonicity of f-holomorphic maps between an almost Hermitian manifold with coclosed Kaehler form and a Riemannian manifold equipped with a f-structure which satisfies condition (A).

In the present paper we show that the condition (A) of Rawnsley is related to the φ-condition, more precisely (cf. Theorem 3.2): if an almost contact semi-Riemannian structure satisfies the condition (A), then it satisfies the φ-condition. The converse does not hold. In Section 4, we compute the tension field $\tau_{\mathcal{H}}(f)$ for a CR map $f : M \to M'$ between two arbitrary almost contact semi-Riemannian manifolds (this result extends Theorem 3.9 of [10]). In Section 5, in analogy with the Kaehlerian case [12], we introduce and study the concept of Levi pluriharmonicity (cf. (5.1)). Of course Levi pluriharmonicity implies Levi harmonicity. In particular we get that if M is an invariant semi-Riemannian submanifold of a contact semi-Riemannian manifold $\overline{M}$, then the inclusion $i : M \to \overline{M}$ is Levi pluriharmonic and a pseudohermitian map. In the last section, we show that any CR map $f : M \to M'$ among two quasi-cosymplectic manifolds is Levi pluriharmonic. Moreover, a CR map $f : M \to M'$ among two quasi-cosymplectic manifolds is a harmonic map if and only if $f_* \xi$ is a geodesic vector field.

2. Preliminaries

2.1. Contact semi-Riemannian manifolds

Let M be a real $(2n+1)$-dimensional C^∞ manifold. An *almost contact structure* (φ, ξ, η) on M consists of a $(1,1)$-tensor field φ, a tangent vector field $\xi \in \mathfrak{X}(M)$ (the *characteristic*, or *Reeb*, *vector field*) and a differential 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

In particular $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. Let $\varepsilon \in \{\pm 1\}$. Given an almost contact structure (φ, ξ, η) on M a *compatible metric* is a semi-Riemannian metric g on M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\, \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.1}$$

Then $\eta(X) = \varepsilon\, g(\xi, X)$ and $g(\xi, \xi) = \varepsilon$. Therefore the characteristic vector field ξ is either spacelike or timelike (ξ is never lightlike). The synthetic object (φ, ξ, η, g) is an *almost contact semi-Riemannian structure*. If in addition the *contact condition*

$$d\eta = g(\cdot, \varphi \cdot) \tag{2.2}$$

is satisfied, then η is a contact form, i.e., $\eta \wedge (d\eta)^n$ is a volume form on M (and (φ, ξ, η, g) is referred to as *contact semi-Riemannian structure* on M). On each contact semi-Riemannian manifold the tensor field $h = (1/2)\, \mathcal{L}_\xi \varphi$ (where $\mathcal{L}$ is the Lie derivative) is symmetric and satisfies

$$\nabla \xi = -\varepsilon\, \varphi - \varphi \circ h, \quad \nabla_\xi \varphi = 0, \quad h \circ \varphi + \varphi \circ h = 0, \quad h(\xi) = 0. \tag{2.3}$$

Here ∇ is the Levi-Civita connection of the semi-Riemannian manifold (M, g). Moreover (by Lemma 4.3 in [6])

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X} \varphi)\varphi Y = 2g(X,Y)\xi - \eta(Y)\big\{\varepsilon\, X + \varepsilon\, \eta(X)\xi + h(X)\big\}. \tag{2.4}$$

We may refer to [6], [15], [16] for more information about the geometry of a contact semi-Riemannian manifold.

2.2. Almost CR structures

Let M be a real $(2n+1)$-dimensional manifold. An *almost CR structure* on M is a complex subbbundle $T_{1,0}(M)$, of complex rank n, of the complexified tangent bundle $T(M) \otimes \mathbb{C}$ such that $T_{1,0}(M) \cap T_{0,1}(M) = (0)$, where $T_{0,1}(M) = \overline{T_{1,0}(M)}$ (overbars denote complex conjugates). The integer n is the *CR dimension*. An almost CR structure $T_{1,0}(M)$ is *integrable*, and then $T_{1,0}(M)$ is referred to as a *CR structure*, if $Z, W \in C^\infty(U, T_{1,0}(M))$ yields $[Z, W] \in C^\infty(U, T_{1,0}(M))$ for any open set $U \subset M$. The *Levi* (or *maximally complex*) *distribution* is the real rank $2n$ distribution on M given by $\mathcal{H} \equiv H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$. It carries the complex structure

$$J : \mathcal{H} \to \mathcal{H}, \quad J(Z + \overline{Z}) = i(Z - \overline{Z}), \quad Z \in T_{1,0}(M) \;\; (i = \sqrt{-1}).$$

Then $T_{1,0}(M) = \{X - iJX : X \in \mathcal{H}\}$, i.e., $T_{1,0}(M)$ is the eigenbundle of J (the $\mathbb{C}$-linear extension of J to $\mathcal{H} \otimes \mathbb{C}$) corresponding to the eigenvalue i. The pair $(\mathcal{H}, J)$ (the real manifestation of $T_{1,0}(M)$) is often referred to as an almost CR structure on M, as well. A *pseudohermitian structure* is a differential 1-form $\theta \in \Omega^1(M)$ such that $\mathrm{Ker}(\theta) = \mathcal{H}$. Given a pseudohermitian structure θ on M the *Levi form* G_θ is given by

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in \mathcal{H}.$$

An almost CR structure $(\mathcal{H}, J)$ is *nondegenerate* if the Levi form G_θ is nondegenerate for some θ. If this is the case θ is a contact form (i.e., $\theta \wedge (d\theta)^n$ is a volume form). An almost CR structure $(\mathcal{H}, J)$ is *strictly pseudoconvex* if G_θ is positive definite for some θ. Let $(M, \mathcal{H}, J)$ be a nondegenerate almost CR manifold and θ a fixed contact form on M. Let us extend J to an endomorphism φ of the tangent bundle by requesting that $\varphi = J$ on $\mathcal{H}$ and $\varphi(T) = 0$. Here $T \in \mathfrak{X}(M)$ is unique nowhere zero tangent vector field on M determined by $\theta(T) = 1$ and $(d\theta)(T, \cdot) = 0$. Then $\varphi^2 = -I + \theta \otimes T$. The *Webster metric* is the semi-Riemannian metric g_θ given by

$$g_\theta(X, Y) = (d\theta)(X, JY), \;\; g_\theta(X, T) = 0, \;\; g_\theta(T, T) = 1$$

for any $X, Y \in \mathcal{H}$. Then $(\varphi,\ \xi = -T, \eta = -\theta,\ g = g_\theta)$ is a contact semi-Riemannian structure on M. If G_θ is positive definite the Webster metric g_θ is a Riemannian metric (and $(\varphi,\ \xi,\ \eta,\ g)$ is a contact metric structure on M). Conversely any almost contact manifold (M, φ, ξ, η) carries the almost CR structure given by $\mathcal{H} = \mathrm{Ker}(\eta)$ and $J = \varphi|_{\mathcal{H}}$.

2.3. Levi harmonic maps

Let $(M, \varphi, \xi, \eta, g)$ be a real $(2n+1)$-dimensional almost contact semi-Riemannian manifold and (M', g') a semi-Riemannian manifold. Let $f : M \to M'$ be a C^∞ map and $f^{-1}T(M') \to M$ the pullback of $T(M')$ by f. Let $\nabla'^f = f^{-1}\nabla'$ be the pullback of the Levi-Civita connection ∇' of (M', g'), i.e., the connection in the vector bundle $f^{-1}T(M') \to M$ induced by ∇'. If (U, x^i) and (V, y^α) are local coordinate systems on M and N such that $f(U) \subset V$, then ∇'^f is locally described by

$$\nabla'^f_{\partial/\partial x^j}(\partial/\partial y^\beta)^f = \frac{\partial f^\alpha}{\partial x^j}\left(\Gamma'^\gamma_{\alpha\beta} \circ f\right)(\partial/\partial y^\gamma)^f,$$

where $Y^f = Y \circ f \in C^\infty(f^{-1}(V),\ f^{-1}T(M'))$ denotes the natural lift of $Y \in \mathfrak{X}(V)$ and $\Gamma'^\gamma_{\alpha\beta}$ are the Christoffel symbols of (M', g'). Let $\mathcal{H} = \mathrm{Ker}(\eta)$ and $J = \varphi|_{\mathcal{H}}$ be the almost CR structure underlying (φ, ξ, η, g). The second fundamental form β_f of f is given by

$$\beta_f(X, Y) = \nabla'^f_X f_* Y - f_* \nabla_X Y, \quad X, Y \in \mathfrak{X}(M). \tag{2.5}$$

Here ∇ is the Levi-Civita connection of (M, g). Also $f_*X \in C^\infty(f^{-1}T(M'))$ is given by $(f_*X)(x) = (d_x f)X_x \in T_{f(x)}(M')$ for any $x \in M$ and any $X \in \mathfrak{X}(M)$. Next let $\tau_{\mathcal{H}}(f) \in C^\infty(f^{-1}T(M'))$ be given by

$$\tau_{\mathcal{H}}(f) = \mathrm{trace}_g\,(\Pi_{\mathcal{H}}\beta_f), \tag{2.6}$$

where $\Pi_{\mathcal{H}}\beta_f$ is the restriction of β_f to $\mathcal{H} \otimes \mathcal{H}$.

Definition 2.1. A C^∞ map $f : M \to M'$ is *Levi harmonic* with respect to $\mathcal{H} = \mathrm{Ker}(\eta)$ if $\tau_{\mathcal{H}}(f) = 0$.

Next, we recall the following:

Definition 2.2. A C^∞ map $f : M \to M'$ of almost CR manifolds is a *CR map* if

$$(d_x f)\mathcal{H}_x \subset \mathcal{H}(M')_{f(x)}, \quad (d_x f) \circ J_x = J'_{f(x)} \circ (d_x f) \tag{2.7}$$

for any $x \in M$.

Typical examples of CR maps are got as traces of holomorphic maps of Kaehlerian manifolds on real hypersurfaces. Precisely let $\overline{M}$ be a Kaehlerian manifold. Any orientable real hypersurface $M \subset \overline{M}$ admits a natural almost contact metric structure (cf. e.g. [5]). If $M' \subset \overline{M}'$ is another oriented real hypersurface in the Kaehlerian manifold $\overline{M}'$ and $F : \overline{M} \to \overline{M}'$ is a holomorphic map such that $F(M) \subset M'$, then $f \equiv F|_M : M \to M'$ is a CR map. It should be emphasized that, in spite of our metric approach [where the wealth of additional first order geometric structure (φ, ξ, η, g) is meant to "compensate" for the lack of integrability of $T_{1,0}(M)$] the property (2.7) is tied to the almost CR structures alone. In particular the statements above hold true for traces of holomorphic maps among indefinite Kaehlerian manifolds (cf. E. Barros and A. Romero, [3], for definitions and examples). Indeed let $\overline{M}$ be an indefinite Kaehlerian manifold and $M \subset \overline{M}$ an orientable real hypersurface. The indefinite Kaehler structure of $\overline{M}$ induces on M an almost contact semi-Riemannian structure (cf. A. Bejancu and K. L. Duggal, [4]).

Let θ and θ' be pseudohermitian structures on the almost CR manifolds M and M', respectively. If $f : M \to M'$ is a CR map, then

$$f^*\theta' = \mu\, \theta$$

for some $\mu \in C^\infty(M)$.

Definition 2.3. A CR map f is *pseudohermitian* if $\mu = c$ for some $c \in \mathbb{R}$. Also f is *isopseudohermitian* if $c = 1$.

3. The φ-condition and the condition (A)

In [10] we adopted the following:

Definition 3.1. We say that an almost contact semi-Riemannian manifold $(M, \varphi, \xi, \eta, g)$ satisfies the *φ-condition* if

$$\nabla_{\varphi X}\varphi X + \nabla_X X = \varphi[\varphi X, X] \tag{3.1}$$

for any $X \in \mathcal{H}$.

Now, we consider the following tensor

$$\mathcal{P}(X, Y) = (\nabla_X \varphi)\varphi Y - (\nabla_{\varphi X}\varphi)Y$$

for any $X, Y \in \mathcal{H} = \ker(\eta)$. $\mathcal{P}$ is a tensor of type $(1, 2)$ which is φ-invariant, i.e., $\mathcal{P}(\varphi X, \varphi Y) = \mathcal{P}(X, Y)$, moreover we note that M satisfies the *φ-condition* if and only if the tensor $\mathcal{P}$ is skew-symmetric, that is $\mathcal{P}(X, X) = 0$. Next, we put

$$\mathcal{H}^+ = T_{1,0}(M) = \{X - i\varphi X,\ X \in \mathcal{H}\}$$

and

$$\mathcal{H}^- = T_{0,1}(M) = \{X + i\varphi X,\ X \in \mathcal{H}\}.$$

Then the complexified tangent bundle of M splits into the direct sum of the tangent bundles $\mathcal{H}^+$, $\mathcal{H}^-$ and $\mathbb{C}\xi$. Besides $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) is the eigenbundle of φ (the $\mathbb{C}$-linear extension to $\mathcal{H} \otimes \mathbb{C}$) corresponding to the eigenvalue i (resp. $-i$).

We note that the tensor field φ of an almost contact semi-Riemannian manifold defines a *f-structure*, that is,

$$\varphi^3 + \varphi = 0.$$

Then, following Rawnsley [17] (cf. p. 91), we say that the almost contact semi-Riemannian structure defines the so-called *condition* (A), if

$$\nabla_V W \in \mathcal{H}^+ \quad \text{for any} \quad V \in \mathcal{H}^- \quad \text{and} \quad W \in \mathcal{H}^+ .$$

Rawnsley introduced a such condition in order to study the harmonicity of f-holomorphic maps between an almost Hermitian manifold with coclosed Kaehler form and a Riemannian manifold equipped with a f-structure which satisfies condition (A).

Since $\big(\nabla_V\varphi\big)W = \nabla_V\varphi W - \varphi\nabla_V W$, *condition* (A) is equivalent to (cf. also [1] Proposition 2.6)

$$\big(\nabla_V\varphi\big)W = 0 \quad \text{for any} \quad V \in \mathcal{H}^- \quad \text{and} \quad W \in \mathcal{H}^+ . \tag{3.2}$$

An almost Hermitian structure satisfying the condition (A) is said to be a $(1,2)$-*symplectic structure* (see also [1] p. 251).

We remark that the condition (A), that is (3.2), is satisfied if and only if

$$(\nabla_X\varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = 0\,, \quad \text{equivalently} \quad \mathcal{P}(X,Y) = 0 \tag{3.3}$$

for any $X, Y \in \mathcal{H}$. Moreover, the condition (3.3) implies the φ-condition, that is $\mathcal{P}(X,X) = 0$. More precisely, one gets

$$\mathcal{P}(X,Y) = 0 \quad \Longleftrightarrow \quad \mathcal{P}(X,X) = 0 \quad \text{and} \quad \mathcal{P}(X,\varphi X) = 0.$$

Therefore, the condition (A) implies the φ-condition. On the other hand, the converse does not hold. In fact, the condition $\mathcal{P}(X,\varphi X) = 0$ means that

$$[X,\varphi X] = \varphi\big(\nabla_{\varphi X}\varphi X + \nabla_X X\big) \in \mathcal{H}$$

for any $X \in \mathcal{H}$. But, if M is a contact semi-Riemannian manifold

$$\begin{aligned}\varepsilon\, g([X,\varphi X],\xi) = \eta([X,\varphi X]) &= -2(\mathrm{d}\eta)(\mathrm{X},\varphi\mathrm{X}) \\ &= -2g(X,\varphi^2 X) = 2g(X,X) \neq 0\end{aligned}$$

for $X \in \mathcal{H}$, $X \neq 0$, X spacelike or timelike.

So contact semi-Riemannian structures do not satisfy the condition (A). However contact semi-Riemannian structures satisfy the φ-condition. In fact, by (2.4), for any $X, Y \in \mathcal{H}$

$$\nabla_X\varphi Y - \varphi\nabla_X Y - \nabla_{\varphi X}Y - \varphi\nabla_{\varphi X}\varphi Y = 2g(X,Y)\xi.$$

In particular for $Y = \varphi X$ one derives (3.1). Hence any contact semi-Riemannian manifold satisfies the φ-condition. Therefore, we get the following:

Theorem 3.2. *Let* (M,φ,ξ,η,g) *be an almost contact semi-Riemannian manifold. If the almost contact semi-Riemannian structure satisfies the condition* (A)*, then it satisfies the* φ*-condition. The converse does not hold.*

So, our φ-condition extends the condition (A) of Rawnsley [17].

Remark 3.3. As emphasized in [10] the class of almost contact semi-Riemannian manifolds obeying to (3.1) is quite large. For instance, contact semi-Riemannian manifolds, orientable real hypersurfaces in an indefinite Kaehler manifold (with the induced almost contact semi-Riemannian structure) and quasi-cosimplectic manifolds (cf. Section 6) satisfy the φ-condition.

4. Harmonicity between almost contact semi-Riemannian manifolds

In this section we establish the following theorem which extends Theorem 3.9 of [10].

Theorem 4.1. *Let* $(M, \varphi, \xi, \eta, g)$ *and* $(M', \varphi', \xi', \eta', g')$ *be two almost contact semi-Riemannian manifolds with* $\dim(M) = 2n+1$*. Then, for each CR map* $f : M \to M'$

$$\tag{4.1} \begin{aligned} \tau_{\mathcal{H}}(f) = & -\operatorname{trace}(\varphi\nabla\xi)\, {\varphi'}^f\, f_*\,\xi + {\varphi'}^f\big(\operatorname{trace}_{|\mathcal{H}} f^*\nabla'\varphi'\big) \\ & - \big(\operatorname{trace}_{|\mathcal{H}} f^*\nabla'\eta'\big)\xi' + f_*\big(\varphi\nabla^*\varphi + (\operatorname{div}\xi)\xi + \varepsilon\nabla_\xi\xi\big), \end{aligned}$$

where $\varepsilon = g(\xi,\xi)$*,* ${\varphi'}^f : f^{-1}T(M') \to f^{-1}T(M')$ *is the pullback of* φ' *by* f*, and* ∇^* *is the operator formal adjoint of* ∇*.*

Proof. The tangent bundle to any $(2n+1)$-dimensional almost contact semi-Riemannian manifold M admits a local semi-orthonormal frame (a φ-*basis*), i.e., a frame of the form $\{\xi,\, E_\alpha\,,\, \varphi E_\alpha : 1 \le \alpha \le n\}$. By (2.1) if E_α is a spacelike (respectively timelike), then φE_α is spacelike (respectively timelike). In particular a semi-Riemannian metric compatible with an almost contact structure has either signature $(2p+1, 2n-2p)$ or signature $(2p, 2n-2p+1)$, according to whether ξ is spacelike or timelike.

Let $\{\xi,\, E_\alpha\,,\, \varphi E_\alpha : 1 \le \alpha \le n\}$ be a φ-basis and let us set $\varepsilon_\alpha = g(E_\alpha\,, E_\alpha) \in \{\pm 1\}$. Then one has

$$\tag{4.2} \tau_{\mathcal{H}}(f) = \sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ {\nabla'}^f_{E_\alpha} f_* E_\alpha - f_*\nabla_{E_\alpha} E_\alpha + {\nabla'}^f_{\varphi E_\alpha} f_*\varphi E_\alpha - f_*\nabla_{\varphi E_\alpha}\varphi E_\alpha \right\}.$$

We consider the operator ∇^*, the formal adjoint of ∇ (see for example [9], pp. 108–110), thus if S is a tensor of type $(1,1)$, $\nabla^* S = -\operatorname{trace}\nabla S$. Then,

$$\begin{aligned} -\nabla^*\varphi &= \sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ (\nabla_{E_\alpha}\varphi)E_\alpha + (\nabla_{\varphi E_\alpha}\varphi)\varphi E_\alpha) \right\} + \varepsilon(\nabla_\xi\varphi)\xi \\ &= \sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ [E_\alpha, \varphi E_\alpha] - \varphi(\nabla_{E_\alpha}E_\alpha + \nabla_{\varphi E_\alpha}\varphi E_\alpha) \right\} - \varepsilon\varphi\nabla_\xi\xi, \end{aligned}$$

and hence

$$-\varphi\nabla^*\varphi = \sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ \varphi[E_\alpha, \varphi E_\alpha] - \varphi^2(\nabla_{E_\alpha}E_\alpha + \nabla_{\varphi E_\alpha}\varphi E_\alpha) \right\} + \varepsilon\nabla_\xi\xi$$

$$= \sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ (\nabla_{E_\alpha} E_\alpha + \nabla_{\varphi E_\alpha} \varphi E_\alpha) - \varphi[\varphi E_\alpha, E_\alpha] \right\} + (\mathrm{div}\xi)\xi + \varepsilon \nabla_\xi \xi,$$

that is,

$$\begin{aligned} &\sum_{\alpha=1}^{n} \varepsilon_\alpha \left(\nabla_{E_\alpha} E_\alpha + \nabla_{\varphi E_\alpha} \varphi E_\alpha \right) \\ &= \sum_{\alpha=1}^{n} \varepsilon_\alpha \, \varphi[\varphi E_\alpha, E_\alpha] - \varphi \nabla^* \varphi - (\mathrm{div}\xi)\xi - \varepsilon \nabla_\xi \xi. \end{aligned} \tag{4.3}$$

Moreover,

$$\begin{aligned} &\varphi'^f \big(\mathrm{trace}_{|\mathcal{H}} f^* \nabla' \varphi'\big) \\ &= \varphi'^f \sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ (\nabla'_{f_* E_\alpha} \varphi') f_* E_\alpha + (\nabla'_{f_* \varphi E_\alpha} \varphi') f_* \varphi E_\alpha \right\} \\ &= \sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ \varphi'^f [f_* E_\alpha, \varphi'^f f_* E_\alpha] - (\varphi'^f)^2 (\nabla'_{f_* E_\alpha} f_* E_\alpha + \nabla'_{f_* \varphi E_\alpha} f_* \varphi E_\alpha) \right\} \\ &= \sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ \varphi'^f [f_* E_\alpha, \varphi'^f f_* E_\alpha] + \nabla'_{f_* E_\alpha} f_* E_\alpha \right. \\ &\quad \left. + \nabla'_{f_* \varphi E_\alpha} f_* \varphi E_\alpha - \eta' (\nabla'_{f_* E_\alpha} f_* E_\alpha + \nabla'_{f_* \varphi E_\alpha} f_* \varphi E_\alpha) \xi' \right\}, \end{aligned}$$

and (as f is a CR map)

$$\begin{aligned} \big(\mathrm{trace}_{|\mathcal{H}} f^* \nabla' \eta'\big) &= \sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ (\nabla'_{f_* E_\alpha} \eta') f_* E_\alpha + (\nabla'_{f_* \varphi E_\alpha} \eta') f_* \varphi E_\alpha) \right\} \\ &= -\sum_{\alpha=1}^{n} \varepsilon_\alpha \, \eta' (\nabla'_{f_* E_\alpha} f_* E_\alpha + \nabla'_{f_* \varphi E_\alpha} f_* \varphi E_\alpha) \end{aligned}$$

give

$$\begin{aligned} &\sum_{\alpha=1}^{n} \varepsilon_\alpha \{ (\nabla'_{f_* E_\alpha} f_* E_\alpha + \nabla'_{f_* \varphi E_\alpha} f_* \varphi E_\alpha) \\ &= \varphi'^f \big(\mathrm{trace}_{|\mathcal{H}} f^* \nabla' \varphi'\big) - \big(\mathrm{trace}_{|\mathcal{H}} f^* \nabla' \eta'\big) \xi' + \sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ \varphi'^f [f_* \varphi E_\alpha, f_* E_\alpha] \right\}. \end{aligned} \tag{4.4}$$

Then, (4.2), (4.3) and (4.4) imply

$$\begin{aligned} \tau_{\mathcal{H}}(f) = &\sum_{\alpha=1}^{n} \varepsilon_\alpha \left\{ (\varphi'^f f_* - f_* \varphi)[\varphi E_\alpha, E_\alpha] \right\} + f_* \big(\varphi \nabla^* \varphi + (\mathrm{div}\xi)\xi + \varepsilon \nabla_\xi \xi \big) \\ &+ \varphi'^f \big(\mathrm{trace}_{|\mathcal{H}} f^* \nabla' \varphi'\big) - \big(\mathrm{trace}_{|\mathcal{H}} f^* \nabla' \eta'\big) \xi' . \end{aligned} \tag{4.5}$$

Note that $\varphi'^f f_* - f_* \varphi = 0$ on $\mathcal{H}$,

$$[\varphi E_\alpha \, , \, E_\alpha] = -\varphi^2 [\varphi E_\alpha \, , \, E_\alpha] + g([\varphi E_\alpha, E_\alpha] \, , \, \xi) \xi$$

and

$$\begin{aligned}\sum_{\alpha=1}^{n}\varepsilon_\alpha\, g([\varphi E_\alpha\,,\,E_\alpha]\,,\,\xi) &= \sum_{\alpha=1}^{n}\varepsilon_\alpha\ \{g(\nabla_{\varphi E_\alpha}E_\alpha\,,\,\xi)-g(\nabla_{E_\alpha}\varphi E_\alpha\,,\,\xi)\}\\ &= \sum_{\alpha=1}^{n}\varepsilon_\alpha\ \{-g(\nabla_{\varphi E_\alpha}\xi\,,\,E_\alpha)+g(\nabla_{E_\alpha}\xi\,,\,\varphi E_\alpha)\}\\ &= -\sum_{\alpha=1}^{n}\varepsilon_\alpha\ \{g(\varphi\nabla_{\varphi E_\alpha}\xi\,,\,\varphi E_\alpha)+g(\varphi\nabla_{E_\alpha}\xi\,,\,E_\alpha)\}\\ &= -\mathrm{trace}(\varphi\,\nabla\xi)+\varepsilon\, g(\varphi\,\nabla_\xi\xi\,,\,\xi) = -\mathrm{trace}(\varphi\,\nabla\xi).\end{aligned}$$

So, from (4.5) we get (4.1). □

Next, if M and M' satisfy the φ-condition, from (4.3) and (4.4) we have

$$\varphi\nabla^*\varphi+(\mathrm{div}\xi)\xi+\varepsilon\nabla_\xi\xi=0$$

and

$$\varphi'^f\big(\mathrm{trace}_{|\mathcal{H}}f^*\nabla'\varphi'\big)-\big(\mathrm{trace}_{|\mathcal{H}}f^*\nabla'\eta'\big)\xi'=0\,.$$

Therefore we obtain:

Corollary 4.2 ([10])**.** *Let (M,φ,ξ,η,g) and $(M',\varphi',\xi',\eta',g')$ be two almost contact semi-Riemannian manifolds with $\dim(M)=2n+1$, satisfying the φ-condition. Then, for any CR map $f:M\to M'$*

$$\tau_{\mathcal{H}}(f)=-\mathrm{trace}(\varphi\nabla\xi)\,\varphi'^f f_*\,\xi\,. \tag{4.6}$$

If additionally (M,φ,ξ,η,g) is a contact semi-Riemannian manifold, then

$$\tau_{\mathcal{H}}(f)=-2n\,\varepsilon\,\varphi'^f f_*\,\xi, \tag{4.7}$$

where $\varepsilon=g(\xi,\xi)$. Hence f is Levi harmonic if and only if $f_*\,\xi$ is collinear to ξ'.

Moreover, since

$$\tau(f)=\tau_{\mathcal{H}}(f)+\varepsilon\left(\nabla'_{f_*\xi}f_*\xi-f_*\nabla_\xi\xi\right),$$

from Theorem 4.1, we get:

Theorem 4.3. *Let (M,φ,ξ,η,g) and $(M',\varphi',\xi',\eta',g')$ be two almost contact semi Riemannian manifolds satisfying the φ-condition, with ξ geodesic and* $\mathrm{trace}(\varphi\nabla\xi)=0$*. Then, a CR map $f:M\to M'$ is a harmonic map if and only if $f_*\,\xi$ is a geodesic vector field.*

An application of this theorem is given in Section 6.

5. Levi pluriharmonicity

Let (M,g,J) be a Kaehler manifold and (M',g') a Riemannian manifold. Following [12] a C^∞ map $f:M\to M'$ is said to be *pluriharmonic* if the second fundamental form β_f satisfies

$$\beta_f(JX,JY)+\beta_f(X,Y)=0,\quad \forall X,Y\in\mathfrak{X}(M).$$

Then, we introduce the following:

Definition 5.1. Let (M,φ,ξ,η,g) be an almost contact semi-Riemannian manifold and $(M,',g')$ a semi-Riemannian manifold. A C^∞ map $f:M\to M'$ is said to be *Levi pluriharmonic* if the second fundamental form β_f satisfies

$$\beta_f(\varphi X,\varphi Y)+\beta_f(X,Y)=0,\quad \forall X,Y\in\mathcal{H}, \tag{5.1}$$

equivalently

$$\beta_f(\varphi X,\varphi X)+\beta_f(X,X)=0,\quad \forall X\in\mathcal{H}.$$

Of course Levi pluriharmonicity implies Levi harmonicity. Now, we show the following.

Theorem 5.2. *Let (M,φ,ξ,η,g) be a contact semi-Riemannian manifold and $(M',\varphi',\xi',\eta',g')$ an almost contact semi-Riemannian manifold satisfying the φ-condition with ξ' geodesic vector field. If $f:M\to M'$ is a CR map, then f is Levi pluriharmonic if and only if f is a pseudohermitian map.*

Proof. Necessity. We suppose that f is a pseudohermitian map. Thus f is a CR map and $f_*\xi=c\xi'$ with $c\in\mathbb{R}$. This implies that

$$f_*\varphi X={\varphi'}^f f_*X,\quad \forall X\in\mathfrak{X}(M). \tag{5.2}$$

Then, as f is a CR map one has

$$\begin{aligned}\beta_f(\varphi X,\varphi X)+\beta_f(X,X)&=\{{\nabla'}^f_X f_*X-f_*\nabla_XX+{\nabla'}^f_{\varphi X}f_*\varphi X-f_*\nabla_{\varphi X}\varphi X\}\\&=\{{\nabla'}^f_X f_*X+{\nabla'}^f_{\varphi X}{\varphi'}^f f_*X-f_*(\nabla_XX+\nabla_{\varphi X}\varphi X)\}.\end{aligned}$$

Next, since both M and M' satisfy the φ-condition, by (5.2)

$$\beta_f(\varphi X,\varphi X)+\beta_f(X,X)=\left({\varphi'}^f f_*-f_*\varphi\right)[\varphi X\,,\,X]=0.$$

Hence f is Levi pluriharmonic. Conversely, we suppose that f is Levi pluriharmonic. Then f is Levi harmonic. On the other hand by Corollary 4.2

$$\tau_{\mathcal{H}}(f)=-\mathrm{trace}(\varphi\nabla\xi)\,{\varphi'}^f\,f_*\,\xi\,.$$

Since for a contact semi-Riemannian manifold $\mathrm{trace}(\varphi\nabla\xi)=2n\varepsilon\neq 0$, we get $\varphi'\,f_*\,\xi=0$ and thus $f_*\,\xi=\lambda\,\xi'$ for some $\lambda\in C^\infty(M,\mathbb{R})$, that is $f^*\eta'=\lambda\,\eta$. Then

$$\begin{aligned}\xi(\lambda)\,\eta-d\lambda&=(d\lambda\wedge\eta)(\xi\,,\,\cdot)=d(\lambda\eta)(\xi\,,\,\cdot)\\&=(d\,f^*\eta')(\xi\,,\,\cdot)=(f^*\,d\,\eta')(\xi\,,\cdot)\\&=(d\eta')(f_*\xi\,,\,f_*\cdot)=\lambda\,(d\eta')(\xi'\,,\,f_*\cdot).\end{aligned}$$

On the other hand, since ξ' is a geodesic vector field, we have

$$\begin{aligned} 2(d\eta')(\xi', X) &= \xi'\eta'(X) - X\eta'(\xi') - \eta'([\xi', X]) \\ &= \varepsilon g'(\nabla'_{\xi'}\xi', X) \\ &= 0. \end{aligned}$$

Therefore $d\lambda = \xi(\lambda)\,\eta$ and consequently

$$\xi(\lambda)\,\eta \wedge d\eta = d\lambda \wedge d\eta = -d(d\lambda \wedge \eta) = -d(\xi(\lambda)\,\eta \wedge \eta) = 0.$$

So, since η is a contact form, we get that $\xi(\lambda) = 0$. This gives that λ is a constant and f is a pseudohermitian map. □

Let $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta})$ be an almost contact manifold. A submanifold M of $\overline{M}$ is called an *invariant submanifold* if $\overline{\varphi}_p T_p(M) \subset T_p(M)$ for any $p \in M$. Then, we have two cases (cf. [18]): I) $\overline{\xi}$ is tangent to M (and then M is odd-dimensional) or II) $\overline{\xi}$ is transverse to M (and then M is even-dimensional). When $\overline{M}$ is a contact semi-Riemannian manifold case II doesn't occur (cf. [5], p. 122). We consider the case I, and then M carries the induced almost contact structure (φ, ξ, η) given by

$$\overline{\varphi} \circ i_* = i_* \circ \varphi, \quad \eta = i^*\,\overline{\eta}, \quad \overline{\xi} = i_*\xi, \tag{5.3}$$

where $i : M \to \overline{M}$ is the inclusion. In particular i is a CR map.

Now, we assume that $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is an almost contact semi-Riemannian manifold, M an odd-dimensional invariant submanifold of $\overline{M}$ equipped with the induced almost contact structure (φ, ξ, η), and let g be a semi-Riemannian metric on M such that $g = i^*\,\overline{g}$. Then g is compatible with (φ, ξ, η), that is (φ, ξ, η, g) is an almost contact semi-Riemannian structure on M. In such case we say that M is an *invariant semi-Riemannian submanifold* of $\overline{M}$.

Suppose that $\overline{M}$ satisfies the $\overline{\varphi}$-condition. Then for any $X \in \mathcal{H}$

$$\overline{\nabla}_{\overline{\varphi}X}\overline{\varphi}X + \overline{\nabla}_X X = \overline{\varphi}[\overline{\varphi}X\,,\,X],$$

and hence (by the Gauss formula)

$$\nabla_{\varphi X}\varphi X + \alpha(\varphi X, \varphi X) + \nabla_X X + \alpha(X, X) = \varphi[\varphi X, X]. \tag{5.4}$$

The tangential and normal components of (5.4) give

$$\nabla_{\varphi X}\varphi X + \nabla_X X = \varphi[\varphi X, X],$$

and

$$\alpha(\varphi X, \varphi X) + \alpha(X, X) = 0 \tag{5.5}$$

for any $X \in \mathcal{H}$. Thus by (5.5) the inclusion i is Levi-pluriharmonic.

Next, we note that if $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a contact semi-Riemannian structure on $\overline{M}$, then from (5.3) follows that

$$\begin{aligned} d\eta = d\,i^*\overline{\eta} = i^*d\overline{\eta} &= \overline{g}(i_*, \overline{\varphi} \circ i_*) \\ &= \overline{g}(i_*, i_* \circ \varphi) = (i^*\overline{g})(\cdot, \varphi) = g(\cdot, \varphi), \end{aligned}$$

that is, $(M, \varphi, \xi, \eta, g)$ is an invariant contact semi-Riemannian manifold of $\overline{M}$.

Therefore, using also Theorem 5.2, we get:

Theorem 5.3. *Let $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be an almost contact semi-Riemannian manifold satisfying the $\overline{\varphi}$-condition. If (M, g) is an odd-dimensional invariant semi-Riemannian submanifold of $\overline{M}$, then the inclusion $i : M \to \overline{M}$ is Levi pluriharmonic. In particular, if $\overline{M}$ is a contact semi-Riemannian manifold and M is an invariant semi-Riemannian submanifold of $\overline{M}$, then the inclusion $i : M \to \overline{M}$ is Levi pluriharmonic and a pseudohermitian map.*

Now we give some examples of Levi pluriharmonic maps.

Sasakian space forms. Let $M^{2n+3}(c)$ be a complete simply connected Sasakian manifold of constant ϕ-sectional curvature c. As well known $M^{2n+3}(c)$ is equivalent to one of the Sasakian manifolds S^{2n+3}, $\mathbb{R}^{2n+3}$ or $D^{n+1}\times\mathbb{R}$ equipped with Sasakian structures of φ-sectional curvature $c > -3$, $c = -3$ and $c < -3$, respectively, where $D^{n+1} \subset \mathbb{C}^{n+1}$ is a simply connected bounded domain. Then $M^{2n+1}(c)$ is an invariant submanifold of $M^{2n+3}(c)$ (cf. [19], p. 328) and, by Theorem 5.3, *the inclusion $i : M^{2n+1}(c) \to M^{2n+3}(c)$ is Levi pluriharmonic and a pseudohermitian map.*

The Brieskorn sphere. Let $\mathbb{C}^{n+1}$ with the Cartesian complex coordinates $z = (z_0, \dots, z_n)$ and $a = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$ such that $a_j \geq 2$. Let us consider the polynomial $P_a(z) = \sum_{j=0}^{n} z_j^{a_j} \in \mathbb{C}[z]$. Then $B^{2n}(a) = \{z \in \mathbb{C}^{n+1} : P_a(z) = 0\}$ is an algebraic hypersurface in $\mathbb{C}^{n+1}$ and $B^{2n}(a) \setminus \{0\}$ is an n-dimensional complex submanifold. Let us set $\Sigma^{2n-1}(a) = B^{2n}(a) \cap S^{2n+1}$ (the *Brieskorn sphere* determined by a). By a result in [19], pp. 303–305, S^{2n+1} admits a Sasakian structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ (distinct from the standard Sasakian structure) such that $\Sigma^{2n-1}(2, \dots, 2)$ is an invariant submanifold of $(S^{2n+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$. Thus, *the inclusion $i : \Sigma^{2n-1}(2, \dots, 2) \to S^{2n+1}$ is Levi pluriharmonic and a pseudohermitian map.*

Let $Q^n = \pi_0(B^{2n+2}(2, \dots, 2))$ be the complex quadric, where $\pi_0 : \mathbb{C}^{n+2} \setminus \{0\} \to \mathbb{C}P^{n+1}$ is the projection. Let $\pi : S^{2n+3} \to \mathbb{C}P^{n+1}$ be the Hopf fibration. The saturated set $P = \pi^{-1}(Q^n)$ is the total space a circle bundle $S^1 \to P \to Q^n$. Then P is a invariant submanifold of the sphere S^{2n+3} equipped with the standard Sasakian structure (cf. [19], p. 328). Hence *the inclusion $P \to S^{2n+3}$ is Levi pluriharmonic and a pseudohermitian map.*

Remark 5.4. Let $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be a contact Riemannian manifold and M an invariant submanifold of $\overline{M}$ equipped with the induced contact Riemannian structure $(M, \varphi, \xi, \eta, g)$. We put

$$\overline{g}_L = \overline{g} - 2\,\overline{\eta} \otimes \overline{\eta} \quad \text{and} \quad g_L = g - 2\,\eta \otimes \eta = i^*\overline{g}_L.$$

Then, $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}_L)$ is a contact Lorentzian structure on $\overline{M}$ with $\overline{\xi}$ time-like, and such structure is Sasakian if and only if the corresponding Riemannian structure is Sasakian [6]. Moreover, M equipped with the contact Lorentzian structure $(\varphi, \xi, \eta, g_L)$ is an *invariant Lorentzian submanifold* of the contact Lorentzian manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}_L)$. Therefore, the inclusion $i : M \to \overline{M}$ is Levi pluriharmonic and a pseudohermitian map between two contact Lorentzian manifolds. In particular, the above examples define Levi pluriharmonic and pseudohermitian maps between contact Lorentz-Sasakian manifolds.

6. Levi harmonicity of quasi-cosymplectic manifolds

In this section we study the harmonicity of a CR map defined on a quasi-cosymplectic manifold. Recall that an almost contact Riemannian manifold M is said to be *quasi-cosymplectic* (cf. [7], and [8] p. 666) if

$$(\nabla_X\varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = \eta(Y)\,\nabla_{\varphi X}\xi, \quad X, Y \in \mathfrak{X}(M). \tag{6.6}$$

By a result of Z. Olszak (cf. [13], Lemma 2.2, p. 240) any almost cosymplectic manifold satisfies (6.6). So, the class of quasi-cosymplectic manifolds is large, it contains the classes of cosymplectic and almost cosymplectic manifolds. In the paper [14] we classified all simply connected homogeneous almost cosymplectic three-manifolds. Moreover, there exist examples of quasi-cosymplectic manifolds which are not almost cosymplectic (see, for example [7], and [8] p. 668).

Next, let M be a quasi-cosymplectic manifold. Then, for $X \in \ker(\eta)$ and $Y = \varphi X$, (6.6) implies

$$(\nabla_X\varphi)\varphi X = (\nabla_{\varphi X}\varphi)X, \tag{6.7}$$

equivalently $\nabla_X X + \nabla_{\varphi X}\varphi X = \varphi[\varphi X, X]$. Thus any quasi-cosymplectic manifold satisfies the φ-condition.

Now, let $f : M \to M'$ be a CR map among two quasi-cosymplectic manifolds. Then one gets

$$\begin{aligned}\beta_f(\varphi X, \varphi X) + \beta_f(X, X) &= \{\nabla'^f_X f_*X - f_*\nabla_X X + \nabla'^f_{\varphi X} f_*\varphi X - f_*\nabla_{\varphi X}\varphi X\} \\ &= \{\nabla'^f_X f_*X + \nabla'^f_{\varphi X}\varphi'^f f_*X - f_*(\nabla_X X + \nabla_{\varphi X}\varphi X)\}.\end{aligned}$$

Consequently, since both M and M' satisfy the φ-condition,

$$\begin{aligned}\beta_f(\varphi X, \varphi X) + \beta_f(X, X) &= \Big(\varphi'^f f_* - f_*\varphi\Big)\,[\varphi X\,,\,X] \\ &= \Big(\varphi'^f f_* - f_*\varphi\Big)\,g\big([\varphi X\,,\,X], \xi\big)\xi.\end{aligned}$$

Besides, $Y = \xi$ in (6.6) gives

$$\varphi\nabla_X\xi = -\nabla_{\varphi X}\xi, \quad \nabla_\xi\xi = 0. \tag{6.8}$$

Using (6.8), we have

$$\begin{aligned}g\big([\varphi X\,,\,X], \xi\big) &= g\big(\nabla_{\varphi X}X, \xi\big) - g\big(\nabla_X\varphi X, \xi\big) \\ &= -g\big(\nabla_{\varphi X}\xi, X\big) + g\big(\nabla_X\xi, \varphi X\big) \\ &= g\big(\varphi\nabla_X\xi, X\big) + g\big(\nabla_X\xi, \varphi X\big) \\ &= 0.\end{aligned}$$

Then, $\beta_f(\varphi X, \varphi X) + \beta_f(X, X) = 0$, and by (5.1), we get:

Theorem 6.1. *Any CR map $f : M \to M'$ among two quasi-cosymplectic manifolds is Levi pluriharmonic.*

Next, by (6.7) and (6.8), we get that any quasi-cosymplectic manifold satisfies

$$\begin{aligned}\mathrm{trace}(\varphi\nabla\xi) &= \sum_{\alpha=1}^{n}\{g(\varphi\nabla_{E_\alpha}\xi, E_\alpha) + g(\varphi\nabla_{\varphi E_\alpha}\xi, \varphi E_\alpha)\}\\ &= \sum_{\alpha=1}^{n}\{g(\varphi\nabla_{E_\alpha}\xi, E_\alpha) + g(\nabla_{E_\alpha}\xi, \varphi E_\alpha)\}\\ &= 0\,.\end{aligned}$$

Consequently, by Theorem 4.3, we have:

Theorem 6.2. *A CR map $f : M \to M'$ among two quasi-cosymplectic manifolds is a harmonic map if and only if $f_*\xi$ is a geodesic vector field. In particular, any CR map $f : M \to M'$ among two quasi-cosymplectic manifolds satisfying $f_*\xi = c\xi'$, for some $c \in \mathbb{R}$, is a harmonic map.*

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