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NORM OF THE COMPOSITION OPERATOR FROM BLOCH SPACE TO BERGMAN SPACE

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Abstract. In this paper, we study some quantity equivalent to the norm of Bloch to A^p_α composition operator where A^p_α is the weighted Bergman space on the unit ball of $\mathbb{C}^n$ ($0 < p < \infty$ and $-1 < \alpha < \infty$).

1. Introduction

Let $n \geq 1$ be an integer. Let $H(B)$ denote the space of all holomorphic functions in the unit ball $B \equiv B_n$ of the complex n-dimensional Euclidean space $\mathbb{C}^n$. Let U stand for B_1. The Bloch space $\mathcal{B} = \mathcal{B}(U)$ is defined by $\mathcal{B} = \{f \in H(U) : \sup_{z\in U}|f'(z)|(1-|z|^2) < \infty\}$. With the norm $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in U}|f'(z)|(1-|z|^2)$, $\mathcal{B}$ is a Banach space. Let ν denote the Lebesgue measure on $\mathbb{C}^n$, so normalized that $\nu(B) = 1$. For $-1 < \alpha < \infty$, we set $c_\alpha = \Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\}$ and $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z),\ z \in B$. Note that $\nu_\alpha(B) = 1$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space $A^p_\alpha \equiv A^p_\alpha(B)$ is defined by $A^p_\alpha = \{f \in H(B) : \int_B |f(z)|^p d\nu_\alpha(z) < \infty\}$. For $f \in A^p_\alpha$, we write $\|f\|_{A^p_\alpha} = \left(\int_B |f(z)|^p d\nu_\alpha(z)\right)^{\frac{1}{p}}$.

In 2003, Kwon and Lee proved the following theorem. This result relates to the pull-back property (see [1], [2], [5], [7] for example).

Theorem 1 ([6]). *Let $f : U \to U$ be a holomorphic function with $f(0) = 0$. For $1 \leq p < \infty$ and $-1 < \alpha < \infty$, the bounded operator $C^0_f : \mathcal{B} \to A^p_\alpha(U)$ defined by $C^0_f h = h \circ f - h(0)$ has its operator norm equivalent to the quantity*

$$\left\{\int_U (1-|z|)^{\alpha+p}\left(\frac{|f'(z)|}{1-|f(z)|^2}\right)^p dxdy\right\}^{\frac{1}{p}}. \tag{1.1}$$

In the present paper we study their analogues in "B and $0 < p < \infty$" in place of "U and $1 \leq p < \infty$". The following is the main result of this paper.

Theorem 2. *Let $f : B \to U$ be a holomorphic function with $f(0) = 0$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the bounded operator $C^0_f : \mathcal{B} \to A^p_\alpha(B)$ defined*

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by $C_f^0 h = h \circ f - h(0)$ *has its operator norm equivalent to the quantity*

$$\left\{ \int_B (1-|z|^2)^{\alpha+p} \left(\frac{|\nabla f(z)|}{1-|f(z)|^2} \right)^p d\nu(z) \right\}^{\frac{1}{p}}, \tag{1.2}$$

where $\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right)$ *is the holomorphic gradient of* f *at* z.

That is, there are positive constants C_1 and C_2 independent of f such that $C_1 \|C_f^0\| \le (1.2) \le C_2 \|C_f^0\|$, where $\|C_f^0\|$ is the operator norm of C_f^0.

Proposition 3 ([3]). *If* $f : B \to U$ *is a holomorphic function, then*

$$|\nabla f(z)| \le \sqrt{n} \cdot \frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in B.$$

Proof. By ([3] page 532, Remark 4),

$$\left| \frac{\partial f(z)}{\partial z_j} \right| \le \frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in B, \quad j = 1, \dots, n. \tag{1.3}$$

By (1.3), the result holds. □

By Proposition 3, (1.2) stays bounded for any holomorphic function $f : B \to U$. In fact, $(1.2) \le c_\alpha^{-\frac{1}{p}} \sqrt{n} < \infty$.

Note that we may see $A^p_{-1} = H^p$, where H^p is the Hardy space on B. When $n = 1$, Kwon and Lee researched the norm of Bloch-H^p pull-back operator ([6]). When $n \ge 1$, Kwon studied their result extensively ([5]).

2. Notation

For $a \in B$, let φ_a be the standard automorphism of B taking 0 to a ([8], page 25). Let λ be the measure on B defined by $d\lambda(z) = \frac{1}{(1-|z|^2)^{n+1}} d\nu(z)$, $z \in B$. For $f \in C^2(B)$ and $a \in B$, define $\tilde{\Delta} f(a) = \frac{1}{n+1} \Delta(f \circ \varphi_a)(0)$, where $\Delta \equiv 4 \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$ is the ordinary Laplacian. We say that a function $f \in C^2(B)$ is $\mathcal{M}$-harmonic in B, if $\tilde{\Delta} f(z) = 0$ for all $z \in B$. Note that $f \in H(B)$ then f is $\mathcal{M}$-harmonic in B ([8], page 59). Let $\widetilde{\nabla}$ denote the gradient with respect to the Bergman metric on B ([10], page 27). For $f \in H(B)$,

$$|\widetilde{\nabla} f(a)|^2 = \frac{2}{n+1}(1-|a|^2) \left[\sum_{j=1}^n \left| \frac{\partial f}{\partial z_j}(a) \right|^2 - \left| \sum_{j=1}^n a_j \frac{\partial f}{\partial z_j}(a) \right|^2 \right], \tag{2.1}$$

$a \in B$ ([10], page 30).

When $n = 1$, we note that $|\widetilde{\nabla} f(a)|^2 = (1-|a|)^2 |f'(a)|^2$, $a \in U$ ([10], page 30).

3. Preliminaries

For the proof of Theorem 2, the following lemmas will be needed.

Lemma 4 ([4], page 78)**.** *Let f be $\mathcal{M}$-harmonic on B.*

(i) *For all p, $0 < p < \infty$, and $s \in \mathbb{R}$, there exists a constant C, independent of f, such that*

$$\int_B (1-|z|^2)^s |\widetilde{\nabla} f(z)|^p d\lambda(z) \le C \int_B (1-|z|^2)^s |f(z)|^p d\lambda(z).$$

(ii) *For all p, $0 < p < \infty$, and $s > n$, there exists a constant C, independent of f, such that*

$$\int_B (1-|z|^2)^s |f(z)|^p d\lambda(z) \le C \left(|f(0)|^p + \int_B (1-|z|^2)^{s+p} |\nabla f(z)|^p d\lambda(z) \right).$$

Lemma 5. *If f is holomorphic in B, then*

$$\frac{\sqrt{2}}{\sqrt{n+1}} (1-|z|^2) |\nabla f(z)| \le |\widetilde{\nabla} f(z)|$$

for all $z \in B$.

Proof. By (2.1) and the Schwarz inequality, the result holds. □

Lemma 6 ([7], page 601)**.** *There exist two functions $h_1, h_2 \in \mathcal{B}(U)$ such that*

$$|h_1'(z)| + |h_2'(z)| \ge \frac{1}{1-|z|}$$

for all $z \in U$.

4. Proof of the main theorem

The essential idea of the proof is the same as the one in [6] except using Lemma 4. For the sake of completeness, we give the proof. Hereafter the letter K stands for a positive constant which changes its value from one appearance to another while remaining independent of the important variables.

By Lemma 4(ii) and $|\nabla(h \circ f)(z)| = |h'(f(z)) \nabla f(z)| \le \|h\|_{\mathcal{B}} \frac{|\nabla f(z)|}{1-|f(z)|^2}$,

$$\begin{aligned}
\|C_f^0\| &= c_\alpha^{\frac{1}{p}} \sup_{h \in \mathcal{B},\ \|h\|_{\mathcal{B}} \le 1} \left\{ \int_B (1-|z|^2)^\alpha |(h \circ f)(z) - h(0)|^p d\nu(z) \right\}^{\frac{1}{p}} \\
&\le K \sup_{h \in \mathcal{B},\ \|h\|_{\mathcal{B}} \le 1} \left\{ \int_B (1-|z|^2)^{\alpha+p} |\nabla(h \circ f)(z)|^p d\nu(z) \right\}^{\frac{1}{p}} \\
&\le K \left\{ \int_B (1-|z|^2)^{\alpha+p} \left(\frac{|\nabla f(z)|}{1-|f(z)|^2} \right)^p d\nu(z) \right\}^{\frac{1}{p}}.
\end{aligned}$$

We note that

$$(a+b)^p \le \gamma_p (a^p + b^p)$$

for $0 < p < \infty$, $a \ge 0$ and $b \ge 0$, where $\gamma_p = \max(1, 2^{p-1})$ ([9], page 73).

Conversely, by Lemma 6, Minkowski's inequality or the above, Lemma 4(i) and Lemma 5,

$$\begin{aligned}
(4.1)\qquad & \left\{\int_B (1-|z|^2)^{\alpha+p}\left(\frac{|\nabla f(z)|}{1-|f(z)|^2}\right)^p d\nu(z)\right\}^{\frac{1}{p}} \\
&\le \left\{\int_B (1-|z|^2)^{\alpha+p}\left(\sum_{j=1}^{2}|\nabla(h_j\circ f)(z)|\right)^p d\nu(z)\right\}^{\frac{1}{p}} \\
&\le K\sum_{j=1}^{2}\left\{\int_B (1-|z|^2)^{\alpha+p}|\nabla(h_j\circ f)(z)|^p d\nu(z)\right\}^{\frac{1}{p}} \\
&\le K\sum_{j=1}^{2}\left\{\int_B (1-|z|^2)^{\alpha}|(h_j\circ f)(z)-h_j(0)|^p d\nu(z)\right\}^{\frac{1}{p}}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left\{\int_B (1-|z|^2)^{\alpha}|(h_j\circ f)(z)-h_j(0)|^p d\nu(z)\right\}^{\frac{1}{p}} \\
&\le \|h_j\|_{\mathcal{B}} \sup_{h\in\mathcal{B},\ \|h\|_{\mathcal{B}}\le 1}\left\{\int_B (1-|z|^2)^{\alpha}|(h\circ f)(z)-h(0)|^p d\nu(z)\right\}^{\frac{1}{p}}, \quad j=1,2,
\end{aligned}$$

from (4.1),

$$\begin{aligned}
&\left\{\int_B (1-|z|^2)^{\alpha+p}\left(\frac{|\nabla f(z)|}{1-|f(z)|^2}\right)^p d\nu(z)\right\}^{\frac{1}{p}} \\
&\le K \sup_{h\in\mathcal{B},\ \|h\|_{\mathcal{B}}\le 1}\left\{\int_B (1-|z|^2)^{\alpha}|(h\circ f)(z)-h(0)|^p d\nu(z)\right\}^{\frac{1}{p}} \\
&= K \sup_{h\in\mathcal{B},\ \|h\|_{\mathcal{B}}\le 1}\|(h\circ f)(z)-h(0)\|_{A^p_\alpha} \\
&= K\|C^0_f\|.
\end{aligned}$$

5. Corollary

Let $\mathcal{M}$ be the group of automporphism of U. For $a\in U$, define $\phi_a(z)=\frac{a-z}{1-\bar{a}z}$. Kwon and Lee proved the following corollary.

Corollary 1 ([6]). *Let $f:U\to U$ be a holomorphic function. For $1\le p<\infty$ and $-1<\alpha<\infty$, the bounded operator $C^0_f:\mathcal{B}\to A^p_\alpha(U)$ defined by $C^0_f h = h\circ\phi_{f(0)}\circ f - h(0)$ has its operator norm equivalent to the quantity* (1.1).

The following is an immediate consequence of Theorem 2.

Corollary 2. *Let $f:B\to U$ be a holomorphic function. For $0<p<\infty$ and $-1<\alpha<\infty$, the bounded operator $C^0_f:\mathcal{B}\to A^p_\alpha(B)$ defined by $C^0_f h = h\circ\phi_{f(0)}\circ f - h(0)$ has its operator norm equivalent to the quantity* (1.2).

Proof. By Theorem 2 and $\mathcal{M}$-invariance of $\frac{|\nabla f(z)|}{1-|f(z)|^2}$, the result holds. $\square$

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