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Roots of Difference Quotient Forms of Chebyshev Polynomials

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Abstract

Let $T_n(x)$ be the Chebyshev polynomials of first kind of degree n. In this paper, we show that for $a>1$, the polynomial with integer coefficients $\dfrac{T_n(z)-T_n(a)}{z-a}$ has all its roots in $|z|\le a$.

Key words : Polynomials, Chebyshev Polynomials, Roots

1. Introduction

Let $f(x)$ and $g(y)$ be polynomials in the single independent variables x and y with coefficients in the field $\mathbb{C}$ of complex numbers. Cassels *et al.*[1,2] studied factorizations of $f(x)-g(y)$ as the polynomial in the pair of variables x and y. They also considered a trivial case when f and g are the same polynomial since $f(x)-f(y)$ is divisible by $x-y$. In this case, obtaining the factors of the polynomial

$$\frac{f(x)-f(y)}{x-y} \qquad (1)$$

is in general rather complicated. The polynomial of the form (1) also arises in Bezout matrices that have appeared in the literature for a long time. Given

$$u(x)=\sum_{j=0}^{n}u_jx^j\ (u_n\neq 0),\qquad v(x)=\sum_{j=0}^{n}v_jx^j,$$

let

$$\pi(x,y)=\frac{u(x)v(y)-u(y)v(x)}{x-y}=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}r_{ij}x^iy^j \qquad (2)$$

Then $B(u,v)=(\gamma_{ij})_{i,j=0}^{n-1}$ is called the Bezout matrix of $u(x)$ and $v(x)$. As a special case of (2) (that is, $v(x)=1$) we have the difference quotient form

$$\frac{u(x)-v(y)}{x-y}$$

that is of the same form with (1). Bezout matrix have many applications in inertia and stability problems of polynomials, control theory and system theory and so on, see [3,4].

Chebyshev polynomials are of great importance in many areas of mathematics, particularly approximation theory. Many papers and books[5,6] have been written about these polynomials. Let $T_n(x)$ and $U_n(x)$ be the Chebyshev polynomials of first kind of degree n and of second kind of degree n, respectively. These polynomials satisfy the recurrence relations

$$T_0(x)=1, T_1(x)=x,\quad T_{n+1}(x)=2xT_n(x)-T_{n-1}(x),$$
$$U_0(x)=1, U_1(x)=2x,\quad U_{n+1}(x)=2xU_n(x)-U_{n-1}(x),$$
$$n\ge 1.$$

The Bezoutian matrix for Chebyshev polynomials of the second kind has been studied in Yang and Cui[4]. They have used Chebyshev polynomials of the second kind to obtain a Barnett-type factorization formula and a triangular-type formula for a generalized Bezout matrix. In this paper, we consider a form of the Bezoutian matrix for Chebyshev polynomials of the first kind. In fact, due to frequent occurrences of Chebyshev polynomials in various branches of mathematics, it is natural to investigate some properties satisfied by polynomials of the form (1) when $f(x)$ is a Chebyshev polynomial.

We particularly study polynomials of z with integer coefficients

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$$QT_n(a,z) := \frac{T_n(z) - T_n(a)}{z - a}$$

for a constant a. For $-1 \le a \le 1$, $QT_n(a,z)$ has obviously all roots real in $[-1,1]$ because $-1 \le T_n(a) \le 1$. We will see in this paper that for $a > 1$, the polynomial $QT_n(a,z)$ has all its roots in $|z| \le a$. This range $|z| \le a$ is sharp in that all roots of $QT_n(a,z)$ seem to be very close to the circle $|z| = a$.

2. Results and Proofs

Let $a > 1$ and

$$\zeta_j = \cos\frac{2j-1}{2n}\pi \quad (1 \le j \le n)$$

the roots of $T_n(z) = 0$. If $T_n(z) = T_n(a)$, then $|T_n(z)| = |T_n(a)|$. So if we prove

$$\prod_{j=1}^{n}|a - \zeta_j| < \prod_{j=1}^{n}|z - \zeta_j|,$$

where $|z| > a$, then all roots of $T_n(z) = T_n(a)$ lie in $|z| \le a$. To prove this, we need two lemmas below.

Lemma 1 The minimum of $\prod_{j=1}^{n}|z - \zeta_j|$ on $|z| = a$ occurs at $z = a$.

Proof Let

$$f(\theta) = \frac{1}{2^{n-1}}\left|T_n(ae^{i\theta})\right|^2 = \prod_{j=1}^{n}\left|ae^{i\theta} - \zeta_j\right|^2.$$

It follows from the symmetry of the roots of T_n that without loss of generality, we assume that $0 \le \theta \le \pi/2$. Now

$$\begin{aligned} f(\theta) &= \prod_{j=1}^{n}|(a\cos\theta - \cos\frac{2j-1}{2n}\pi) + ia\sin\theta|^2 \\ &= \prod_{j=1}^{n}(a^2 - 2a\cos\frac{2j-1}{2n}\pi\cos\theta + \cos^2\frac{2j-1}{2n}\pi + a^2\sin^2\theta) \end{aligned}$$

For n even,

$$\begin{aligned} f(\theta) &= \prod_{j=1}^{n/2}(a^2 - 2a\cos\frac{2j-1}{2n}\pi\cos\theta + \cos^2\frac{2j-1}{2n}\pi + a^2\sin^2\theta) \\ &\quad (a^2 + 2a\cos\frac{2j-1}{2n}\pi\cos\theta + \cos^2\frac{2j-1}{2n}\pi + a^2\sin^2\theta) \\ &= \prod_{j=1}^{n/2}(a^2 + \cos^2\frac{2j-1}{2n}\pi + a^2\sin^2\theta)^2 - 4a^2\cos^2\frac{2j-1}{2n}\pi\cos^2\theta) \end{aligned}$$

and

$$\begin{aligned} f'(\theta) = \sum_{k=1}^{n/2}\Bigg[&\left(2(a^2 + \cos^2\frac{2j-1}{2n}\pi + a^2\sin^2\theta)(a^2\sin 2\theta) + 8a^2\cos\frac{(2k-1)\pi}{2n}\sin 2\theta\right)\cdot \\ &\prod_{j\ne k}(a^2 + \cos^2\frac{2j-1}{2n}\pi + a^2\sin^2\theta)^2 - 4a^2\cos^2\frac{2j-1}{2n}\pi\cos^2\theta)\Bigg]. \end{aligned}$$

But for $0 \le \theta \le \dfrac{\pi}{2}$,

$$\begin{aligned} &2\left(a^2 + \cos^2\frac{2j-1}{2n}\pi + a^2\sin^2\theta\right)(a^2\sin 2\theta) \\ &+ 8a^2\cos\frac{(2k-1)\pi}{2n}\sin 2\theta \ge 0 \end{aligned}$$

and

$$\begin{aligned} &(a^2 + \cos^2\frac{2j-1}{2n}\pi + a^2\sin^2\theta)^2 - 4a^2\cos^2\frac{2j-1}{2n}\pi\cos^2\theta \\ &\ge (a^2 + \cos^2\frac{2j-1}{2n}\pi)^2 - 4a^2\cos^2\frac{2j-1}{2n}\pi \\ &= \left(a - \cos\frac{2j-1}{2n}\pi\right)^2\left(a + \cos\frac{2j-1}{2n}\pi\right)^2 \ge 0 \end{aligned}$$

So $f(\theta)$ is increasing on $[0,\pi/2]$ and hence $f(\theta)$ has its minimum at $\theta = 0$. The case n odd can be proved in the same way.

Lemma 2 For $1 < a < b$,

$$\min_{|z|=a}\prod_{j=1}^{n}|z - \zeta_j| < \min_{|z|=b}\prod_{j=1}^{n}|z - \zeta_j|.$$

Proof By Lemma 1, $\min_{|z|=a}\prod_{j=1}^{n}|z - \zeta_j|$ and $\min_{|z|=b}\prod_{j=1}^{n}|z - \zeta_j|$ occur at $z = a$ and $z = b$, respectively. Since $1 < a < b$ and $\zeta_j \le 1$,

$$\prod_{j=1}^{n}|a - \zeta_j| < \prod_{j=1}^{n}|b - \zeta_j|$$

which proves the result.

Theorem 3 For $a > 1$, the polynomial $\dfrac{T_n(z) - T_n(a)}{z - a}$ has all its roots in $|z| \le a$.

Proof By Lemmas 1 and 2, for $|z| = b > a$,

$$\prod_{j=1}^{n}|a - \zeta_j| < \prod_{j=1}^{n}|b - \zeta_j| \le \prod_{j=1}^{n}|z - \zeta_j|.$$

Hence

$$\prod_{j=1}^{n}|a - \zeta_j| < \prod_{j=1}^{n}|z - \zeta_j|,$$

where $|z| > a$,

The range $|z| \leq a$ of Theorem 3 is sharp in that all roots seem to be very close to the circle $|z| = a$. We finally give an example when $n = 13$ and $a = 3$.

Example 4 The modulii of the twelve roots of $QT_{13}(3,x)$ are approximately

$$\begin{array}{llll} 2.99044, & 2.99044, & 2.92579, & 2.92579, \\ 2.85057, & 2.85057, & 2.83099, & 2.83099, \\ 2.88491, & 2.88491, & 2.96379, & 2.96379. \end{array}$$

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