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Lattices of Interval-Valued Fuzzy Subgroups

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Abstract

We discuss some interesting sublattices of interval-valued fuzzy subgroups. In our main result, we consider the set of all interval-valued fuzzy normal subgroups with finite range that attain the same value at the identity element of the group. We then prove that this set forms a modular sublattice of the lattice of interval-valued fuzzy subgroups. In fact, this is an interval-valued fuzzy version of a well-known result from classical lattice theory. Finally, we employ a lattice diagram to exhibit the interrelationship among these sublattices.

Keywords: Interval-valued fuzzy set, Interval-valued fuzzy subgroup, Interval-valued fuzzy normal subgroup, Level subset, Modular lattice

1. Introduction

In 1965, Zadeh [1] introduced the concept of a fuzzy set, and later generalized this to the notion of an interval-valued fuzzy set [2]. Since then, there has been tremendous interest in this subject because of the diverse range of applications, from engineering and computer science to social behavior studies. In particular, Gorzalczany [3] developed an inference method using interval-valued fuzzy sets.

In 1995, Biswas [4] studied interval-valued fuzzy subgroups. Subsequently, a number of researchers applied interval-valued fuzzy sets to algebra [5-11], and Lee et al. [12] furthered the investigation of interval-valued fuzzy subgroups in the sense of a lattice.

Later, in 1999, Mondal and Samanta [13] applied interval-valued fuzzy sets to topology, and Jun et al. [14] studied interval-valued fuzzy strong semi-openness and interval-valued fuzzy strong semicontinuity. Furthermore, Min [15-17] investigated interval-valued fuzzy almost M-continuity, the characterization of interval-valued fuzzy m-semicontinuity and interval-valued fuzzy mβ-continuity, and then Min and Yoo [18] researched interval-valued fuzzy mα-continuity. In particular, Choi et al. [19] introduced the concept of an interval-valued smooth topology, and described some relevant properties.

In this paper, we discuss some interesting sublattices of the lattice of interval-valued fuzzy subgroups of a group.

In the main result of our paper, we consider the set of all interval-valued fuzzy normal subgroups with finite range that attain the same value at the identity element of the group. We prove that this set forms a modular sublattice of the lattice of interval-valued fuzzy subgroups. In fact, this is an interval-valued fuzzy version of a well-known result from classical lattice theory. Finally, we use a lattice diagram to exhibit the interrelationship among these sublattices.

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2. Preliminaries

In this section, we list some basic concepts and well-known results which are needed in the later sections. Throughout this paper, we will denote the unit interval $[0,1]$ as I. For any ordinary subset A on a set X, we will denote the characteristic function of A as χ_A.

Let $D(I)$ be the set of all closed subintervals of the unit interval $[0,1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$, and a = $[a,a]$ for every $a \in (0,1)$. We also note that

(i) $(\forall M, N \in D(I))$ $(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))$ $(M = N \leq M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^C, is defined by $M^C = 1 - M = [1 - M^U, 1 - M^L]$(See [13]).

Definition 2.1 [2,3]. A mapping $A : X \to D(I)$ is called an interval-valued fuzzy set (IVFS) in X, denoted by $A = [A^L, A^U]$, if $A^L, A^L \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$[resp $A^U(x)$] is called the *lower*[resp *upper*] *end point of* x *to* A. For any $[a,b] \in D(I)$, the interval-valued fuzzy A in X defined by $A(x) = [A^L(x), A^U(x)] = [a,b]$ for each $x \in X$ is denoted by $\widetilde{[a,b]}$ and if $a = b$, then the IVFS $\widetilde{[a,b]}$ is denoted by simply $\widetilde{a}$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVFSs in X as $D(I)^X$. It is clear that set $A = [A, A] \in D(I)^X$ for each $A \in I^X$.

Definition 2.2 [13]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

(iii) $A^C = [1 - A^U, 1 - A^L]$.

(iv) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

(iv)′ $\bigcup_{\alpha\in\Gamma} A_\alpha = [\bigvee_{\alpha\in\Gamma} A^L_\alpha, \bigvee_{\alpha\in\Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

(v)′ $\bigcap_{\alpha\in\Gamma} A_\alpha = [\bigwedge_{\alpha\in\Gamma} A^L_\alpha, \bigwedge_{\alpha\in\Gamma} A^U_\alpha]$.

Result 2.A[13, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then

(a) $\widetilde{0} \subset A \subset \widetilde{1}$.

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} (A \cap A_\alpha)$.

(f) $A \cup (\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma} (A \cup A_\alpha)$.

(g) $(\widetilde{0})^c = \widetilde{1}$, $(\widetilde{1})^c = \widetilde{0}$.

(h) $(A^c)^c = A$.

(i) $(\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A^c_\alpha$, $(\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A^c_\alpha$.

Definition 2.3 [8]. Let $(X, \cdot)$ be a groupoid and let $A \in D(I)^X$. Then A is called an interval-valued fuzzy subgroupoid (IVGP) in X if

$$A^L(xy) \geq A^L(x) \wedge A^L(y)$$

and

$$A^U(xy) \geq A^U(x) \wedge A^U(y), \forall x, y \in X.$$

It is clear that $\widetilde{0}, \widetilde{1} \in$ IVGP(X).

Definition 2.4 [4]. Let A be an IVFs in a group G. Then A is called an interval-valued fuzzy subgroup (IVG) in G if it satisfies the conditions : For any $x, y \in G$,

(i) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$.

(ii) $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$.

We will denote the set of all IVGs of G as IVG(G).

Result 2.A [8, Proposition 4.3]. Let G be a group and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset$ IVG(G). Then $\bigcap_{\alpha\in\Gamma} A_\alpha \in$ IVG(G).

Result 2.B [4, Proposition 3.1]. Let A be an IVG in a group G. Then

(a) $A(x^{-1}) = A(x), \forall x \in G$.

(b) $A^L(e) \geq A^L(x)$ and $A^U(e) \geq A^U(x), \forall x \in G$, where e is the identity of G.

Result 2.C [8, Proposition 4.2]. Let G be a group and let $A \subset G$. Then A is a subgroup of G if and only if $[\chi_A, \chi_A] \in$ IVG(G).

Definition 2.5 [8]. Let A be an IVFS in a set X and let $\lambda, \mu \in I$ with $\lambda \leq \mu$. Then the set $A^{[\lambda,\mu]} = \{x \in X : A^L(x) \geq \lambda$ and $A^U(x) \geq \mu\}$ is called a $[\lambda, \mu]$-*level subset* of A.

3. Lattices of Interval-Valued Fuzzy Subgroups

In this section, we study the lattice structure of the set of interval-valued fuzzy subgroups of a given group. From Definitions 2.1 and 2.2, we can see that for a set X, $D(I)^X$ forms a complete lattice under the usual ordering of interval-valued fuzzy inclusion $\subset$, where the inf and the sup are the intersection and the union of interval-valued fuzzy sets, respectively. To construct the lattice of interval-valued fuzzy subgroups, we define the inf of a family A_α of interval-valued fuzzy subgroups to be the intersection $\bigcap A_\alpha$. However, the sup is defined as the interval-valued fuzzy subgroup generated by the union $\bigcup A_\alpha$ and denoted by $(\bigcup A_\alpha)$. Thus we have the following result.

Proposition 3.1. Let G be a group. Then IVG(G) forms a complete lattice under the usual ordering of interval-valued fuzzy set inclusion $\subset$.

Proof. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be any subset of IVG(G). Then, by Result 2.A, $\bigcap_{\alpha\in\Gamma} \in$ IVG(G). Moreover, it is clear that $\bigcap_{\alpha\in\Gamma} A_\alpha$ is the largest interval-valued fuzzy subgroup contained in A_α for each $\alpha \in \Gamma$. So $\bigwedge_{\alpha\in\Gamma} A_\alpha = \bigcap_{\alpha\in\Gamma} A_\alpha$. On the other hand, we can easily see that $(\bigcup_{\alpha\in\Gamma} A_\alpha)$ is the least interval-valued fuzzy subgroup containing A_α for each $\alpha \in \Gamma$. So $\bigvee_{\alpha\in\Gamma} A_\alpha = (\bigcup_{\alpha\in\Gamma} A_\alpha)$. Hence IVG$(G)$ is a complete lattice.

Next we construct certain sublattice of the lattice IVG(G). In fact, these sublattices reflect certain peculiarities of the interval-valued fuzzy setting. For a group G, let IVG$_f(G) = \{A \in$ IVG(G) : Im A is finite $\}$ and let IVG$_{[s,\ t]}(G) = \{A \in$ IVG$(G) : A(e) = [s, t]\}$, where e is the identity of G. Then it is clear that IVG$_f(G)$[resp. IVG$_{[s,\ t]}(G)$] is a sublattice of IVG(G). Moreover, IVG$_f(G)\cap$ IVG$_{[s,\ t]}(G)$ is also a sublattice of IVG(G).

Definition 3.2[11]. Let $(X, \cdot)$ be a groupoid and let $A, B \in D(I)^X$. Then the *interval-valued fuzzy product of A and B*, denoted by $A \circ B$, is an IVFS in X defined as follows : For each $x \in X$,

$$(A \circ B)(x) = \begin{cases} [\bigvee_{yz=x} [A^L(y) \wedge B^L(z)], \\ \bigvee_{yz=x} [A^U(y) \wedge B^U(z)] & \text{if yz = x}, \\ [0, 0] & \text{otherwise.} \end{cases}$$

Now to obtain our main results, we start with following two lemmas.

Lemma 3.3. Let G be a group and let $A, B \in$ IVG(G). Then for each $[\lambda, \mu] \in D(I)$, $A^{[\lambda,\ \mu]} \cdot B^{[\lambda,\ \mu]} \subset (A \circ B)^{[\lambda,\ \mu]}$.

Proof. Let $z \in A^{[\lambda,\ \mu]} \cdot B^{[\lambda,\ \mu]}$. Then there exist $x_0, y_0 \in G$ such that $z = x_0y_0$. Thus $A^L(x_0) \geq \lambda$, $A^U(x_0) \geq \mu$ and $A^L(y_0) \geq \lambda$, $A^U(y_0) \geq \mu$. So

$$\begin{aligned}(A \circ B)^L(z) &= \bigvee_{z=xy} [A^L(x) \wedge B^L(y)] \\ &\geq A^L(x_0) \wedge B^L(y_0) \geq \lambda\end{aligned}$$

and

$$\begin{aligned}(A \circ B)^U(z) &= \bigvee_{z=xy} [A^U(x) \wedge B^U(y)] \\ &\geq A^U(x_0) \wedge B^U(y_0) \geq \mu.\end{aligned}$$

Thus $z \in (A \circ B)^{[\lambda,\ \mu]}$. Hence $A^{[\lambda,\ \mu]} \cdot B^{[\lambda,\ \mu]} \subset (A \circ B)^{[\lambda,\ \mu]}$.

The following is the converse of Lemma 3.2.

Lemma 3.4. Let G be a group and let $A, B \in$IVG(G). If Im A and Im B are finite, then for each $[\lambda,\ \mu] \in D(I)$, $(A \circ B)^{[\lambda,\ \mu]} \subset A^{[\lambda,\ \mu]} \cdot B^{[\lambda,\ \mu]}$.

Proof. Let $z \in (A \cdot B)^{[\lambda,\ \mu]}$. Then

$A \circ B^L(z) = \bigvee_{z=xy}[A^L(x) \wedge B^L(y)] \geq \lambda$

and

$A \circ B^U(z) = \bigvee_{z=xy}[A^U(x) \wedge B^U(y)] \geq \mu$.

Since Im A and Im B are finite, there exist $x_0, y_0 \in G$ with $z = x_0y_0$ such that

$\bigvee_{z=xy}[A^L(x) \wedge B^L(y)] = A^L(x_0) \wedge B^L(y_0) \geq \lambda$

and

$\bigvee_{z=xy}[A^U(x) \wedge B^U(y)] = A^L(x_0) \wedge B^U(y_0) \geq \lambda$.

Thus $A^L(x_0) \geq \lambda$, $A^U(x_0) \geq \mu$ and $B^L(y_0) \geq \lambda$, $B^L(y_0) \geq$

μ. So $x_0 \in A^{[\lambda,\ \mu]}$ and $y_0 \in B^{[\lambda,\ \mu]}$, i.e., $z = x_0 y_0 \in A^{[\lambda,\ \mu]} \cdot B^{[\lambda,\ \mu]}$. Hence $(A \circ B)^{[\lambda,\ \mu]} \subset A^{[\lambda,\ \mu]} \cdot B^{[\lambda,\ \mu]}$. This completes the proof.

The following is the immediate result of Lemmas 3.3 and 3.4.

Proposition 3.5. Let G be a group and let $A, B \in \text{IVG}(G)$. If Im A and Im B are finite, then for each $[\lambda,\ \mu] \in D(I)$,

$$(A \circ B)^{[\lambda,\ \mu]} = A^{[\lambda,\ \mu]} \cdot B^{[\lambda,\ \mu]}.$$

Definition 3.6 [8]. Let G be a group and let $A \in \text{IVG}(G)$. Then A is called interval-valued fuzzy normal subgroup (IVNG) of G if $A(xy) = A(yx)$ for any $x, y \in G$.

We will denote the set of all IVNGs of G as IFNG(G). It is clear that if G is abelian, then every IVG of G is an IVNG of G.

Result 3.A [6, Proposition 2.13]. Let G be a group, let $A \in \text{IFNG}(G)$ and let $[\lambda,\ \mu] \in D(I)$ such that $\lambda \leq A^L(e)$ and $\mu \leq A^U(e)$. Then $A^{[\lambda,\ \mu]} \lhd G$, where $A^{[\lambda,\ \mu]} \lhd G$ means that $A^{[\lambda,\ \mu]}$ is a normal subgroup of G.

Result 3.B [6, Proposition 2.17]. Let G be a group and let $A \in \text{IVG}(G)$. If $A^{[\lambda,\ \mu]} \lhd G$ for each $[\lambda,\ \mu] \in$ Im A, Then $A \in \text{IVNG}(G)$.

The following is the immediate result of Results 3.A and 3.B.

Theorem 3.7. Let G be a group and let $A \in \text{IVG}(G)$. Then $A \in \text{IVNG}(G)$ if and only if for each $[\lambda,\ \mu] \in$ Im A, $A^{[\lambda,\ \mu]} \lhd G$.

Result 3.C[8, Proposition 5.3]. Let G be a group and let $A \in \text{IVNG}(G)$. If $B \in \text{IVG}(G)$, then $B \circ A \in$IVG(G).

The following is the immediate result of Result 2.A and Definition 3.6.

Proposition 3.8. Let G be a group and let $A, B \in \text{IVNG}(G)$. Then $A \cap B \in \text{IVNG}(G)$.

It is well-known that the set of all normal subgroups of a group forms a sublattice of the lattice of its subgroups. As an interval-valued fuzzy analog of this classical result we obtain the following result.

Theorem 3.9. Let G be a group and let $\text{IVN}_{f[s,\ t]}(G) = \{A \in \text{IVNG}(G)$: Im A is finite and $A(e) = [s, t]\}$. Then $\text{IVN}_{f[s,\ t]}(G)$ is a sublattice of $\text{IVG}_f(G) \cap \text{IVG}_{[s,\ t]}(G)$. Hence $\text{IVN}_{f[s,\ t]}(G)$ is a sublattice of IVG(G).

Proof. Let $A, B \in \text{IVN}_{f[s,\ t]}(G)$. Then, by Result 3.C, $A \circ B \in \text{IVG}(G)$. Let $z \in G$. Then

$$\begin{aligned}(A \circ B)^L(z) &= \bigvee_{z=xy} [A^L(x) \wedge B^L(y)] \\ &\geq A^L(z) \wedge B^L(e) = A^L(z) \wedge A^L(e) \\ \text{[Since } A(e) &= (s, t) = B(e)] \\ &= A^L(z). \quad \text{[By Result 2.B]}\end{aligned}$$

Similarly, we have $(A \circ B)^U(z) \geq A^U(z)$. Thus $A \subset A \circ B$. By the similar arguments, we have $B \subset A \circ B$.

Let $C \in \text{IVG}(G)$ such that $A \subset C$ and $B \subset C$. Let $z \in G$. Then

$(A \circ B)^L(z) = \bigvee_{z=xy} [A^L(x) \wedge B^L(y)]$
$\leq \bigvee_{z=xy} [C^L(x) \wedge C^L(y)]$ [Since $A \subset C$ and $B \subset C$]
$\leq C^L(xy)$ [Since $C \in \text{IVG}(G)$]
$= C^L(z)$.

Similarly, we have $(A \circ B)^U(z) \leq C^U(z)$. Thus $A \circ B \subset C$. So $A \circ B = A \vee B$.

Now let $[\lambda,\ \mu] \in D(I)$. Since $A, B \in \text{IVNG}(G)$, $A^{[\lambda,\ \mu]} \lhd G$ and $B^{[\lambda,\ \mu]} \lhd G$. Then $A^{(\lambda, \mu)} \circ B^{[\lambda,\ \mu]} \lhd G$. By Proposition 3.5, $(A \circ B)^{[\lambda,\ \mu]} \lhd G$. Thus, by Theorem 3.7, $A \circ B \in \text{IVNG}(G)$. So $A \vee B \in \text{IVN}_{f[s,\ t]}(G)$. From Proposition 3.8, it is clear that $A \wedge B \in \text{IVNG}(G)$. Thus $A \wedge B \in \text{IVN}_{f[s,t]}(G)$. Hence $\text{IVN}_{f[s,t]}(G)$ is a sublattice of $\text{IVG}_f \cap \text{IVG}_{[s,t]}(G)$, and therefore of IVG$(G)$. This complete the proof.

The relationship of different sublattice of the lattice of interval-valued fuzzy subgroup discussed herein can be visualized by the lattice diagram in Figure 1.

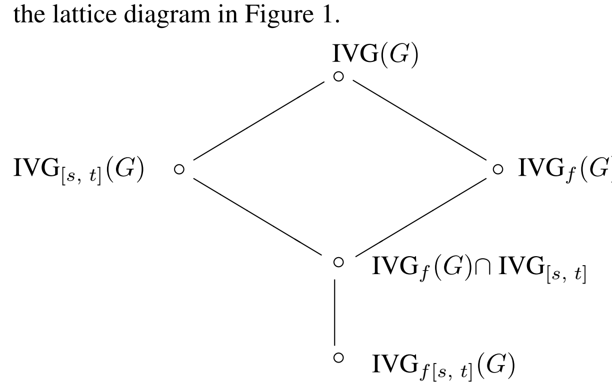

Figure 1.

It is also well-known[20, Theorem I.11] that the sublattice of normal subgroups of a group is modular. As an interval-valued fuzzy version to the classical theoretic result, we prove that $\mathrm{IVN}_{([s,\,t]}(G)$ forms a modular lattice.

Result 3.D [11, Lemma 3.2]. Let G be a group and let $A \in \mathrm{IVG}(G)$. If for any $x, y \in G$, $A^L(x) < A^L(y)$ and $A^U(x) < A^U(y)$, then $A(xy) = A(x) = A(yx)$.

Definition 3.10 [20,21]. A lattice $(L, \wedge, \vee)$ is said to be *modular* if for any $x, y, z \in L$ with $x \leq z$[resp. $x \geq z$], $x \vee (y \wedge z) = (x \vee y) \wedge z$[resp. $x \wedge (y \vee z) = (x \wedge y) \vee z$].

In any lattice L, it is well-known[21, Lemma I.4.9] that for any $x, y, z \in L$ if $x \leq z$[resp. $x \geq z$], then $x \vee (y \wedge z) \leq (x \vee y) \wedge z$[resp. $x \wedge (y \vee z) \geq (x \wedge y) \vee z$]. The inequality is called the *modular inequality*.

Theorem 3.11. The lattice $\mathrm{IVN}_{f[s,\,t]}(G)$ is modular.
Proof. Let $A, B, C \in \mathrm{IVN}_{f[s,\,t]}(G)$ such that $A \supset C$. Then, by the modular inequality, $(A \wedge B) \vee C \subset A \wedge (B \vee C)$. Assume that $A \wedge (B \vee C) \not\subset (A \wedge B) \vee C$, i.e., there exists $z \in G$ such that

$$[A \wedge (B \vee C)]^L(z) > [(A \wedge B) \vee C]^L(z)$$

and

$$[A \wedge (B \vee C)]^U(z) > [(A \wedge B) \vee C]^U(z).$$

Since Im B and Im C are finite, there exist $x_0, y_0 \in G$ with $z = x_0 y_0$ such that

$(B \vee C)(z) = (B \circ C)(z)$
(By the process of the proof of Theorem 3.9)
$= (\bigvee_{z=xy}[B^L(x) \wedge C^L(y)], \bigvee_{z=xy}[B^U(x) \wedge C^U(y)])$
$= [B^L(x_0) \wedge C^L(y_0), B^U(x_0) \wedge C^U(y_0)]$.

Thus
$[A \wedge (B \vee C)](z)$
$= [A^L(z) \wedge (B^L(x_0) \wedge C^L(y_0)),$
$A^U(z) \wedge (B^U(x_0) \wedge C^U(y_0))]. \quad (3.1)$

On the other hand,
$[(A \wedge B) \vee C]^L(z)$
$= \bigvee_{z=xy}[(A \wedge B)^L(x) \wedge C^L(y)]$
$\geq (A \wedge B)^L(x_0) \wedge C^L(y_0)$
$= A^L(x_0) \wedge B^L(x_0) \wedge C^L(y_0) \quad (3.2)$

and
$[(A \wedge B) \vee C]^U(z)$
$= \bigvee_{z=xy}[(A \wedge B)^U(x) \wedge C^U(y)]$
$\geq (A \wedge B)^U(x_0) \wedge C^U(y_0)$
$= A^U(x_0) \wedge B^U(x_0) \wedge C^U(y_0) \quad (3.3)$

By (3.1), (3.2) and (3.3),
$A^L(z) \wedge B^L(x_0) \wedge C^L(y_0) > A^L(x_0) \wedge B^L(x_0) \wedge C^L(y_0)$
and
$A^U(z) \wedge B^U(x_0) \wedge C^U(y_0) > A^U(x_0) \wedge B^U(x_0) \wedge C^U(y_0)$.
Then
$A^L(z), B^L(x_0), C^L(y_0) > A^L(x_0) \wedge B^L(x_0) \wedge C^L(y_0)$
and
$A^U(z), B^U(x_0), C^U(y_0) > A^U(x_0) \wedge B^U(x_0) \wedge C^U(y_0)$.
Thus
$A^L(x_0) \wedge B^L(x_0) \wedge C^L(y_0) = A^L(x_0)$
and
$A^U(x_0) \wedge B^U(x_0) \wedge C^U(y_0) = A^U(x_0)$.
So
$A^L(z) > A^L(x_0),\ A^U(z) > A^U(x_0)$
and
$C^L(y_0) > A^L(x_0),\ C^U(y_0) > A^U(x_0)$.
By Result 2.B,
$A^L(x_0^{-1}) = A^L(x_0) < A^L(x_0 y_0)$
and
$A^U(x_0^{-1}) = A^U(x_0) < A^U(x_0 y_0)$.
By Result 3.D, $A(x_0) = A(x_0^{-1} x_0 y_0) = A(y_0)$.
Thus
$C^L(y_0) > A^L(y_0)$ and $C^U(y_0) > A^U(y_0)$.
This contradicts the fact that $A \supset C$. So $A \wedge (B \vee C) \subset (A \wedge B) \vee C$. Hence $A \wedge (B \vee C) = (A \wedge B) \vee C$. Therefore $\mathrm{IVN}_{f[s,t]}(G)$ is modular. This completes the proof.

We discuss some interesting facts concerning a special class of interval-valued fuzzy subgroups that attain the value $[1, 1]$ at the identity element of G.

Lemma 3.12. Let A be a subset of a group G. Then

$$< [\chi_A, \chi_A] >= [\chi_{<A>}, \chi_{<A>}],$$

where $< A >$ is the subgroup generated by A.
Proof. Let $\mathcal{B} = \{B \in \mathrm{IVG}(G) : [\chi_A, \chi_A] \subset B\}$, let $B \in \mathcal{B}$ and let $x \in A$. Then

$$\chi_A(x) = 1 \leq B^L(x) \text{ and } \chi_A(x) = 1 \leq B^U(x).$$

Thus $B(x) = [1, 1]$. Since $B \in \mathrm{IVG}(G)$, $B = \widetilde{1}$ for any composite of elements of A. So $[\chi_{<A>}, \chi_{<A>}] \subset B$. Hence $[\chi_{<A>}, \chi_{<A>}] \subset \bigcap \mathcal{B}$. By Result 2.C, $[\chi_{<A>}, \chi_{<A>}] \in \mathrm{IVG}(G)$. Moreover, $[\chi_{<A>}, \chi_{<A>}] \in \mathcal{B}$.

Therefore $[\chi_{<A>}, \chi_{<A>}] = \bigcap \mathcal{B} =< [\chi_A, \chi_A] >$.

The following can be easily seen.

Lemma 3.13. Let A and B subgroups of a group G. Then

(a) $A \lhd G$ if and only if $[\chi_A, \chi_A] \in$IVN(G).

(b) $[\chi_A, \chi_A] \circ [\chi_B, \chi_B] = [\chi_{A \cdot B}, \chi_{A \cdot B}]$.

Proposition 3.14. Let $S(G)$ be the set of all subgroup of a group G and let IVG$(S(G)) = \{[\chi_A, \chi_A] : A \in S(G)\}$. Then IVG$(S(G))$ forms a sublattice of IVG$_f(G) \cap$ IVG$_{[1,1]}(G)$ and hence of IVG(G).

Proof. Let $A, B \in S(G)$. Then it is clear that $[\chi_A, \chi_A] \cap [\chi_B, \chi_B] = [\chi_{A \cap B}, \chi_{A \cap B}] \in$ IVG$(S(G))$. By Lemma 3.12,

$$< [\chi_A, \chi_A] \cup [\chi_B, \chi_B] > \; = \; < [\chi_{A \cup B}, \chi_{A \cup B}] > \; = [\chi_{<A \cup B>}, \chi_{<A \cup B>}].$$

Thus

$[\chi_A, \chi_A] \vee [\chi_B, \chi_B] =< [\chi_A, \chi_A] \cup [\chi_B, \chi_B] > \in$ IVG$(S(G))$.

Moreover, IVG$(S(G)) \subset$ IVG$_f(G) \cap$ IVG$_{[1,1]}(G)$.

Hence IVG$(S(G))$ is a sublattice of IVG$_f(G) \cap$IVG$_{[1,1]}(G)$.

Proposition 3.14 allows us to consider the lattice of subgroups $S(G)$ of G a group G as a sublattice of the lattice of all interval-valued fuzzy subgroups IVG(G) of G.

Now, in view of Theorems 3.9 and 3.11, for each fixed $[s, t] \in D(I)$, IVN$_{f[s,\, t]}(G)$ forms a modular sublattice of IVG$_f(G) \cap$ IVG$_{[s,\, t]}(G)$. Therefore, for $[s, t] = [1, 1]$, the sublattice IVN$_{f[1,\, 1]}(G)$ is also modular. It is clear that

$$\text{IVN}_{f[1,\, 1]}(G) \cap \text{IVG}(S(G)) = \text{IVN}(N(G)),$$

where $N(G)$ denotes the set of all normal subgroups of G and IVN$(N(G)) = \{[\chi_N, \chi_N] : N \in N(G)\}$. Moreover, IVG$(N(G))$ is also modular.

The lattice structure of these sublattices can be visualized by the diagram in Figure 2,

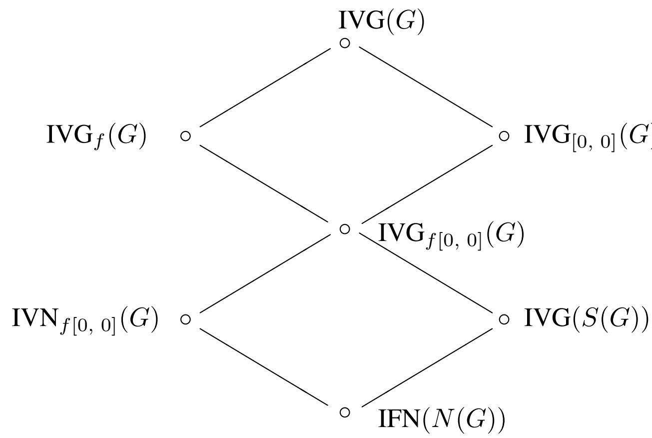

Figure 2.

By using Lemmas 3.12 and 3.13, we obtain a well-known classical result.

Corollary 3.15. Let G be a group. Then $N(G)$ forms a modular sublattice of $S(G)$.

4. Conclusion

Lee et al. [11] studied interval-valued fuzzy subgroup in the sense of a lattice. Cheong and Hur [5], Lee et al. [10], Jang et al. [6], Kang and Hur [8] investigated interval-valued fuzzy ideals/(generalized) bi-ideals, subgroup and ring, respectively.

In this paper, we mainly study sublattices of the lattice of interval-valued fuzzy subgroups of a group. In particular, we prove that the lattice IVN$_{f[s,\, t]}(G)$ is modular lattice (See Theorem 3.11). Finally, for subgroup $S(G)$ of a group G, IVG$(S(G))$ forms a sublattice of IVG$_f(G) \cap$ IVG$_{[1,1]}(G)$ and hence of IVG(G) (See Proposition 3.14).

In the future, we will investigate sublattices of the lattice of interval-valued fuzzy subrings of a ring.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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