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Exact Controllability for Abstract Fuzzy Differential Equations in Credibility Space

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Abstract

With reasonable control selections on the space of functions, various application models can take the shape of a well-defined control system on mathematics. In the credibility space, controlability management of fuzzy differential equation is as much important issue as stability. This paper addresses exact controllability for abstract fuzzy differential equations in the credibility space in the perspective of Liu process. This is an extension of the controllability results of Park et al. (Controllability for the semilinear fuzzy integro-differential equations with nonlocal conditions) to fuzzy differential equations driven by Liu process.

Keywords: Abstract fuzzy differential equations, Credibility space, Liu process, Fuzzy process

1. Introduction

The concept of fuzzy set was initiated by Zadeh via membership function in 1965. Fuzzy differential equations are a field of increasing interest, due to their applicability to the analysis of phenomena where imprecision in inherent. Kwun et al. [1-4] and Lee et al. [5] have studied the existence and uniqueness for solutions of fuzzy equations.

The theory of controlled processes is one of the most recent mathematical concepts to enable very important applications in modern engineering. However, actual systems subject to control do not admit a strictly deterministic analysis in view of various random factors that influence their behavior. The theory of controlled processes takes the random nature of a systems behavior into account. Many researchers have studied controlled processes. With regard to fuzzy systems, Kwun and Park [6] proved controllability for the impulsive semilinear fuzzy differential equation in n-dimension fuzzy vector space. Park et al. [7] studied the controllability of semilinear fuzzy integrodifferential equations with nonlocal conditions. Park et al. [8] demonstrated the controllability of impulsive semilinear fuzzy integrodifferential equations, while Phu and Dung [9] studied the stability and controllability of fuzzy control set differential equations. Lee et al. [10] examined the controllability of a nonlinear fuzzy control system with nonlocal initial conditions in n-dimensional fuzzy vector space E_N^n.

In terms of the controllability of stochastic systems, P. Balasubramaniam [11] studied quasilinear stochastic evolution equations in Hilbert spaces, and the controllability of stochastic control systems with time-variant coefficients was proved by Yuhu [12]. Arapostathis et al. [13] studied the controllability properties of stochastic differential systems that are characterized by a linear controlled diffusion perturbed by a smooth, bounded, uniformly Lipschitz nonlinearity.

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Stochastic differential equations driven by Brownian motion have been studied for a long time, and are a mature branch of modern mathematics. A new kind of fuzzy differential equation driven by a Liu process was defined as follows by Liu [14]

$$dX_t = f(X_t, t)dt + g(X_t, t)dC_t$$

where C_t is a standard Liu process, and f, g are some given functions. The solution of such equation is a fuzzy process. You [15] discussed the solutions of some special fuzzy differential equations, and derived an existence and uniqueness theorem for homogeneous fuzzy differential equations. Chen [16] for fuzzy differential equations. Liu [17] studied an analytic method for solving uncertain differential equations. In this paper, we extend the result of Liu [17] to fuzzy differential equations driven by a Liu process within a controlled system.

We study the exact controllability of abstract fuzzy differential equations in a credibility space:

$$\begin{cases} dx(t,\theta) = Ax(t,\theta)dt + f(t, x(t,\theta))dC_t \\ \qquad\qquad +Bu(t)dt, \ t \in [0,T], \\ x(0) = x_0 \in E_N, \end{cases} \tag{1}$$

where the state $x(t,\theta)$ takes values in $X(\subset E_N)$ and another bounded space $Y(\subset E_N)$. We use the following notation: E_N is the set of all upper semi-continuously convex fuzzy numbers on R, $(\Theta, \mathcal{P}, Cr)$ is the credibility space, A is a fuzzy coefficient, the state function $x : [0,T] \times (\Theta, \mathcal{P}, Cr) \to X$ is a fuzzy process, $f : [0,T] \times X \to X$ is a fuzzy function, $u : [0,T] \times (\Theta, \mathcal{P}, Cr) \to Y$ is a control function, B is a linear bounded operator from Y to X, C_t is a standard Liu process and $x_0 \in E_N$ is an initial value.

In Section 2, we discuss some basic concepts related to fuzzy sets and Liu processes.

In Section 3, we show the existence of solutions to the free fuzzy differential equation (1)$(u \equiv 0)$.

Finally, in Section 4, we prove the exact controllability of the fuzzy differential Eq. (1).

2. Preliminaries

In this section, we give some basic definitions, terminology, notation, and Lemmas that are relevant to our investigation and are needed in latter sections. All undefined concepts and notions used here are standard.

We consider E_N to be the space of one-dimensional fuzzy numbers $u : R \to [0,1]$, satisfying the following properties:

(1) u is normal, i.e., there exists an $u_0 \in R$ such that $u(t_o) = 1$;

(2) u is fuzzy convex, i.e., $u(\lambda t+(1-\lambda)s) \geq \min\{u(t), u(s)\}$ for any $t, s \in R$, $0 \leq \lambda \leq 1$;

(3) $u(t)$ is upper semi-continuous, i.e., $u(t_0) \geq \overline{\lim}_{k\to\infty} u(t_k)$ for any $t_k \in R$ $(k = 0, 1, 2, \cdots)$, $t_k \to t_0$;

(4) $[u]^0$ is compact.

The level sets of u, $[u]^\alpha = \{t \in R : u(t) \geq \alpha\}, \alpha \in (0,1]$, and $[u]^0$ are nonempty compact convex sets in R [8].

Definition 2.1 [19] We define a complete metric D_L on E_N by

$$\begin{aligned} D_L(u,v) &= \sup_{0\leq\alpha\leq 1} d_L([u]^\alpha, [v]^\alpha) \\ &= \sup_{0\leq\alpha\leq 1} \max\{|u_l^\alpha - v_l^\alpha|, |u_r^\alpha - v_r^\alpha|\}, \end{aligned}$$

for any $u, v \in E_N$, which satisfies $D_L(u + w, v + w) = D_L(u, v)$ for each $w \in E_N$, and $[u]^\alpha = [u_l^\alpha, u_r^\alpha]$, for every $\alpha \in [0,1]$ where $u_l^\alpha, u_r^\alpha \in R$ with $u_l^\alpha \leq u_r^\alpha$.

Definition 2.2 [20] For any $u, v \in C([0,T], E_N)$, the metric $H_1(u,v)$ on $C([0,T], E_N)$ is defined by

$$H_1(u,v) = \sup_{0\leq t\leq T} D_L(u(t), v(t)).$$

Let Θ be a nonempty set, and let $\mathcal{P}$ be the power set of Θ. Each element in $\mathcal{P}$ is called an event. To present an axiomatic definition of credibility, it is necessary to assign a number $Cr\{A\}$ to each event A indicating the credibility that A will occur. To ensure that the number $Cr\{A\}$ has certain mathematical properties that we intuitively expect, we accept the following four axioms:

(1) $(Normality)$ $Cr\{\Theta\} = 1$.

(2) $(Monotonicity)$ $Cr\{A\} \leq Cr\{B\}$ whenever $A \subset B$.

(3) $(Self-Duality)$ $Cr\{A\} + Cr\{A^c\} = 1$ for any event A.

(4) $(Maximality)$ $Cr\{\cup_i A_i\} = \sup_i Cr\{A_i\}$ for any events $\{A_i\}$ with $\sup_i Cr\{A_i\} < 0.5$.

Definition 2.5 [21] Let Θ be a nonempty set, $\mathcal{P}$ be the power set of Θ, and Cr be a credibility measure. Then the triplet $(\Theta, \mathcal{P}, C_r)$ is called a credibility space.

Definition 2.6 [14] A fuzzy variable is a function from a credibility space $(\Theta, \mathcal{P}, C_r)$ to the set of real numbers.

Definition 2.7 [14] Let T be an index set and $(\Theta, \mathcal{P}, C_r)$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, C_r)$ to the set of real numbers.

That is, a fuzzy process $x(t, \theta)$ is a function of two variables such that the function $x(t^*, \theta)$ is a fuzzy variable for each t^*. For each fixed θ^*, the function $x(t, \theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $x(t, \theta)$ is said to be sample-continuous if the sample ping is continuous for almost all θ. Instead of writing $x(t, \theta)$, we sometimes we use the symbol x_t.

Definition 2.8 Let $(\Theta, \mathcal{P}, C_r)$ be a credibility space. For fuzzy random variable x_t in credibility space, for each $\alpha \in [0, 1]$, the α-level set $[x_t]^\alpha = [(x_t)_l^\alpha, (x_t)_r^\alpha]$ is defined by

$$(x_t)_l^\alpha = \inf (x_t)^\alpha = \inf\{a \in R\,;\; x_t(a) \geq \alpha\},$$

$$(x_t)_r^\alpha = \sup (x_t)^\alpha = \sup\{a \in R\,;\; x_t(a) \geq \alpha\},$$

where $(x_t)_l^\alpha, (x_t)_r^\alpha \in R$ with $(x_t)_l^\alpha \leq (x_t)_r^\alpha$ when $\alpha \in [0, 1]$.

Definition 2.9 [22] Let ξ be a fuzzy variable and r is real number. Then the expected value of ξ is defined by

$$E\xi = \int_0^{+\infty} Cr\{\xi \geq r\}dr - \int_{-\infty}^{0} Cr\{\xi \leq r\}dr$$

provided that at least one of the integrals is finite.

Lemma 2.1 [22] Let ξ be a fuzzy vector. The expected value operator E has the following properties:
(i) if $f \leq g$, then $E[f(\xi)] \leq E[g(\xi)]$,
(ii) $E[-f(\xi)] = -E[f(\xi)]$,
(iii) if functions f and g are comonotonic, then for any nonnegative real numbers a and b, we have

$$E[af(\xi) + bg(\xi)] = aE[f(\xi)] + bE[g(\xi)].$$

Where $f(\xi)$ and $g(\xi)$ are fuzzy variables.

Definition 2.10 [?] A fuzzy process C_t is said to be a Liu process if
(i) $C_0 = 0$,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{t+s} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$, whose membership function is

$$\mu(x) = 2\Big(1 + \exp\Big(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\Big)\Big)^{-1}, \; x \in R.$$

The parameters e and σ are called the $drift$ and $diffusion$ coefficients, respectively. Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

Definition 2.11 [23] Let x_t be a fuzzy process and let C_t be a standard Liu process. For any partition of closed interval $[c, d]$ with $c = t_0 < \cdots < t_n = d$, the mesh is written as $\triangle = \max_{1 \leq i \leq n}(t_i - t_{i-1})$. Then the fuzzy integral of x_t with respect to C_t is

$$\int_c^d x_t dC_t = \lim_{\triangle \to 0} \sum_{i=1}^{n} x(t_{i-1})(C_{t_i} - C_{t_{i-1}})$$

provided that the limit exists almost surely and is a fuzzy variable.

Lemma 2.2 [23] Let C_t be a standard Liu process. For any given θ with $Cr\{\theta\} > 0$, the path C_t is Lipschitz continuous, that is, the following inequality holds

$$|C_{t_1} - C_{t_2}| < K(\theta)|t_1 - t_2|,$$

where K is a fuzzy variable called the Lipschitz constant of a Liu process with

$$K(\theta) = \begin{cases} \sup_{0 \leq s < t} \frac{|C_t - C_s|}{t - s}, & Cr\{\theta\} > 0, \\ \infty, & \text{otherwise}, \end{cases}$$

and $E[K^p] < \infty$, $\forall p > 0$.

Lemma 2.3 [23] Let C_t be a standard Liu process, and let $h(t; c)$ be a continuously differentiable function. Define $x_t = h(t; C_t)$. Then we have the following chain rule

$$dx_t = \frac{\partial h(t; C_t)}{\partial t}dt + \frac{\partial h(t; C_t)}{\partial C}dC_t.$$

Lemma 2.4 [23] Let $f(t)$ be continuous fuzzy process, the following inequality of fuzzy integral holds

$$\Big|\int_c^d f(t)dC_t\Big| \leq K \int_c^d |f(t)|dt,$$

where $K = K(\theta)$ is defined in Lemma 2.2.

3. Existence of Solutions for Abstract Fuzzy Differential Equations

In this section, by Definition 2.7, instead of longer notation $x(t, \theta)$, sometimes we use the symbol x_t. We consider the existence and uniquencess of solutions for the fuzzy differential Eq (1)($u \equiv 0$).

$$\begin{cases} dx_t = Ax_t dt + f(t, x_t)dC_t, \ t \in [0, T], \\ x(0) = x_0 \in E_N, \end{cases} \tag{2}$$

where the state x_t takes values in $X(\subset E_N)$. E_N is the set of all upper semi-continuously convex fuzzy numbers on R, $(\Theta, \mathcal{P}, Cr)$ is credibility space, A is fuzzy coefficient, the state function $x : [0, T] \times (\Theta, \mathcal{P}, Cr) \to X$ is a fuzzy process, $f : [0, T] \times X \to X$ is regular fuzzy function, C_t is a standard Liu process, $x_0 \in E_N$ is initial value.

Lemma 3.1 [19] Let g be a function of two variables and let a_t be an integrable uncertain process. Then a given uncertain differential equation by

$$dX_t = a_t X_t dt + g(t, X_t)dC_t$$

has a solution

$$X_t = Y_t^{-1} Z_t$$

where

$$Y_t = \exp\Big(-\int_0^t a_s ds\Big)$$

and Z_t is the solution of uncertain differential equation

$$dZ_t = Y_t g(t, Y^{-1} Z_t)dC_t$$

with initial value $Z_0 = X_0$.

Using Lemma 3.1, we show that, for fuzzy coefficient A, the Eq. (2) have a solution.

Lemma 3.2 For $x(0) = x_0$, if x_t is solution of the Eq. (2), then the solution x_t is given by

$$x_t = S(t)x_0 + \int_0^t S(t - s)f(s, x_s)dC_s, \ t \in [0, T],$$

where $S(t)$ is continuous with $S(0) = I, \ |S(t)| \leq c, \ c > 0$, for all $t \in [0, T]$.

Proof For fuzzy coefficient A, the following define inverse of $S(t)$

$$S^{-1}(t) = e^{-At}.$$

Then it follows that

$$dS^{-1}(t) = -Ae^{-At}dt = -AS^{-1}(t)dt.$$

Applying the integration by parts to the above equation provides

$$\begin{aligned} &d(S^{-1}(t)x_t) \\ &\quad = d(S^{-1}(t))x_t + S^{-1}(t)dx_t \\ &\quad = -AS^{-1}(t)dtx_t + S^{-1}(t)Ax_t dt \\ &\qquad\qquad + S^{-1}(t)f(t, x_t)dC_t. \end{aligned}$$

That is,

$$d(S^{-1}(t)x_t) = S^{-1}(t)f(t, x_t)dC_t.$$

Defining $z_t = S^{-1}(t)x_t$, we obtain $x_t = S(t)z_t$ and

$$dz_t = S^{-1}(t)f(t, S(t)z_t)dC_t.$$

Furthermore, we get in virtue of $S(0) = I$, and $z_0 = x_0$,

$$z_t = x_0 + \int_0^t S^{-1}(s)f(s, S(s)z_s)dC_s.$$

Therefore the Eq. (2) has the following solution

$$x_t = S(t)x_0 + \int_0^t S(t - s)f(s, x_s)dC_s, \ t \in [0, T],$$

where we write $S(t - s)$ instead of $S(t)S^{-1}(s)$.

Assume the following statements:

(H1) For $x_t, y_t \in C([0, T] \times (\Theta, \mathcal{P}, C_r), X), \ t \in [0, T]$, there exists positive number m such that

$$d_L([f(t, x_t)]^\alpha, [f(t, y_t)]^\alpha) \leq m d_L([x_t]^\alpha, [y_t]^\alpha)$$

and $f(0, \mathcal{X}_{\{0\}}(0)) \equiv 0$.

(H2) $2cmKT \leq 1$.

By Lemma 3.2, we know that the Eq. (2) have a solution x_t. Thus in Theorem 3.1, we show that uniqueness of solution for Eq. (2).

Theorem 3.1 For every $x_0 \in E_N$, if hypotheses (H1), (H2) are hold, then the eEq. (2) have a unique solution $x_t \in C([0, T] \times (\Theta, \mathcal{P}, C_r), X)$.

Proof For each $\xi_t \in C([0, T] \times (\Theta, \mathcal{P}, C_r), X), \ t \in [0, T]$ define

$$\Psi\xi_t = S(t)x_0 + \int_0^t S(t - s)f(s, \xi_s)dC_s.$$

Thus, one can show that $\Psi\xi : [0,T]\times(\Theta,\mathcal{P},C_r) \to C([0,T]\times(\Theta,\mathcal{P},C_r),X)$ is continuous, then

$$\Psi : C([0,T]\times(\Theta,\mathcal{P},C_r),X) \to C([0,T]\times(\Theta,\mathcal{P},C_r),X).$$

It is also obvious that a fixed point of Ψ is solution for the Eq. (2). For $\xi_t,\ \eta_t \in C([0,T]\times(\Theta,\mathcal{P},C_r),X)$, by Lemma 2.4 and hypothesis (H1), we have

$$\begin{aligned}
&d_L([\Psi\xi_t]^\alpha,[\Psi\eta_t]^\alpha)\\
&= d_L\Big(\Big[\int_0^t S(t-s)f(s,\xi_s)dC_s\Big]^\alpha,\\
&\qquad\qquad \Big[\int_0^t S(t-s)f(s,\eta_s)dC_s\Big]^\alpha\Big)\\
&\le cmK\int_0^t d_L([\xi_s]^\alpha,[\eta_s]^\alpha)ds.
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
D_L(\Psi\xi_t,\Psi\eta_t) &= \sup_{\alpha\in(0,1]} d_L([\Psi\xi_t]^\alpha,[\Psi\eta_t]^\alpha)\\
&\le cmK\int_0^t \sup_{\alpha\in(0,1]} d_L([\xi_s]^\alpha,[\eta_s]^\alpha)ds\\
&= cmK\int_0^t D_L(\xi_s,\eta_s)ds.
\end{aligned}$$

Hence, for a.s. $\theta\in\Theta$, by Lemma 2.1,

$$\begin{aligned}
E\Big(H_1(\Psi\xi,\Psi\eta)\Big) &= E\Big(\sup_{t\in[0,T]} D_L(\Psi\xi_t,\Psi\eta_t)\Big)\\
&\le E\Big(cmK\sup_{t\in[0,T]}\int_0^t D_L(\xi_s,\eta_s)ds\Big)\\
&\le cmKTE\Big(H_1(\xi,\eta)\Big).
\end{aligned}$$

By hypotheses (H2), Ψ is a contraction mapping. By the Banach fixed point theorem, Eq. (2) have a unique fixed point $x_t \in C([0,T]\times(\Theta,\mathcal{P},C_r),X)$.

4. Exact Controllability for Abstract Fuzzy Differential Equations

In this section, we study exact controllability for abstract fuzzy differential Eq. (1).

We consider solution for the Eq. (1), for each u in $Y(\subset E_N)$.

$$\begin{cases} x_t = S(t)x_0 + \int_0^t S(t-s)f(s,x_s)dC_s \\ \qquad\qquad\qquad + \int_0^t S(t-s)Bu_sds, \\ x(0) = x_0 \in E_N, \end{cases} \tag{3}$$

where $S(t)$ is continuous with $S(0)=I$, $|S(t)|\le c,\ c>0$, for all $t\in[0,T]$.

We define the controllability concept for abstract fuzzy differential equations.

Definition 4.1 The Eq. (1) are said to be controllable on [0,T], if for every $x_0 \in E_N$ there exists a control $u_t \in Y$ such that the solution x of (1) satisfies $x_T = x^1 \in X$, a.s. θ (i.e., $[x_T]^\alpha = [x^1]^\alpha$).

Define the fuzzy mapping $\widetilde{G} : \widetilde{P}(R) \to X$

$$\widetilde{G}^\alpha(v) = \begin{cases} \int_0^T S^\alpha(T-s)Bv_sds, & v\subset\overline{\Gamma}_u, \\ 0, & \text{otherwise}, \end{cases}$$

where $\widetilde{P}(R)$ is a nonempty fuzzy subset of R and $\overline{\Gamma}_u$ is closure of support u. Then there exists $\widetilde{G}_i^\alpha(i=l,r)$ such that

$$\begin{aligned}
\widetilde{G}_l^\alpha(v_l) &= \int_0^T S_l^\alpha(T-s)B(v_s)_l ds,\\
&\qquad\qquad (v_s)_l \in [(u_s)_l^\alpha,(u_s)^1],\\
\widetilde{G}_r^\alpha(v_r) &= \int_0^T S_r^\alpha(T-s)B(v_s)_r ds,\\
&\qquad\qquad (v_s)_r \in [(u_s)^1,(u_s)_r^\alpha].
\end{aligned}$$

We assume that $\widetilde{G}_l^\alpha,\widetilde{G}_r^\alpha$ are bijective mappings.

We can introduce α-level set of u_s defined by

$$\begin{aligned}
&[u_s]^\alpha\\
&= [(u_s)_l^\alpha,(u_s)_r^\alpha]\\
&= \Big[(\widetilde{G}_l^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha\\
&\qquad - \int_0^T S_l^\alpha(T-s)f_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\},\\
&\quad (\widetilde{G}_r^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha\\
&\qquad - \int_0^T S_r^\alpha(T-s)f_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big].
\end{aligned}$$

Then substitute this expression into the Eq. (3) yields α-level of x_T.

$$
\begin{aligned}
&[x_T]^\alpha \\
&= \Big[S(T)x_0 + \int_0^T S(T-s)f(s,x_s)dC_s \\
&\qquad\qquad + \int_0^T S(T-s)Bu_s ds\Big]^\alpha \\
&= \Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)f_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad + \int_0^T S_l^\alpha(T-s)B(\widetilde{G}_l^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha \\
&\quad - \int_0^T S_l^\alpha(T-s)f_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}ds, \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)f_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad + \int_0^T S_r^\alpha(T-s)B(\widetilde{G}_r^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)x_{0r}^\alpha \\
&\quad - \int_0^T S_r^\alpha(T-s)f_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}ds\Big] \\
&= \Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)f_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad + \widetilde{G}_l^\alpha(\widetilde{G}_l^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha \\
&\quad - \int_0^T S_l^\alpha(T-s)f_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}, \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)f_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad + \widetilde{G}_r^\alpha(\widetilde{G}_r^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha \\
&\quad - \int_0^T S_r^\alpha(T-s)f_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big] \\
&= [(x^1)_l^\alpha, (x^1)_r^\alpha] = [x^1]^\alpha.
\end{aligned}
$$

Hence this control u_t satisfis $x_T = x^1$, a.s. θ.

We now set

$$
\begin{aligned}
&\Phi x_t \\
&= S(t)x_0 + \int_0^t S(t-s)f(s,x_s)dC_s \\
&\quad + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big\{x^1 - S(T)x_0 \\
&\qquad\qquad - \int_0^T S(T-\tau)f(\tau,x_\tau)dC_\tau\Big\}ds,
\end{aligned}
$$

where the fuzzy mappings $\widetilde{G}^{-1}$ satisfies above statements.

(H3) Assume that the linear system of Eq. (1) ($f \equiv 0$) is controllable.

Theorem 4.1 If Lemma 2.4 and the hypotheses (H1), (H2) and (H3) are satisfied, then the Eq. (1) are controllable on $[0,T]$.

Proof We can easily check that Φ is continuous from $C([0,T]\times(\Theta,\mathcal{P},Cr),X)$ to itself. By Lemma 2.4 and hypotheses (H1) and (H2), for any given θ with $Cr\{\theta\}>0$, $x_t,\ y_t \in C([0,T]\times(\Theta,\mathcal{P},Cr),X)$, we have

$$
\begin{aligned}
&d_L\Big([\Phi x_t]^\alpha, [\Phi y_t]^\alpha\Big) \\
&= d_L\Big(\Big[S(t)x_0 + \int_0^t S(t-s)f(s,x_s)dC_s \\
&\qquad + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big\{x^1 - S(T)x_0 \\
&\qquad\quad - \int_0^T S(T-\tau)f(\tau,x_\tau)dC_\tau\Big\}ds\Big]^\alpha, \\
&\quad \Big[S(t)x_0 + \int_0^t S(t-s)f(s,y_s)dC_s \\
&\qquad + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big\{x^1 - S(T)x_0 \\
&\qquad\quad - \int_0^T S(T-\tau)f(\tau,y_\tau)dC_\tau\Big\}ds\Big]^\alpha\Big) \\
&\le d_L\Big(\Big[\int_0^t S(t-s)f(s,x_s)dC_s\Big]^\alpha, \\
&\qquad \Big[\int_0^t S(t-s)f(s,y_s)dC_s\Big]^\alpha\Big) \\
&\quad + d_L\Big(\Big[\int_0^t S(t-s)B\widetilde{G}^{-1} \\
&\qquad \times \int_0^T S(T-\tau)f(\tau,x_\tau)dC_\tau(s)ds\Big]^\alpha, \\
&\quad \Big[\int_0^t S(t-s)B\widetilde{G}^{-1} \\
&\qquad \times \int_0^T S(T-\tau)f(\tau,y_\tau)dC_\tau(s)ds\Big]^\alpha\Big) \\
&\le cmK\int_0^t d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds \\
&\quad + d_L\Big(\Big[\widetilde{G}\widetilde{G}^{-1}\int_0^T S(T-\tau)f(\tau,x_\tau)dC_\tau\Big]^\alpha, \\
&\qquad \Big[\widetilde{G}\widetilde{G}^{-1}\int_0^T S(T-\tau)f(\tau,y_\tau)dC_\tau\Big]^\alpha\Big) \\
&\le cmK\int_0^t d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds \\
&\qquad + cK\int_0^t d_L\Big([f(s,x_s)]^\alpha,[f(s,y_s)]^\alpha\Big)ds
\end{aligned}
$$

$$\leq 2cmK \int_0^t d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds.$$

Therefore by Lemma 2.1,

$$\begin{aligned}
&E\Big(H_1(\Phi x, \Phi y)\Big)\\
&= E\Big(\sup_{t\in[0,T]} D_L(\Phi x_t, \Phi y_t)\Big)\\
&= E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\leq 1} d_L\Big([\Phi x_t]^\alpha, [\Phi y_t]^\alpha\Big)\Big)\\
&\leq E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\leq 1} 2cmK\int_0^t d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds\Big)\\
&\leq E\Big(\sup_{t\in[0,T]} 2cmK\int_0^t D_L(x_s, y_s)ds\Big)\\
&\leq 2cmKTE\Big(H_1(x,y)\Big).
\end{aligned}$$

We take sufficiently small T, $(2cmKT) < 1$. Hence Φ is a contraction mapping. We now apply the Banach fixed point theorem to show that the Eq. (3) have a unique fixed point.

Consequently, the Eq. (1) are controllable on $[0,T]$.

Example 4.1 We consider the following abstract fuzzy differential equations in credibility space

$$\begin{cases} dx_t = Ax_tdt + f(t,x_t)dC_t + Bu_tdt,\\ x(0) = x_0 \in E_N, \end{cases} \tag{4}$$

where the state x_t takes values in $X(\subset E_N)$ and another bounded space $Y(\subset E_N)$. E_N is the set of all upper semi-continuously convex fuzzy numbers on R, $(\Theta, \mathcal{P}, Cr)$ is credibility space, A is a fuzzy coefficient, the state function $x : [0,T]\times(\Theta,\mathcal{P},Cr) \to X$ is a fuzzy process, $f : [0,T]\times X \to X$ is a regular fuzzy function, $u : [0,T]\times(\Theta,\mathcal{P},Cr)\to Y$ is a control function, B is a linear bounded operator from Y to X. C_t is a standard Liu process, $x_0 \in E_N$ is an initial value.

Let $f(t,x_t) = \widetilde{2}tx_t$, $S^{-1}(t) = e^{-\widetilde{2}t}$, defining $z_t = S^{-1}(t)x_t$, then the balance equations become

$$\begin{cases} x_t = S(t)x_0 + \int_0^t S(t-s)\widetilde{2}tx_tdC_s\\ \qquad\qquad + \int_0^t S(t-s)Bu_sds,\\ x(0) = x_0 \in E_N. \end{cases} \tag{5}$$

Therefore Lemma 3.2 is satisfy.

The α-level set of fuzzy number $\widetilde{2}$ is $[2]^\alpha = [\alpha+1, 3-\alpha]$ for all $\alpha\in[0,1]$. Then α-level sets of $f(t,x_t)$ is $[f(t,x_t)]^\alpha = t[(\alpha+1)(x_t)_l^\alpha, (3-\alpha)(x_t)_r^\alpha]$. Further, we have

$$\begin{aligned}
&d_L\Big([f(t,x_t)]^\alpha, [f(t,y_t)]^\alpha\Big)\\
&= d_L\Big(t[(\alpha+1)(x_t)_l^\alpha, (3-\alpha)(x_t)_r^\alpha],\\
&\qquad t[(\alpha+1)(y_t)_l^\alpha, (3-\alpha)(y_t)_r^\alpha]\Big)\\
&= t\max\Big\{(\alpha+1)|(x_t)_l^\alpha - (y_t)_l^\alpha|,\\
&\qquad (3-\alpha)|(x_t)_r^\alpha - (y_t)_r^\alpha|\Big\}\\
&\leq 3T\max\Big\{|(x_t)_l^\alpha - (y_t)_l^\alpha|, |(x_t)_r^\alpha - (y_t)_r^\alpha|\Big\}\\
&= md_L([x_t]^\alpha, [y_t]^\alpha),
\end{aligned}$$

where $m = 3T$ satisfies the inequality in hypothesis (H1), (H2). Then all the conditions stated in Theorem 3.1 are satisfied.

Let an initial value x_0 is $\widetilde{0}$. Target set is $x^1 = \widetilde{2}$. The α-level set of fuzzy number $\widetilde{0}$ is $[\widetilde{0}] = [\alpha-1, 1-\alpha]$, $\alpha\in(0,1]$. We introduce the α-level set of u_s of Eq. (4).

$$\begin{aligned}
&[u_s]^\alpha\\
&= [(u_s)_l^\alpha, (u_s)_r^\alpha]\\
&= \Big[(\widetilde{G}_l^\alpha)^{-1}\Big\{(\alpha+1) - S_l^\alpha(T)(\alpha-1)\\
&\qquad - \int_0^T S_l^\alpha(T-s)s(\alpha+1)(x_s)_l^\alpha dC_s\Big\},\\
&\quad (\widetilde{G}_r^\alpha)^{-1}\Big\{(3-\alpha) - S_r^\alpha(T)(3-\alpha)\\
&\qquad - \int_0^T S_r^\alpha(T-s)s(3-\alpha)(x_s)_r^\alpha dC_s\Big\}\Big].
\end{aligned}$$

Then substituting this expression into the Eq. (5) yields α-level of x_T.

$$\begin{aligned}
&[x_T]^\alpha\\
&= \Big[S_l^\alpha(T)(\alpha-1)\\
&\qquad + \int_0^T S_l^\alpha(T-s)s(\alpha+1)(x_s)_l^\alpha dC_s\\
&\qquad + \int_0^T S_l^\alpha(T-s)B(\widetilde{G}_l^\alpha)^{-1}\Big\{(\alpha+1)\\
&\qquad\qquad - S_l^\alpha(T)(\alpha-1)\\
&\qquad - \int_0^T S_l^\alpha(T-s)s(\alpha+1)(x_s)_l^\alpha dC_s\Big\}ds,\\
&\quad S_r^\alpha(T)(1-\alpha)\\
&\qquad + \int_0^T S_r^\alpha(T-s)s(3-\alpha)(x_s)_r^\alpha dC_s\\
&\qquad + \int_0^T S_r^\alpha(T-s)B(\widetilde{G}_r^\alpha)^{-1}\Big\{(3-\alpha)
\end{aligned}$$

$$
\begin{aligned}
&-S_r^\alpha(T)(1-\alpha)\\
&-\int_0^T S_r^\alpha(T-s)s(3-\alpha)(x_s)_r^\alpha dC_s\Big\}ds\Big]\\
&=[(\alpha+1),(3-\alpha)]\\
&=[\widetilde{2}]^\alpha.
\end{aligned}
$$

Then all the conditions stated in Theorem 4.1 are satisfied. So the Eq. (4) are controllable on $[0,T]$.

5. Conclusions

If there is an exact controllability encouraged for the abstract fuzzy differential equations, it can provide a benchmark for an approach to handle controllability about the equations such as fuzzy semilinear integrodifferential equations, fuzzy delay integrodifferential equations on the credibility space. Therefore, the theoretical result of this study can be used to make stochastic extension on the credibility space.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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