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Hong JeongHa's Tianyuanshu and Zhengcheng Kaifangfa

洪正夏의 天元術과 增乘開方法

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Tianyuanshu and Zengcheng Kaifangfa introduced in the Song–Yuan dynasties and their contribution to the theory of equations are one of the most important achievements in the history of Chinese mathematics. Furthermore, they became the most fundamental subject in the history of East Asian mathematics as well. The operations, or the mathematical structure of polynomials have been overlooked by traditional mathematics books. Investigation of GuIlJib (九一集) of Joseon mathematician Hong JeongHa reveals that Hong's approach to polynomials is highly structural. For the expansion of $\prod_{k=1}^{n}(x+a_k)$, Hong invented a new method which we name Hong JeongHa's synthetic expansion. Using this, he reveals that the processes in Zhengcheng Kaifangfa is not synthetic division but synthetic expansion.

Keywords: Hong JeongHa, GuIlJib, Hong JeongHa's synthetic expansion, Tianyuanshu, Structure of polynomials, Binomial coefficients, Zhengcheng Kaifangfa, Shisuo Kaifangfa; 洪正夏, 九一集, 洪正夏의 組立展開, 天元術, 多項式의 構造, 二項係數, 增乘開方法, 釋鎖開方法.

MSC: 01A13, 01A25, 01A45, 01A50, 12–03, 12E05, 12E12

1 Introduction

The theory of equations in Eastern mathematics has as long a history as that in the West and divides into two parts, namely constructing equations and solving them. For the former, Tianyuanshu (天元術) was introduced in the early period of the Song dynasty (960–1279) and then extended up to Siyuanshu (四元術) to represent polynomials of four indeterminates by Zhu Shijie (朱世傑) in his Siyuan Yujian (四元玉鑑, 1303). In about the same time, Zengcheng Kaifangfa (增乘開方法) was introduced for the latter part. The early history about them was mostly lost and only

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fragments of the history remain in the books published in the 13th century. Li Ye (李冶, 1192–1279) extensively used Tianyuanshu in his books Ceyuan Haijing (測圓海鏡, 1248) and Yigu Yanduan (益古演段, 1259) and then Zhu Shijie in Suanxue Qimeng (算學啓蒙, 1299) and Siyuan Yujian. Qin Jiushao (秦九韶, 1202–1261) included Zhengcheng Kaifangfa in his book Shushu Jiuzhang (數書九章, 1247) and Yang Hui (楊輝) in Xiangjie Jiuzhang Suanfa (詳解九章算法, 1261) and YangHui Suanfa (楊輝算法, 1274–1275) [2].

Beginning in the Ming dynasty (1368–1644), Tianyuanshu became almost forgotten and it reappeared in the late 18th century in China. On the other hand, Suanxue Qimeng and YangHui Suanfa were brought in Joseon (1392–1910) in the 15th century and they formed the basic references for the development of Joseon mathematics. We note that the other books mentioned above were brought into Joseon in the mid 19th century and hence Joseon mathematicians built their theory of equations based only on Suanxue Qimeng and YangHui Suanfa. Although Yang Hui used Tianyuanshu representations of polynomials in YangHui Suanfa to explain the process of Zhengcheng Kaifangfa, he did not use them for constructing equations. Thus Suanxue Qimeng is the only reference in Joseon for constructing equations via Tianyuanshu before the mid 19th century. They were also transmitted to Japan during her invasion to Joseon in the late 16th century and gave a great impetus for the Japanese mathematics, or Wasan (和算) in the Edo period.

Republication of Suanxue Qimeng by Kim SiJin (金始振, 1618–1667) in 1660 revived Joseon mathematics which had been dormant for about two centuries. This led Park Yul (朴繘, 1621–1668) to write SanHak WonBon (算學原本, 1700), which is the first book to treat a good deal of Tianyuanshu after Siyuan Yujian in East Asia [6, 12].

Hong JeongHa (洪正夏, 1684–?) completed GuIlJib (九一集) in 1724 [3, 4]. He is the greatest mathematician in the Joseon dynasty and served as an official in HoJo (戶曹) (see [9, 16] for the detail). GuIlJib consists of 8 books (卷) and an appendix. Hong has finished the first 8 books before 1713 and added the appendix in 1724. His main concern in GuIlJib is the theory of equations although the book contains many interesting new results in other fields [10, 11, 15]. For the theory of equations, he included Tianyuanshu and Zengcheng Kaifangfa in detail together with a great deal of applications to various subjects [8, 10].

The aim of this paper is to show Hong's contributions to Tianyuanshu. To do this, first we discuss Tianyuanshu briefly. Tianyuanshu is a method to represent a rational polynomial, $\sum_{k=-m}^{n} a_k x^k$ $(0 \leq m, n)$ by its coefficients

$$a_{-m}, a_{-(m-1)}, \cdots, a_0, a_1, \cdots, a_n,$$

which are arranged vertically. To indicate proper degrees of terms, a_0 or a_1, i.e., the constant term or the linear coefficient of the polynomial, is attached with symbols tai (太) or yuan (元), respectively. Suanxue Qimeng deals with polynomials $\sum_{k=0}^{n} a_k x^k$ only so that the top one in the Tianyuanshu representation indicates the constant term and hence Zhu Shijie omitted the symbols. Furthermore, coefficients in the representation are denoted by the counting rods so that operations of polynomials are easily manipulated and polynomials became abstract entities [14]. As in Li Ye's books, only final results of operations of polynomials are included in Suanxue Qimeng and their processes are completely omitted. Zhu Shijie and Li Ye concerned themselves mainly about constructing equations and regarded the processes of operations as trivial. Indeed, there in Suanxue Qimeng are all together 28 problems which use expansions of polynomials. Among them, 25 are solved using the binomial expansions and merely 3 problems need the expansions of type $(ax+b)(cx+d)$ as in Yigu Yanduan [5]. We do not know the reason why Zhu Shijie did not use Tianyuanshu to solve the problems dealing with equations in the section Duiji Huanyuanmen (堆積還源門). In particular, the last problem in the section involves the expansion $(x+7)(x+8)(2x+15)$ and Zhu's explanation for that is not systematic (see also [7]). A possible reason for the overlook may be the lack of papers so that the authors could not include every detail as in the original manuscript of Siyuan Yujian. We note that Li Rui (李銳, 1773–1817) is the first in China to define properly Tianyuanshu and its operations in his commentary to Ceyuan Haijing (1797). In all, the algebraic structure of polynomials were overlooked by traditional Chinese mathematicians and historians.

We now return to Hong JeongHa's GuIlJib. Unlike the Chinese authors as mentioned above, Hong JungHa is the first author in East who extensively deals with the operations, particularly multiplication of polynomials. We note that the addition, subtraction in general, and division by x^n in Tianyuanshu representations are trivial. First he noticed that Tianyuanshu representations are basically a numeral system like decimal one, if one considers its base not as 10 but as infinity ∞ with the exponential law $x^n \times x^m = x^{n+m}$. Thus he could use the associative and distributive laws together with Shenwai Jiafa (身外加法). The most important contribution of Hong to the operations of polynomials is the invention of a new method for the expansion of $\prod_{k=1}^{n}(x+a_k)$. In the following the new method is called *Hong JeongHa's synthetic expansion*. Using this, Hong JeongHa obtains the binomial coefficients of $a(x+\alpha)^n$ as a special case of the synthetic expansion. We recall that in the traditional East Asian mathematics except Shuli Jingyun (數理精蘊, 1723), the long divisions have not appeared so that the process involved in Zhengcheng Kaifangfa cannot be synthetic divisions. Using Hong's synthetic expansion, we show that the process in Zhengcheng Kaifangfa is indeed the synthetic expansion as in

Shisuo Kaifangfa (釋鎖開方法).

In the following, we will use the horizontal Tianyuanshu representation of polynomials for the sake of typing and also for the readers. For a polynomial $\sum_{k=0}^{n} a_k x^k$, its Tianyuanshu representation becomes

$$a_n, a_{n-1}, \cdots, a_1, a_0,$$

where counting rods in Tianyuanshu are all replaced by numerals. We note that sequences of binomial expansions in Gufa Qichengfangtu (古法七乘方圖), the Zia Xian (賈憲) triangle in Siyuan Yujian are given by horizontal arrays. Traditional ordering of the horizontal representation of a polynomial is the *reverse* of our notion as in Gufa Qichengfangtu for it is natural for brush writings. It is a wild guess but it also seems plausible that the vertical representation is just for the book format and the actual computations are carried out by the horizontal one as those for numbers with counting rods.

The reader may find all the Chinese sources of this paper in ZhongGuo KeXue JiShu DianJi TongHui ShuXueJuan (中國科學技術典籍通彙 數學卷) [1] and hence they will not be numbered as an individual reference.

2 Tianyuanshu in GuIlJib

The first appearance of Tianyuanshu in GuIlJib is in Problem 8 of the section GuCheokHaeEunMun (毬隻解隱門) in Book 4. He uses a typical process like one in Suanxue Qimeng with the expansion of $(x-4)^3$. Hong JeongHa has shown his characteristic use of Tianyuanshu in constructing equations, which will be discussed in another paper. In this paper we will restrict ourselves to only the algebraic structure of Tianyuanshu revealed in GuIlJib. Furthermore, we will not discuss the usual binomial expansions in GuIlJib for they do not belong to Hong's characteristic.

2.1 Associative law, Distributive law, Shenwai Jiafa (身外加法)

A really interesting problem dealing with expansions of polynomials appears in Problem 16 of the section BuByeongToeTaMun (缶瓶堆垜門), section on finite series in Book 4. The problem is exactly like the last problem in Suanxue Qimeng which we mentioned above but has the different sum. Hong JeongHa used the formula $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(n+(1/2))}{3}$ instead of $\dfrac{n(n+1)(2n+1)}{6}$ and hence he should have expanded $(x+7)(x+8)(x+7.5)$. For this, he first expanded

$$(x+8)(x+7.5) = x^2 + (8+7.5)x + 8 \times 7.5 = x^2 + 15.5x + 60$$

and then expanded

$$(x+7)(x^2+15.5x+60) = x^3 + 22.5x^2 + 168.5x + 420$$

as the sum of $x(x^2+15.5x+60)$ and $7(x^2+15.5x+60)$ which is done as follows:

1	15.5	60	0
	7	108.5	420
1	22.5	168.5	420

The above process indicates that Hong first used associative law and then distributive law to expand the multiplication. Further, the last process is precisely Shenwai Jiafa. For the expansion of $(x+7)(x+8)(2x+8)$ mentioned in the previous section, Hong might have computed the expansion as follows: He first used Shenwai Jiafa to get $(x+8)(2x+15) = 2x^2+31x+120$ and then applied Shenwai Jiafa once again to get $(x+7)(2x^2+31x+120) = 2x^3+45x^2+337x+840$.

One will find a significant difference when he compares the computations first as above and then using the process in Suanxue Qimeng.

In the following, the methods in each subsections 2.1, 2.2, 2.3 will be denoted by type I, II, III, respectively.

2.2 YuSeung (維乘), Combined Multiplications

Problem 18 in the section BuByeongToeTaMun of Book 4 involves the expansion of $(-x+32)(-x+33)(-x+32.5)$, or equivalently $-(x-32)(x-33)(x-32.5)$ which can be expanded by type I, i.e., the method in the above subsection 2.1. Hong JeongHa takes a similar method to the one in Suanxue Qimeng but his method is more systematic than Zhu's. He first arranges the three constants $32, 33, 32.5$ using counting rods and then find the product of the three numbers ($= 34,320$). Hong then introduces the terminology YuSeung (維乘) of the array which means three products $32\times 33, 32\times 32.5, 33\times 32.5$. Thus YuSeung means the products of 2-combinations out of the three numbers, so we call them the combined multiplications of the three numbers. He then finds the sum of the combined multiplications ($= 3,168$) and finally sum of the three numbers ($= 97.5$). In the end, Hong has the expansion $-x^3+97.5x^2-3,168.5x+34,320$. The process indicates that Hong uses the sum of products of k-combinations ($k=1,2,3$) of the three numbers and the appropriate signs for the expansion.

2.3 Hong JeongHa's Synthetic Expansion

Hong JeongHa included many problems which use the method of type I and II in many sections in GuIlJib. In Problem 34 of the section GaeBangGakSulMun Ha (開方各術門 下) in Book 8, Hong introduced a new method for the expansion of four terms, $(x+4)(x+8)(x+12)(x+16)$. We note that in type II, products of 2-combinations of three terms may be understood through the sum of three fractions, $\frac{b_1}{a_1}+\frac{b_2}{a_2}+\frac{b_3}{a_3} = \frac{a_1(b_2b_3)+a_2(b_1b_3)+a_3(b_1b_2)}{a_1a_2a_3}$. Indeed, Hong called the numerator

1	0	0	0	0	$\alpha, \beta, \gamma, \delta$
	α	$\alpha\beta$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	
1	α	$\alpha\beta$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	
	β	$(\alpha+\beta)\gamma$	$(\alpha\beta+(\alpha+\beta)\gamma)\delta$		
1	$\alpha+\beta$	$\alpha\beta+(\alpha+\beta)\gamma$	$\alpha\beta\gamma+(\alpha\beta+(\alpha+\beta)\gamma)\delta$		
	γ	$(\alpha+\beta+\gamma)\delta$			
1	$\alpha+\beta+\gamma$	$\alpha\beta+(\alpha+\beta)\gamma+(\alpha+\beta+\gamma)\delta$			
	δ				
1	$\alpha+\beta+\gamma+\delta$				

Figure 1. Hong JeongHa's synthetic expansion process.

of the fraction HoSeung (互乘) [11] or used the process, MoHoSeungJa (母互乘子) as in Jiuzhang Suanshu, and it is indeed the sum of combined multiplications when $a_1 = a_2 = a_3 = 1$. For four terms, products of 3-combinations may be YuSeung but he may have had difficulty in describing products of 2-combinations of four terms as a special case of fractions.

In the following, we call the new method *Hong JeongHa's synthetic expansion* or simply *synthetic expansion*. For the convenience and generality, we will explain Hong JeongHa's synthetic expansion for the expansion of $(x+\alpha)(x+\beta)(x+\gamma)(x+\delta)$ as a diagram in the Figure 1.

One can easily figure out the above process. Indeed, the first row is the array of 1 and four 0's, where 4 is the number of factors of the multiplication together with the array of α, β, γ and δ instead of the first guess (初商) in the Zhengcheng Kaifangfa. Beginning with α in Zhengcheng (增乘) process, the role of the first guess α is replaced in succession by β, γ and δ in each process of Zhengcheng.

Using Hong JeongHa's synthetic expansion, one is led to

$$\begin{aligned} & x^4 + (\alpha+\beta+\gamma+\delta)\, x^3 \\ & \quad + [\alpha\beta + (\alpha+\beta)\,\gamma + (\alpha+\beta+\gamma)\,\delta\,]\, x^2 \\ & \quad + [\alpha\beta\gamma + (\alpha\beta + (\alpha+\beta)\gamma\,)\delta\,]\, x \\ & \quad + \alpha\beta\gamma\delta. \end{aligned}$$

Hong JeongHa's synthetic expansion also implies that the coefficients of the expansion of $\prod_{k=1}^{n}(x+a_k)$ are precisely the elementary symmetric polynomials in $a_1, a_2, \cdots, a_n$.

We change the ordering in Hong JeongHa's synthetic expansion. His ordering is determined by the order of α, β, γ and δ. The processes are clearly apparent when one sees the diagram in Figure 1 diagonally from the top right to the bottom left. The first diagonal deals with α, and in succession with β, γ and δ. There are two reasons for the change of ordering. Firstly, our ordering is more familiar to readers. More important reason is that Hong JeongHa actually adopted our ordering,

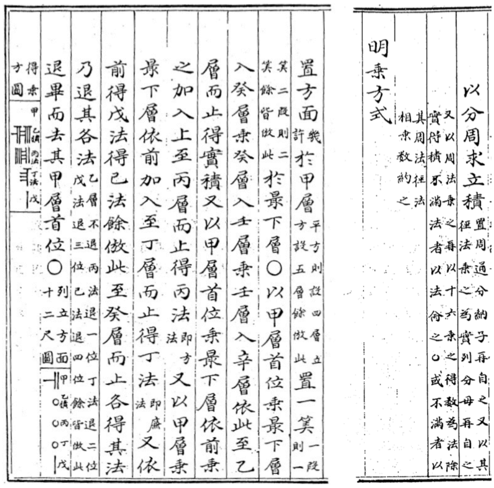

Figure 2. Binomial expansion (明乘方式) by synthetic expansion in GuIlJIb.

which will be discussed in the next subsection. Indeed, Hong states the following in the very next problem, Problem 35 in the same section involving the binomial expansion $(x+28)^3$.

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The above quote means that one can also have the binomial expansion by Hong's synthetic expansion. This leads to the next discussions.

2.4 Hong JeongHa's binomial expansions

As mentioned in the previous section, the most frequently used manipulation in Tianyuanshu is binomial expansions. We now discuss the relation between binomial expansions and Hong JeongHa's methods. We first show that binomial expansions can be obtained by Shenwai Jiafa. We assume that $(x+1)^n = \sum_{k=0}^{n} a_k^n x^k$. Then one has Tianyuanshu representation of $(x+1)^{n+1}$ as follows.

$$\begin{array}{ccccccccccc}
1 & a_{n-1}^n & . & \cdots & a_k^n & a_{k-1}^n & . & \cdots & a_1^n & 1 & \\
 & 1 & a_{n-1}^n & \cdots & . & a_k^n & a_{k-1}^n & \cdots & . & a_1^n & 1 \\
\hline
1 & a_{n-1}^n+1 & . & \cdots & . & a_{k-1}^n+a_k^n & . & \cdots & . & 1+a_1^n & 1
\end{array}$$

Thus one has $(x+1)^{n+1} = \sum_{k=0}^{n+1}(a_{k-1}^n + a_k^n)x^k$, in other words,

$$a_k^{n+1} = a_{k-1}^n + a_k^n.$$

We recall that the above fact is indicated in Gufa Qichengfangtu in Siyuan Yujian by connecting lines between the upper two terms and the lower term, which do not appear in YangHui's triangle in Yongle Dadian (永樂大典, 1408).

Using Type II for the expansion of $\prod_{k=1}^{n}(x+a_k)$, one can also have the binomial coefficients of $(x+1)^n$. In this case, $a_k = 1$ for all k and hence the array of a_k's become the array consisting of 1. Thus, the multiplication of each k-combination of 1's is simply 1 so that the sum of multiplications of all the k-combinations is simply the number of k-combinations of n terms, i.e., $\binom{n}{k}$. In all, $(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$.

As mentioned above, we now discuss Hong JeongHa's binomial expansion by his synthetic expansion (Figure 2). Hong put this discussion as MyungSeungBangSik (明乘方式) at the preliminary remarks (凡例) in Book 9, titled miscellaneous records (雜錄) (see also [15] for the organization of two versions of GuIlJib). Hong first noticed that for the expansion of $a\prod_{k=1}^{n}(x+a_k)$, one has the expansion simply replacing 1 of the first row in the process of type III by a and that the expansion $a(x+\alpha)^n$ is a special case of $a_k = \alpha$ for all k. Hong explained the process for the case of $n = 10$, any α (幾許) and natural number a. We just include an example for the expansion of $a(x+\alpha)^4$ using Hong's synthetic expansion.

a	0	0	0	0	α
	$a\alpha$	$a\alpha^2$	$a\alpha^3$	$a\alpha^4$	
a	$a\alpha$	$a\alpha^2$	$a\alpha^3$	$a\alpha^4$	
	$a\alpha$	$2a\alpha^2$	$3a\alpha^3$		
a	$2a\alpha$	$3a\alpha^2$	$4a\alpha^3$		
	$a\alpha$	$3a\alpha^2$			
a	$3a\alpha$	$6a\alpha^2$			
	$a\alpha$				
a	$4a\alpha$				

The above process is precisely the ones in Zhengheng Kaifangfa, i.e., the successive synthetic divisions of ax^4 by $x-\alpha$, or the representation

$$ax^4 = a(x-\alpha)^4 + 4a\alpha(x-\alpha)^3 + 6a\alpha^2(x-\alpha)^2 + 4a\alpha^3(x-\alpha) + a\alpha^4.$$

If we put $y = x-\alpha$, then the above representation is precisely the expansion of $a(y+\alpha)^4$. We explained Hong JeongHa's binomial expansions by this view [15] but it is not the case. Indeed, the representation of ax^n by $x-\alpha$ is indeed resulted through the expansion but not by divisions. This view leads to the real structure of Zhengcheng Kaifangfa and the relation between Zhengcheng Kaifangfa and Shisuo Kaifangfa.

2.5 Hong JeongHa's Synthetic Expansion and Zhengcheng Kaifangfa

We just recall Zengcheng Kaifangfa for a polynomial equation $p(x) = 0$, where $p(x) = \sum_{k=0}^{n} a_k x^k$. First one has a suitable first choice α for the solution, called Chushang (初商) and then finds an equation for the remaining part y, called Chishang (次商). Since $x = y + \alpha$, we have the equation for y:

$$\sum_{k=0}^{n} a_k (y+\alpha)^k = 0$$

Here one has to expand $a_k(y+\alpha)^k$ for each k. But these expansions of $a_k(y+\alpha)^k$ are obtained by Hong JeongHa's synthetic expansion with the first array starting with a_k, and k zeros and α. Combining all those processes according to the degrees or the number of zeros in the first array into one process, one has the processes in Zhengcheng Kaifangfa, which are known as successive synthetic divisions so far, but Zhengcheng Kaifangfa is exactly Shisuo Kaifangfa with Hong JeongHa's synthetic expansion. In other words, Shisuo Kaifangfa is originated from the geometric observation and Zhengcheng Kaifangfa from its algebraic version through synthetic expansion (see [13]).

3 Conclusions

Although Tianyuanshu is one of the most important concepts in the history of Chinese mathematics, most of mathematicians have not paid any attention to its mathematical structures in East Asian mathematics. Hong JeongHa studied the operations of polynomials using Tianyuanshu and then obtained unprecedented results on the mathematical structure of polynomials, which also illustrates Hong's structural approaches to mathematics [15].

Along with basic structures of polynomials, Hong JeongHa introduced a concept of synthetic expansion and then was able to unify Shisuo Kaifangfa and Zhengcheng Kaifangfa through the synthetic expansion. Indeed, the processes in Zhengcheng Kaifangfa are not successive synthetic division but synthetic expansion. We note that the early history of the two Kaifangfa has not been clear for it was essentially lost. We have succeeded in explaining the processes in Zhengcheng Kaifangfa by expansions [17] and presented it in the annual meeting of our society and now find that it is essentially the same with Hong JeongHa's results. We conclude that Hong JeongHa's synthetic expansion gives a historically proper insight to Kaifangfa in East Asian mathematics.

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