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COMPACTNESS AND DIRICHLET'S PRINCIPLE

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ABSTRACT. In this paper we explore the emergence of the notion of compactness within its historical beginning through rigor versus intuition modes in the treatment of Dirichlet's principle. We emphasize on the intuition in Riemann's statement on the principle criticized by Weierstrass'requirement of rigor followed by Hilbert's restatement again criticized by Hadamard, which pushed the ascension of the notion of compactness in the analysis of PDEs. A brief overview of some techniques and problems involving compactness is presented illustrating the importance of this notion.

Compactness is discussed here to raise educational issues regarding rigor vs intuition in mathematical studies. The concept of compactness advanced rapidly after Weierstrass's famous criticism of Riemann's use of the Dirichlet principle. The rigor of Weierstrass contributed to establishment of the concept of compactness, but such a focus on rigor blinded mathematicians to big pictures. Fortunately, Poincaré and Hilbert defended Riemann's use of the Dirichlet principle and found a balance between rigor and intuition. There is no theorem without rigor, but we should not be a slave of rigor. Rigor (highly detailed examination with toy models) and intuition (broader view with real models) are essentially complementary to each other.

1. INTRODUCTION

The purpose of this article is to provide motivations, intuition, and applications for the notion of compactness. Rigor, intuition, and applications in mathematics are each important to the advancement of scientific knowledge. Intuition helps to invent theories, and methodological rigor ascertains their reliability. Emphasis on rigor should not overshadow a more conceptual overview or a work's physical motivation. Simmons remarked [27] "Mathematical rigor is like clothing: in its style it ought to suit the occasion, and it diminishes comfort and restricts freedom of movement if it is either too loose or too tight." Indeed, students majoring in mathematics learn the concept of compactness without knowing its physical or historical

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background. This type of educational approach prevents students from expanding their ability to conduct very complex or advanced science.

The concept of compactness and its introduction was highlighted by the famous debate between Riemann(1826-1866) and Weierstrass(1815-1897) regarding the convergence issue of the minimization problem in Dirichlet's principle. Riemann used the Dirichlet principle as follows: If u is the solution of Dirichlet's problem $\Delta u = 0$ in a bounded smooth domain $\Omega \subset \mathbb{R}^3$ with boundary data $u|_{\partial\Omega} = \phi \in C(\partial\Omega)$, then u can be obtained by the limit of the minimizing sequence $\{v_n\}$ of the energy functional

$$f(v) := \int_\Omega |\nabla v|^2 dx \tag{1.1}$$

within an admissible set such as $\mathcal{A} := \{v \in C^2(\Omega) \cap C(\overline{\Omega}) \ : \ v|_{\partial\Omega} = \phi\}$. In short, Riemann's observation was that f attains its minimum at a function u in the admissible class.

Riemann's use of Dirichlet's principle had been attacked by many mathematicians until the work of Hilbert [18, 24, 25], which was also insufficiently rigorous. In particular, Weierstrass strongly criticized Riemann's use of the Dirichlet principle, because this method had no rigorous evidence of the existence of the limit of the sequence $\{v_n\}$ within the admissible set. Moreover, in 1870, he presented an example of a 1D minimization problem whose minimizers do not exist: He considered the following energy functional over the set $\mathcal{A} = \{v \in C^1([-1,1]) \ : \ v(-1) = 0 \ \& \ v(1) = 1\}$:

$$f(v) = \int_{-1}^{1} \left| x \frac{\partial}{\partial x} v(x) \right|^2 dx$$

To be precise, the sequence given by

$$v_n(x) = (\sin n\pi x/2)^2 \chi_{[0,1/n]}(x) + \chi_{(1/n,1]}(x)$$

is contained in the set $\mathcal{A}$. Given that $\lim_{n\to\infty} f(v_n) \to 0$, it is a minimizing sequence. However, the sequence v_n is bounded in $W^{1,1}([-1,1])$ because $||\frac{\partial v}{dx}||_{L^1} = 1$; thus, it will converge inside the larger space $BV([-1,1])$ of functions with bounded variation to the Heaviside function, which does not belong to $\mathcal{A}$.

This famous riposte is not a proper counterexample attacking Riemann's use of the Dirichlet principle, because it corresponds to the degenerate partial differential equation(PDE)

$$\frac{d}{dx}\left(x^2 \frac{d}{dx} u \right) = 0$$

whose coefficient is not away from zero. (See Lax-Milgram theorem.) This degenerate PDE is very different from the Laplace equation having coercivity.

The problem lays with Riemann's incorrect use of the Dirichlet principle. He had not understood a suitable hypothesis, and was therefore considered to be wrong. Weierstrass chose to attack and attempt to discredit Riemann on that principle, rather than correct the principle by considering the right hypothesis. Had he done so, he might have noticed that it is necessary for the space on which the minimizing problem is set to have some specific properties that guarantee the limit of the minimizing sequence to stay inside it. Weierstrass demonstrated his prowess through his counter-example disproving Riemann's result on the Dirichlet principle, but by failing to attempt to re-formulate the hypothesis in order to correct it, he also demonstrated his own limitations.

Weierstrass's focused criticism of Riemann's work - which sought a fatal blow rather than constructive refinement - prevented him from finding the right correction. His studies on analytic functions would lead to his achievements regarding the compactness of the minimizing sequence. His works on analytic functions and compactness were some of his main achievements in that period, along with the Bolzano–Weierstrass theorem.

Therefore, Riemann and Weierstrass's confrontation over the tradeoff between rigor and intuition during their work on the Dirichlet principle was an early demonstration of the importance of the notion of compactness. Weierstrass's meticulous approach might be connected to his prior teaching work, while Riemann's earlier theological studies suggest an intuitive character able to forgo rigor for a more conceptual outlook.

Weierstrass may also have had a personal interest in discrediting Riemann, because Riemann had published a paper on the Abelians functions, taking the same title as an earlier work on that subject by Weierstrass and writing a substantially better contribution. Thus aggrieved, Weierstrass would later focus on undermining Riemann. Weierstrass criticized [6] *"Riemann's disciples are making the mistake of attributing everything to their master, while many had already been made by and are due to Cauchy, etc. Riemann did nothing more than to dress them in his manner for his convenience"*. Weierstrass also pointed out the incorrectness of Cauchy's belief that a continuous function is differentiable except at some isolated points. Many of Weierstrass' results, including his example of a continuous non-differentiable function as well as his counterexample to Dirichlet's principle, were motivated by his criticism of Riemann's methods, and his distrust in Riemann's results [6]. Weierstrass provided the following continuous function that is nowhere differentiable (except a set of measure zero): $\phi(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{\pi n^2}$.

In the early 20th century, mathematicians again looked at the Dirichlet principle after long neglecting it. Hilbert improved it, correcting the statement by considering the right hypothesis on the domain and the boundary values.

Weierstrass's rigor certainly contributed to establishing the concept of compactness. However, such focusing on rigor can blind mathematicians to the broader context of their work. Rigor should not prevent researchers looking at the big picture to find new approaches. In 1900, some leading mathematicians warned of narcissists practicing "malignant self-love in narrow-scope mathematics". Poincaré mentioned [15]: *"For half a century there has been springing up a host of weird functions, which seem to strive to have as little resemblance as possible to honest functions that are of some use. No more continuity, or else continuity but no derivatives, etc. ... Formerly, when a new function was invented, it was in view of some practical end. Today they are invented on purpose to show our ancestors reasonings at fault, and we shall never get anything more out of them."*

There is no theorem without rigor, but we should not be a slave of rigor. Mathematicians' work should not lend credence to the following well-known joke: *Physicists think that the real world approximates equations; engineers think that equations approximate the real world;*

mathematicians simply don't see the relation between them. We should find a balance between rigor (detailed analysis using a toy model) and intuition (broader overview using a real model).

The following sections summarize some theories related to compactness, illustrating the importance of this notion in analysis. The examples are presented with limited details to maintain ease of reading.

2. DIRICHLET'S PRINCIPLE IN THE ABSENCE OF RIGOR

In the mid-nineteenth century, the Dirichlet principle was presented by Gauss and Dirichlet for the study of potential theory in electrostatics and Newtonian gravity. Let Ω be a bounded domain in $\mathbb{R}^3$ (or $\mathbb{R}^2$) with its connected Lipschitz boundary. For a given $\phi \in C(\partial\Omega)$, they considered a function in $\mathcal{A}_\phi := \{v \in C^2(\Omega) \cap C(\overline{\Omega}) \ : \ v|_{\partial\Omega} = \phi\}$ which minimizes the integral

$$f(v) = \int_\Omega |\nabla v|^2. \tag{2.1}$$

The following Dirichlet principle was stated without rigorous proof.

- Any function u minimizing $f(v)$ satisfies $\Delta u = 0$ in Ω.
- Any function u which satisfies $\Delta u = 0$ in Ω with $u|_{\partial\Omega} = \phi \in C(\partial\Omega)$ is a minimizer.

However, the first statement is not completely accurate in terms of the existence of a minimizer. Indeed, the space $C^2(\Omega)$ is not the right space in which to set the admissible set for the existence of a minimizer due to the lack of completeness with respect to the distance $f(u_n - u_m)$ in terms of energy between two functions u_n and u_m. In practice, the existence of a minimizer is obvious provided that the given system is physically existing or concerns observable quantities. Hence, it requires to find the correct space of physically meaningful functions with a proper distance concept. The correct space for solutions of the Laplace equation is the Sobolev space $\{v \in H^1(\Omega) \ : \ v|_{\partial\Omega} = \phi\}$ equipped with the norm $\|v\| = \sqrt{\int_\Omega |v|^2 + |\nabla v|^2}$, which measures both the size and regularity of a function. Before the twentieth century, the Hilbert space $H^1(\Omega)$ and measure theory had not been introduced; thus, there was insufficient knowledge to validate Dirichlet's principle in a rigorous way. This issue of ascertaining the existence of a minimizer might have been a possible motivation for the development of the notion of compactness.

The second statement is also not correct. (Non-physical solutions should be excluded for the uniqueness.) Hadamard [23] gave the following counter example of a solution of Dirichlet's problem in the unit disk Ω whose energy blows up:

$$u(x) = \sum_{n=1}^{\infty} 2^{-n} |x|^{2^{2n}} \cos(2^{2n}\theta) \quad (\tan\theta = x_2/x_1). \tag{2.2}$$

In this example, $u \in \mathcal{A}_\phi$ has the boundary data $\phi = \sum_{n=1}^{\infty} 2^{-n} \cos(2^{2n}\theta)$ which is continuous but not differentiable almost everywhere. For a matter of fact, the following is an example of an harmonic function which does not have the standard maximum principle. Consider the two dimensional function [32]:

$$u(x) = (|x|^{-3/2} - |x|^{3/2})\sin(\frac{3}{2}\theta), \qquad (\tan\theta = \frac{x_2}{x_1})$$

defined in the domain

$$\Omega := \{(r\cos\theta, r\sin\theta) \ : \ 0 < r < 1, \ 0 < \theta < \frac{3\pi}{2}\}.$$

Clearly, u is not a minimizer ($f(u) = \infty$) although it satisfies

$$\nabla^2 u = 0 \quad \text{in } \Omega\,, \qquad u|_{\partial\Omega} = 0.$$

In this example, $u \in C^\infty(\overline{\Omega} \setminus \{0\})$ but $u \notin C(\overline{\Omega})$.

Let us deal with this issue of the existence of a minimizer from a mid-nineteenth century perspective by considering a minimizing sequence $\{u_n\}$ in $\mathcal{A}_\phi$ such that

$$\lim_{n\to\infty} f(u_n) = \inf_{v\in\mathcal{A}_\phi} f(v).$$

Clearly, the existence of a minimizer essentially depends on whether a limit function of the set $\{u_n : n = 1, 2\cdots\}$ exists in $\mathcal{A}_\phi$. The L^2-norm or the H^1-norm were not known in the nineteenth century; thus, the distance between u_n and u_m was considered to be $\|u_n - u_m\| = \sup_{x\in\Omega}|u_n(x) - u_m(x)|$. (According to the Pythagorean formula, the L^2-distance between two functions resembles the Euclidean distance between two points in $\mathbb{R}^n$. The sup-norm is an inappropriate distance to deal with the convergence issue.) Hence, $\{u_n\}$ must be contained in the closed and bounded set

$$K = \{v \in \mathcal{A} \ : \ \|v\| \le a\}, \tag{2.3}$$

where $a = \sup_{x\in\partial\Omega}|\phi(x)|$. This leads us to ask the following questions about the compactness properties of K.

- Does every sequence in the set K have a limit in K?
- Does f defined in K attain its minimum in K?

Before the development of Ascoli-Arzelà theorem in the late nineteenth century, the concepts of a sequence of functions being uniformly equicontinuous or uniformly convergent were unknown. (The sequence $\{u_n\}$ has a limit in K provided the gradient ∇u_n is uniformly bounded.) Without knowledge of H^1-norm, it is very difficult to extract a suitable compactness for the function space.

The difficulties facing nineteenth century mathematicians trying to handle rigorously the space of functions led them to study oversimplified models (called the toy models) of the admissible set. They studied the closed interval $[0, a]$ instead of the admissible set K in (2.3). The following basic compactness properties were obtained using the toy models.

- Every continuous function f on the closed bounded interval $[0, a]$ attains its minimum because of its compactness property (Bolzano-Weierstrass [37, 31]).
  - The $[0, a]$ has no hole, that is, every open cover has a finite subcover (Heine-Borel [31]).
  - Every sequence in $[0, a]$ has at least one limit point in K (Fréchet's approach involving nested intersection).

- This compactness concept can be extended to any closed and bounded set in n- dimensional Euclidean space. Basically, any sequence in a closed and bounded domain contains a convergent (Cauchy) subsequence because any compact domain can be divided into a finite number of subdomains whose diameters are less than the half of its diameter.

However, this compactness property cannot be applied to the closed and bounded set K in (2.3) which lies in infinite dimensional space. Indeed, K is not compact, that is, there exist a sequence $\{v_n\}$ in K that does not have a limit in K. Moreover, there exist many examples of bounded sequence $\{v_n\} \subset K$ such that $f(v_n) \to \infty$. (For example, $v_n := \cos n|x|$ is bounded by 1, but $f(v_n) = \int_\Omega |\nabla \cos n|x||^2 dx \to \infty$.)

Hilbert restated Dirichlet's principle in 1900, and by the following year he published two papers that effectively revived it. His statement was based on the hypothesis that both the boundary data and the boundary $\partial\Omega$ are analytic. Nevertheless, Hadamard criticized the instability of his statement since his approach cannot be extended to the case of only continuous boundary data ϕ. To be precise, let Ω be the unit disc and ϕ the boundary data in Hadamard's example (2.2). There exist a sequence of analytic functions $\{\phi_n\}$ such that $\phi_n \to \phi$ in L^∞ sense. Clearly, ϕ_n has an extension $\widetilde{\phi}_n \in C^2(\overline{\Omega})$ to the domain Ω. Writing $g_n = \Delta\widetilde{\phi}_n$, the Dirichlet problem is changed into the following Poisson's equation:

$$-\Delta(u_n - \widetilde{\phi}_n) = g_n \quad \text{in } \Omega \quad \text{with} \quad (u_n - \widetilde{\phi}_n)|_{\partial\Omega} = 0 \tag{2.4}$$

The corresponding energy functional is

$$f(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega g_n v$$

within the set $\{v \in C^2(\overline{\Omega}) \ : \ v|_{\partial\Omega} = 0\}$. According to Hilbert's statement, there exists a solution $u_n - \widetilde{\phi}_n$ to Poisson's equation which is a minimizer of the corresponding energy functional with the source term g_n. This solution can be written as

$$u_n - \widetilde{\phi}_n = \Delta^{-1} g_n.$$

Since Δ^{-1} is a compact operator in the Sobolev space $H^{-1}(\Omega)$, the sequence u_n seems to have a limit as far as g_n is bounded in $H^{-1}(\Omega)$. Due to Ascoli-Arzelà theorem, boundedness of g_n leads to the uniform equicontinuity of the sequence $\{u_n - \widetilde{\phi}_n\}$. However, in the example of Hadamard, ϕ does not have an extension to $H^1(\Omega)$, and therefore g_n can not be bounded in $H^{-1}(\Omega)$. A more complete proof requires the condition $\phi \in H^{1/2}(\partial\Omega)$ in order to have $\|\widetilde{\phi}_n - \widetilde{\phi}\|_{H^1(\Omega)} \to 0$, whereas the Sobolev spaces were not introduced yet until that period.

Before ending this section, let us mention the mathematical term *well-posedness* stemed from Hadamard [22]. Mathematical models of physical phenomena require to meet the following three properties: (1) Existence: at least one solution exists. (2) Uniqueness: only one solution exists. (3) Continuity or stability: a solution depends continuously on the data.

3. Sobolev compact embeddings and compensated compactness

Sobolev spaces, named for the Russian mathematician Sergei Lvovich Sobolev [34], constitute one of the main function spaces used in the study of PDEs. The space $W^{k,p}(\Omega)$ when $k \in \mathbb{N}$ and $p \geq 1$ is the space of $L^p(\Omega)$ functions for which any distributional derivative of the order α is in $L^p(\Omega)$ for any multi-index α such that $|\alpha| \leq k$. When endowed with the norm

$$||u||_{W^{k,p}} = (\sum_{|\alpha|\leq k} ||\partial^\alpha u||^p_{L^p})^{\frac{1}{p}},$$

they are Banach spaces, and in the special case $p = 2$ where $W^{k,2}(\Omega) = H^k(\Omega)$, it is an Hilbert space endowed with the scalar product

$$\langle u, v\rangle_{H^k} = \sum_{|\alpha|\leq k} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2}.$$

When considering minimizing sequences in the resolution of a PDE, belonging to a Sobolev space, some compactness properties are required for the convergence of these sequences. Using the space $W^{1,p}(\Omega)$, with $1 < p < +\infty$ being reflexive, we can thus extract from any bounded sequence a weak convergent subsequence.

By definition, weak convergence is the convergence of a sequence to some limit that holds through the dual space when multiplying in an appropriate way by any element of the dual space. For Sobolev spaces, the dual space may be difficult to identify with another function space. Therefore, we consider the following characterization of weak convergence: a sequence u_n converges weakly to u in $W^{1,p}(\Omega)$, $1 < p < +\infty$, if and only if $u_n \rightharpoonup u$ weakly in $L^p(\Omega)$ and all the partial derivatives $\frac{\partial u_n}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i}$ weakly in $L^p(\Omega)$. Note that the weak convergence in $L^p(\Omega)$ is clearly defined via Riesz theorem, because we have that the dual of $L^p(\Omega)$ is $L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We also notice analogous weak convergence in the case $p = +\infty$, thus adding a star, due to $L^1(\Omega)$ being a strict subspace of the dual of $L^\infty(\Omega)$, which gives the weak$\star$ convergence of a sequence u_n to u in $L^\infty(\Omega)$, denoted as $u_n \overset{\star}{\rightharpoonup} u$, if and only if

$$\int_\Omega u_n\varphi \to \int_\Omega u\varphi$$

for every $\varphi \in L^1(\Omega)$. The case of bounded sequences in $W^{1,1}(\Omega)$ will be evoked in the following sections when discussing the behavior of energies with linear growth.

Moreover, the following Sobolev compact embeddings are given by the Rellich-Kondrachov theorem [30, 28], which says that when Ω is a bounded regular domain in $\mathbb{R}^N$ then the embeddings of $W^{1,p}(\Omega)$ into $L^q(\Omega)$ with $1 \leq q < \frac{NP}{N-p}$ for $1 \leq p < N$ or $1 \leq q < +\infty$ for $p = N$, are compact. When $p > N$, then $W^{1,p}(\Omega)$ is compactly embedded in $C^{0,\gamma}(\overline{\Omega})$, the space of Hölderian functions of order $0 \leq \gamma < 1 - \frac{N}{p}$. The compact embedding meaning that any bounded sequence in $W^{1,p}(\Omega)$ will contain a convergent subsequence in the larger space.

As a consequence of the Rellich-Kondrachov theorem in the case $p = 2$, if a sequence u_n is bounded in $H^1(\Omega)$ converging weakly to some u in $L^2(\Omega)$ and if v_n is bounded in $L^2(\Omega)$

converging weakly to some v in $L^2(\Omega)$, then the product $u_n v_n$ converges to uv weakly in $\mathcal{D}'(\Omega)$, the dual space of $C_c^\infty(\Omega)$, i.e.

$$\int_\Omega u_n v_n \varphi \to \int_\Omega uv\varphi$$

for every $\varphi \in C_c^\infty(\Omega)$. Murat-Tartar theory [29, 36] of compensated compactness permits to obtain the same result when we do not control all the partial derivatives of u_n but controlling some other partial derivatives of v_n in a complementary way. For example, if $u_n, v_n : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ satisfy

$$||u_n||_{L^2(\Omega;\mathbb{R}^2)} + ||\mathrm{div} u_n||_{L^2(\Omega)} \le \alpha \text{ and } ||v_n||_{L^2(\Omega;\mathbb{R}^2)} + ||\mathrm{rot}\, v_n||_{L^2(\Omega;\mathbb{R}^2)} \le \beta$$

for constants $\alpha, \beta > 0$, and if $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $L^2(\Omega;\mathbb{R}^2)$, then the product $u_n v_n$ converges to uv weakly in $\mathcal{D}'(\Omega)$.

The compensated compactness was used also in the study of the convergence of Dirichlet problems with strongly oscillating coefficients, namely the convergence of the solutions of

$$\begin{cases} \mathrm{div}(A(\frac{x}{\varepsilon})\nabla u_\varepsilon(x)) = f \text{ for } x \in \Omega \\ u_\varepsilon = 0 \text{ on } \partial\Omega, \end{cases}$$

when ε goes to zero. For $f \in H^{-1}(\Omega)$ and A verifying some growth and coercivity conditions, we have $\sup_{\varepsilon>0} ||u_\varepsilon||_{H^1(\Omega)} \le ||f||_{H^{-1}(\Omega)}$ and u_ε has a weak-limit. However, $A(\frac{x}{\varepsilon})\nabla u_\varepsilon(x)$ may not have any weak-limit. The compensated compactness theory can be used to get a weak-limit.

4. Stone-Weierstrass, Ascoli-Arzelà and Shannon sampling Theorems

Let us look into the main result behind the Sobolev compact embeddings. Two of the fundamental theorems in analysis are the Stone-Weierstrass Theorem and the Ascoli-Arzelà theorems. These results are very important in the study of sets of continuous functions. The first theorem permits the assertion that the polynomials are dense in the space of the continuous functions and thus enables the possibility to approximate a continuous function with a sequence of polynomial functions that are very regular and easy to handle.

The Stone-Weierstrass [42, 35] theorem states that if $\mathcal{A}$ is an algebra containing the constant function 1 in the space $C(X)$ of continuous functions defined on a compact metric space X, then $\mathcal{A}$ is dense in $C(X)$ if and only if for every $x, y \in X$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. It is then easy to verify that the family of polynomials has the right assumptions to conclude the result mentioned above.

The Ascoli-Arzelà [3, 2] theorem states that a subset of $C(X)$ is relatively compact if and only if it is bounded and equicontinuous. This theorem could be useful in resolving some kind of problems with some compactness properties, which can be approximated by a sequence of problems for which we have more information. The extraction of a convergent subsequence would then give the solution for the initial problem.

It is the case in Péano theorem for the resolution of the Cauchy problem that consists in considering two open domains $I \in \mathbb{R}$ and $\Omega \in \mathbb{R}^N$, $(t_0, y_0) \in I \times \Omega$. Then, using Ascoli-Arzelà compactness theorem, we can find a neighborhood J of t_0 in I, and a function $y : J \to \Omega$ solution of the equation

$$\begin{cases} y'(t) = F(t, y(t)), & t \in J \\ y(t_0) = y_0, \end{cases}$$

where $F : I \times \Omega \to \mathbb{R}^N$ is continuous.

Ascoli-Arzelà theorem is also the basis of the Sobolev compact embeddings. In fact, the case $p > N$ is a direct consequence, while the case $p \leq N$ uses the L^p-version of it which is the Riez-Fréchet-Kolmogorov theorem. This theorem asserts that if a bounded family $\mathcal{F}$ in $L^p(\Omega)$ is such that for an $\omega \subset\subset \Omega$ we have $\forall \varepsilon > 0,\ \exists 0 < \delta < \text{dist}(\omega, \Omega)$ such that $||\tau_h f - f||_{L^p(\omega)} < \varepsilon\ \forall |h| < \delta, f \in \mathcal{F}$, with $\tau_h f(x) = f(x+h)$, then $\mathcal{F}_{|\omega}$ is relatively compact in $L^p(\omega)$. Its proof being in its turn based on the Ascoli-Arzelà theorem.

The proof in Ascoli-Arzelà theorem relies on the fact that every compact metric set contains a countable dense subset. It is on this set that the uniformly continuous limit application and the extracted sequence are constructed. There is, therefore, some discrete-to-continuous procedure behind this compactness theory. It appears also in completely non-related problems where we expect similar ideas of compactness to be hidden.

Sampling theory in signal processing was first derived for communication signal processing in the first half of the twentieth century. The Shannon sampling theorem [33] states that if a continuous L^1 function defined on $\mathbb{R}$ has bounded width B, that is the support of the Fourier transform supp $\hat{u} \subset [-B, B]$ (no frequencies higher than B), u is completely determined by giving its ordinates at a series of points, called a sample, whose points are separated by a distance smaller than $\frac{1}{2B}$ (2B being the sample rate called Nyquist rate). In fact, when the frequencies are less than B, the ordinates of u in points spaced by $\frac{1}{2B}$ gives the Fourier coefficients of $\hat{u}$ which gives $\hat{u}$ itself and thus u.

The set $\mathcal{A} = \{v \in C([0,1])\ :\ \text{supp}\, \hat{v} \subset [-B, B],\ \sup |v| \leq 1\}$ is compact according to Bolzano-Weierstrass and Shannon sampling theorem. Since any function in the set $\mathcal{A}$ can be viewed as a vector in a finite dimensional space with the dimension higher than $\frac{1}{2B}$, $\mathcal{A}$ can be considered as a closed bounded set in the finite dimensional space. The assumption on the bandwidth enable us to control the oscillations of the function preventing them from being too fast (or too high) within a distance smaller than $\frac{1}{2B}$. Exactly the same principle applies to compactness. The equicontinuity assumption in the Ascoli-Arzelà theorem also prevents such behavior of the oscillations. This means that the notion of compactness is also behind the sampling process by identifying infinite-dimensional function spaces as finite dimensional vector spaces when controlling the oscillations or the variations.

5. Compact operator and layer potentials for Laplace equation

In the 1830s, Gauss and Weber [20] began collaborating on the theory of terrestrial magnetism. Weber was a physics professor in Göttingen who wrote important contributions on electricity. They used the harmonic property of magnetic potential to compute the horizontal intensity of the magnetic field; the magnetic potential is dictated by the Laplace equation from

Gauss's law ($\nabla \cdot \mathbf{B} = 0$ for magnetic field $\mathbf{B}$) and divergence-free magnetic induction. They mentioned Dirichlet's principle without proof.

Gauss used a layer potential approach to handle the Dirichlet problem. An electrical potential u in electrostatics can be expressed as a single layer potential

$$\mathcal{S}\rho(\mathbf{r}) = \int_{\partial\Omega} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\, g(\mathbf{r}')\, ds_{\mathbf{r}'} \qquad (\mathbf{r} \in \mathbb{R}^3 \setminus \partial\Omega) \tag{5.1}$$

where ρ represents the distribution of electric charge. Given that $\mathcal{S}\rho$ satisfies $\Delta u = 0$ in $\mathbb{R}^3 \setminus \partial\Omega$, we have

$$\int_{\partial\Omega} \rho\, \mathcal{S}\rho\, ds = \int_{\mathbb{R}^3\setminus\partial\Omega} |\nabla \mathcal{S}\rho|^2.$$

Indeed, the Dirichlet problem with the boundary data ϕ can be reduced to finding ρ such that $\mathcal{S}\rho = \phi$ on $\partial\Omega$. However, solving this integral equation computationally is not simple at all due to compactness of the operator $\mathcal{S}$ on $L^2(\partial\Omega)$.

Let us consider the computational issue of solving the Dirichlet problem. According to the Riemann mapping theorem, for any simply connected open set Ω in complex domain $\mathbb{C}$, there exists a conformal mapping Φ from Ω to the unit disk. Noting that complex analytic functions have their real and imaginary parts obeying the Cauchy-Riemann equations, the two dimensional Dirichlet problem in any simply connected domain ($\nabla^2 u = 0$ in Ω with $u|_{\partial\Omega} = \phi$) can be reduced (through the conformal mapping) to solve the much simpler Dirichlet problem in the unit disk; due to the rectangular geometry of the unit disc in polar coordinates, Fourier analysis and separable variable technique enable us to solve the corresponding Dirichlet problem constructively. At a first glance, it appears that conformal mappings can be effectively used for computing solutions to the two dimensional Dirichlet problem in a simply connected domain. Unfortunately, this is not true when considering the fact that the Poisson kernel (the normal derivative of Green's function or Randon-Nikodym derivative of the harmonic measure with respect to the surface measure over $\partial\Omega$) is very sensitive (and highly nonlinear) to any perturbation of the geometry of $\partial\Omega$. Except in some special cases, it is impossible to find a reliable method of identifying conformal mapping. In short, difficulty in identifying conformal mapping is equivalent in difficulty to finding the Green function.

In the contrast to Gauss's use of single layer potential in solving Dirichlet problem, Neumann used double layer potential:

$$\mathcal{D}\rho(\mathbf{r}) = \int_{\partial\Omega} \frac{\langle \mathbf{r}'-\mathbf{r}, \mathbf{n}(\mathbf{r}')\rangle}{4\pi|\mathbf{r}-\mathbf{r}'|^3}\, \rho(\mathbf{r}')\, dS_{\mathbf{r}'} \qquad (\mathbf{r} \in \mathbb{R}^3 \setminus \partial\Omega) \tag{5.2}$$

The reason is that $\mathcal{D}_\Omega\phi$ satisfies the Laplace equation $\nabla^2 u = 0$ in Ω and solving the Dirichlet problem is reduced to find ϕ such that

$$\rho(\mathbf{r}) = \lim_{t\to 0^+} \mathcal{D}_\Omega \rho(\mathbf{r} - t\mathbf{n}(\mathbf{r})) \;=\; (\frac{1}{2}I + \mathcal{K})\rho(\mathbf{r}), \qquad (\mathbf{r} \in \partial\Omega) \tag{5.3}$$

where I is the identity operator and $\mathcal{K}$ is the trace operator given by

$$\mathcal{K}\rho(\mathbf{r}) = \int_{\partial\Omega} \frac{\langle \mathbf{r}' - \mathbf{r}, \mathbf{n}(\mathbf{r}')\rangle}{4\pi|\mathbf{r} - \mathbf{r}'|^3}\, \rho(\mathbf{r}')\, dS_{\mathbf{r}'} \quad \text{for } \mathbf{r} \in \partial\Omega. \tag{5.4}$$

In the case where Ω is smooth and convex, the solution u can be expressed as Neumann series

$$u(\mathbf{r}) = 2\sum_{n=0}^{\infty} \mathcal{D}\left((-2\mathcal{K})^n \phi\right)(\mathbf{r}) \quad (\mathbf{r} \in \Omega). \tag{5.5}$$

The Calderón-Zygmund school of harmonic analysis [9] made a significant contribution in developing layer potential techniques for solving Dirchlet problem. For the ease of explanation, Ω is assumed to be three dimensional domain with its connected Lipschitz boundary $\partial\Omega$. The boundary value problem of the Laplace equation can be solved by double layer potential.

If the operator $\frac{1}{2}I + \mathcal{K}$ is invertible in a "proper" Banach space, the solution of the Dirichlet problem

$$\nabla^2 u = 0 \quad \text{in } \Omega \qquad \text{with } u|_{\partial\Omega} = \phi$$

can be expressed as

$$u(x) = \mathcal{D}(\frac{1}{2}I + \mathcal{K})^{-1}\phi(x) \qquad \text{in } \Omega.$$

If $\partial\Omega$ is C^1, $\mathcal{K}$ is a compact operator on $L^2(\partial\Omega)$ so that we can apply Fredholm theory[14].

- Since the operator $\mathcal{K} : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is compact, for every bounded sequence $\{\rho_n\}$ in $L^2(\partial\Omega)$, the sequence $\{\mathcal{K}\rho_n\}$ has a limit in $L^2(\partial\Omega)$.
- There exist eigenvalues $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ and the corresponding orthonormal eigenfunctions $\{\rho_n : n = 1, 2, \cdots\}$ of $\mathcal{K}^*\mathcal{K}$ such that
$$\mathcal{K}^*\mathcal{K}\rho = \sum_n \lambda_n^2 \langle \rho, \rho_n\rangle \rho_n.$$
- If the range of $\mathcal{K}$ is infinite dimensional, then $\lim_{n\to\infty} \sigma_n = 0$.
- Hence, $\frac{1}{2}I + \mathcal{K} \approx \frac{1}{2}I$ except a finite dimensional space X_N spanned by eigenfunctions $\{\rho_n : n = 1, 2, \cdots, N\}$ for large N. The invertibility of $\frac{1}{2}I + \mathcal{K}$ on X_N is like invertibility of $N \times N$ matrix.

When $\partial\Omega$ is only in $C^{0,1}$, $\mathcal{K}$ is no more compact so that Fredholm theory can not be applied. Until 1980, the Calderón-Zygmund school worked toward the proof of the boundedness of the trace operator $\mathcal{K}_\Omega$ on $L^2(\partial\Omega)$ which was solved by a deep knowledge on the harmonic analysis [11]. It turns out that the boundedness of $\mathcal{K}$ on $L^2(\Omega)$ is equivalent to $\mathcal{K}1 \in BMO$ [12]. In 1984, the invertibility of $\frac{1}{2}I + \mathcal{K}$ on $L^2(\partial\Omega)$ was proven using Rellich type identity that substitutes the compactness [41].

6. Γ-CONVERGENCE AND COMPACTNESS

Another example where compactness is involved in variational problem is illustrated with the Γ-convergence theory. In the 1970s, Ennio de Giorgi [13] introduced the notion of Γ-convergence, a kind of convergence that was mainly used in problems related to the calculus of variations. This convergence gives the limit of a sequence of minimizing problems instead of computing only the limit of a sequence of functional. It was very useful in treating several variety of problems like relaxation, homogenization, dimension reduction etc.. The limit problem having the advantage of being relaxed which guarantees the existence of solution even when there is no solution for the initial minimizing problems.

We recall that a sequence of functions F_n from a metric space X into $\bar{\mathbb{R}}$ is said to Γ-converge into F for the topology of X if the following two conditions are satisfied for every $x \in X$:

$$\begin{cases} \forall x_n \to x, \ \liminf F_n(x_n) \geq F(x) \\ \exists y_n \to x, \ \lim F_n(y_n) = F(x). \end{cases} \tag{6.1}$$

The Γ-convergence has mainly two compactness properties. The first concerns the set of functions from X into $\bar{\mathbb{R}}$ which have a sequential compactness property with respect to Γ-convergence, in the sense that any sequence $F_n : X \to \bar{\mathbb{R}}$ admits a Γ-convergent subsequence. The Γ-convergence verifies also the Urysohn property of convergence structures, in the sense that a sequence of functions $F_n : X \to \bar{\mathbb{R}}$ Γ-converge to a function F if and only if, every subsequence of F_n contains a further subsequence which Γ-converges to F. In the practice, once we obtain the Γ-limit for the subsequences and verify that it does not depend on the chosen subsequence, we can conclude to the Γ-convergence of the initial sequence.

The second property concerns the minimizers of a sequence $F_n : X \to \bar{\mathbb{R}}$. It states that if the minimizers stay in a compact set of X for all n, then, their limit points are minimizer of the Γ-limit F. The proof of this result is very simple and uses only the definition (6.1).

The computation of the Γ-limit of a sequence of functions consists in two fundamental steps. The first dealing with the lower bound proving the first statement in the definition (6.1) for any sequence $x_n \to x$. The second consists in constructing the recovery sequence y_n appearing in the second statement of the definition (6.1) which permits to have the upper bound for the Γ-limit and thus reaching the result combining the two steps.

In the first step, we consider a sequence $x_n \to x$ in the topology of X and we suppose that $F_n(x_n)$ is bounded otherwise the result is obvious. This boundedness combined with eventual coercivity properties of the functions F_n implies the compactness of the sequence x_n in some function space, thus the convergence of the sequence in some sense that permits to obtain the desired inequality and then the lower bound. Thus, the importance of having some coercivity properties realized by the sequence of functions F_n.

It is worthwhile to mention that in some problems, as in dimension reduction, the Γ-convergence technique replaced the usual method of formal asymptotic expansion which lies on an Ansatz supposing that the deformations has special form. The Γ-convergence procedure does not suppose such an Ansatz and thus gives more rigorous results.

An example underlying the importance of compactness in problems involving Γ-convergence techniques, is about relaxation or dimension reduction for energies of the type

$$E(u) = \int_\Omega f(x, u(x), \nabla u(x))dx$$

with $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^m$, ∇u its gradient, when f has linear growth, i.e. there exist $\alpha, \beta > 0$ such that

$$\alpha||A|| \leq f(x, u, A) \leq \beta(1 + ||A||),$$

for every $(x, u, A) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{m \times N}$. Minimizing sequences with finite energy will be bounded in the Sobolev space $W^{1,1}(\Omega; \mathbb{R}^m)$. However, because of the lack of reflexivity of $W^{1,1}(\Omega; \mathbb{R}^m)$, such minimizing sequences will only be relatively compact in the larger space $BV(\Omega; \mathbb{R}^m)$ of functions with bounded variation firstly envoked by Jordan [26] in the scalar case, then by Tonelli [40] in the vectorial case. It is the space of L^1 functions whose first distributional derivatives are Radon measures. Thus, the limit energy, in both cases of relaxation or dimension reduction, will be decomposed for every $u \in BV(\Omega; \mathbb{R}^m)$ into three parts, according to the Besicovitch decomposition Theorem, written with respect to the three mutually singular measures; $\mathcal{L}^N$, $\mathcal{H}^{N-1}_{|J_u}$ and $D^c u$, where $\mathcal{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$, $\mathcal{H}^{N-1}_{|J_u}$ is the $(N-1)$-dimensional Hausdorff measure restricted to the jump set of u and $D^c u$ the Cantor part of Du. Thus, underlying how much the compactness can change the issue of the result.

7. Conclusion

A broad conclusion of this work is that rigor and intuition are both fundamental to proper mathematical computing, both in teaching and in research. They should never be separated, but they must be used in the right way. As we have seen, the intuition in the work of Riemann led to several important discoveries and results, while the rigorous approach taken by Weierstrass in criticizing Riemann's result led to other interesting studies and results. Weierstrass's meticulous criticism not only identified a wrong result, eventually prompting its correction, but its implications also aided other works, some in unrelated areas. Compactness is one of the most important notions that emerged from the study of the Dirichlet principle and its correction. It has since found applicability in several mathematical fields related to many important problems.

In treating mathematical problems, especially new ones, and when trying to advance on them, the first step should be intuitive. Intuition should provide either the correct result or something closely resembling it. If the result is not perfect, rigorous study should point out its flaws and identify the necessary corrections. This should also be the way in which mathematics is studied. There is often a key point in a long proof of a result. This key point is usually not reached by systematic rigor, but often through intuition. Good intuition is honed by practice, but needs curiosity and imagination. Therefore, it is important for scientists to take an interest in fields not necessarily related to their work. This interest can lend a perspective on the broader context of their work, thus stimulating the innovative ideas required to solve their own difficult problems.

This article, however, should not give a wrong message that rigor is relatively less important than intuition. Mathematical rigor is important tool to destroy wrong intuition where mathematicians have advantage over physicists and engineers. Intuition can be obtained by everlasting effort to try to find connection between mathematical output and relevant physical problems based on rigorous mathematics.

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