

EVALUATION OF CERTAIN ALTERNATING SERIES

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Abstract. Ever since Euler solved the so-called Basler problem of $\zeta(2) = \sum_{n=1}^{\infty} 1/n^2$, numerous evaluations of $\zeta(2n)$ ($n \in \mathbb{N}$) as well as $\zeta(2)$ have been presented. Very recently, Ritelli [61] used a double integral to evaluate $\zeta(2)$. Modifying mainly Ritelli's double integral, here, we aim at evaluating certain interesting alternating series.

The Riemann Zeta function $\zeta(s)$ is defined by (see, *e.g.*, [70, p. 164])

$$(1) \quad \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \Re(s) > 1.$$

The Riemann Zeta function $\zeta(s)$ in (1) plays a central rôle in the applications of complex analysis to number theory. The number-theoretic properties of $\zeta(s)$ are exhibited by the following result known as *Euler's formula*, which gives a relationship between the set of primes and the set of positive integers:

$$(2) \quad \zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad \text{for } \Re(s) > 1,$$

where the product is taken over all primes.

The solution of the so-called *Basler problem* (*cf.*, *e.g.*, [16], [28, p. xxii], Spiess [67, p. 66], Stark [75, pp. 197-198] and Zygmund [90, p. 364]):

$$(3) \quad \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

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was first found in 1735 by Leonhard Euler (1707-1783) [32], although Jakob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748) did their utmost to sum the series in (3). The former of these Bernoulli brothers did not live to see the solution of the problem, and the solution became known to the latter soon after Euler found (see, for details, Knopp [48, p. 238]). Five years later in 1740, Euler (see [33]; see also [34, pp. 137-153]) succeeded in evaluating all of $\zeta(2n)$ ($n \in \mathbb{N} := \{1, 2, 3, \dots\}$):

$$(4) \quad \zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \quad \text{for } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where B_n ($n \in \mathbb{N}_0$) are the n th Bernoulli numbers defined by the following generating function (see, *e.g.*, [70, p. 81]):

$$(5) \quad \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad \text{for } |z| < 2\pi.$$

The following recursion formula

$$(6) \quad B_n = \sum_{k=0}^n \binom{n}{k} B_k \quad (n \in \mathbb{N} \setminus \{1\}) \quad \text{and} \quad B_0 = 1$$

can be used for computing Bernoulli numbers. Ever since Euler first evaluated $\zeta(2)$ and $\zeta(2n)$, numerous interesting solutions of the problem of evaluating the $\zeta(2n)$ ($n \in \mathbb{N}$) have appeared in the mathematical literature. Even though there were certain earlier works which gave a rather long list of papers and books together with some useful comments on the methods of evaluation of $\zeta(2)$ and $\zeta(2n)$ (see, *e.g.*, [9], [25], [45] and [75]), the reader may be referred to the very recent work [19] which contains an extensive literature of as many as more than 70 papers.

Among many different ways to prove (3), several authors have taken advantage of the nice interplay between a double integral and a geometric series (see, *e.g.*, [5, 11, 39]). Very recently, Ritelli [61] chose to use the following double integral

$$(7) \quad \int_0^{\infty} \int_0^{\infty} \frac{dx dy}{(1+y)(1+x^2 y)}$$

to evaluate (3). Indeed, Ritelli [61] evaluated the double integral (7) in two ways by changing the order of integration regarding the two variables x and y and incorporate the two resulting expressions to prove (3).

Modifying the double integral (7) by the following ones

$$(8) \quad \mathcal{I}_p =: \int_0^\infty \int_0^\infty \frac{dx dy}{(1+y)(1+x^2 \sqrt[p]{y})} \quad (p \in \mathbb{N} \setminus \{1\})$$

and evaluating the integrals (8) in two ways by changing the order of integration regarding the two variables x and y , we present (presumably) new formulas for certain interesting alternating series asserted by the following theorem.

Theorem. *Each of the following formulas holds true.*

$$(9) \quad \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{1}{(4n+1)^2} - \frac{1}{(4n+3)^2} \right\} = \frac{\pi^2}{8\sqrt{2}}.$$

$$(10) \quad \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{1}{(8n+1)^2} - \frac{1}{(8n+7)^2} \right\} = \frac{1}{32} \sqrt{5 + \frac{7}{\sqrt{2}}} \pi^2.$$

$$(11) \quad \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{1}{(12n+1)^2} - \frac{1}{(12n+11)^2} \right\} = \frac{1}{72} \sqrt{26 + 15\sqrt{3}} \pi^2.$$

$$(12) \quad \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{1}{(16n+1)^2} - \frac{1}{(16n+15)^2} \right\} \\ = \frac{1}{128} \sqrt{42 + 29\sqrt{2} + \sqrt{3445 + \frac{4871}{\sqrt{2}}}} \pi^2.$$

$$(13) \quad \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{1}{(20n+1)^2} - \frac{1}{(20n+19)^2} \right\} = \frac{\pi^2}{400} \cot\left(\frac{\pi}{20}\right) \csc\left(\frac{\pi}{20}\right).$$

Proof. We will prove only (9) by evaluating \mathcal{I}_2 in (8). Beginning by integrating \mathcal{I}_2 with respect to the variable x , we have

$$\begin{aligned}
\mathcal{I}_2 &= \int_0^{\infty} \left(\frac{1}{1+y} \int_0^{\infty} \frac{dx}{1+x^2\sqrt{y}} \right) dy \\
&= \int_0^{\infty} \left(\frac{1}{\sqrt[4]{y}(1+y)} \arctan(\sqrt[4]{y}x) \Big|_{x=0}^{x=\infty} \right) dy \\
&= \frac{\pi}{2} \int_0^{\infty} \frac{dy}{\sqrt[4]{y}(1+y)}.
\end{aligned}$$

Setting $t = \sqrt[4]{y}$ in the last integral, we have

$$\mathcal{I}_2 = 2\pi \int_0^{\infty} \frac{x^2}{1+x^4} dx = \pi \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx.$$

The last improper integral can be easily evaluated by applying the residue theorem (see, *e.g.*, [14, Chapter]):

$$(14) \quad \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = 2 \int_0^{\infty} \frac{x^2}{1+x^4} dx = \frac{\pi}{\sqrt{2}}.$$

We therefore obtain

$$(15) \quad \mathcal{I}_2 = \frac{\pi^2}{\sqrt{2}}.$$

On the other hand, we start with integrating \mathcal{I}_2 with respect to the variable y :

$$\mathcal{I}_2 = \int_0^{\infty} \mathcal{I}_y dx,$$

where, for convenience,

$$\mathcal{I}_y := \int_0^{\infty} \frac{dx}{(1+y)(1+x^2\sqrt{y})} dy.$$

Setting $y = u^2$ in the integral \mathcal{I}_y , we find

$$\begin{aligned}\mathcal{I}_y &= \int_0^{\infty} \frac{2u}{(1+u^2)(1+x^2u)} du \\ &= \frac{1}{x^4+1} \int_0^{\infty} \left(-2 \frac{x^2}{1+x^2u} + \frac{2u}{1+u^2} + 2x^2 \frac{1}{1+u^2} \right) du\end{aligned}$$

We thus obtain

$$\begin{aligned}\mathcal{I}_y &= \frac{1}{x^4+1} \left[-2 \ln(1+x^2u) + \ln(1+u^2) + 2x^2 \arctan u \right] \Big|_{u=0}^{u=\infty} \\ &= -\frac{4}{x^4+1} \ln x + \frac{\pi x^2}{x^4+1}.\end{aligned}$$

By applying (14) to the following integral

$$(16) \quad \mathcal{I}_2 = \int_0^{\infty} \mathcal{I}_y dx = -4 \int_0^{\infty} \frac{\ln x}{x^4+1} dx + \frac{\pi^2}{2\sqrt{2}}.$$

Equating the two formulas in (15) and (16), we get

$$(17) \quad \int_0^{\infty} \frac{\ln x}{1+x^4} dx = -\frac{\pi^2}{8\sqrt{2}}.$$

Now we find that

$$\begin{aligned}-\frac{\pi^2}{8\sqrt{2}} &= \int_0^1 \frac{\ln x}{1+x^4} dx + \int_1^{\infty} \frac{\ln x}{1+x^4} dx = \int_0^1 \frac{\ln x}{1+x^4} dx - \int_0^1 \frac{x^2 \ln x}{1+x^4} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(4n+1)^2} - \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(4n+3)^2}.\end{aligned}$$

Finally it is easy to see that the last formula proves (9). A similar argument will establish the other formulas in the theorem. \square

Remark. In order to get the formulas in Theorem, including the formula (17), we finally need to obtain the following ones:

$$(18) \quad \int_0^{\infty} \frac{\ln x}{1+x^8} dx = -\frac{1}{32} \sqrt{5 + \frac{7}{\sqrt{2}}} \pi^2.$$

$$(19) \quad \int_0^{\infty} \frac{\ln x}{1+x^{12}} dx = -\frac{1}{72} \sqrt{26+15\sqrt{3}} \pi^2.$$

$$(20) \quad \int_0^{\infty} \frac{\ln x}{1+x^{16}} dx = -\frac{1}{128} \sqrt{42+29\sqrt{2} + \sqrt{3445 + \frac{4871}{\sqrt{2}}}} \pi^2.$$

$$(21) \quad \int_0^{\infty} \frac{\ln x}{1+x^{20}} dx = -\frac{\pi^2}{400} \cot\left(\frac{\pi}{20}\right) \csc\left(\frac{\pi}{20}\right).$$

It is noted that the integral formulas (18) to (21) as well as (17) can be directly evaluated by applying the residue calculus to the functions

$$\frac{\log z}{1+z^{2n}} \quad \left(|z| > 0; -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right),$$

for $n = 2, 4, 6, 8, 10$, on the upper complex plane indented upward at the origin 0 (see, *e.g.*, [14, pp. 280–283]).

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