East Asian Math. J.
Vol. 30 (2014), No. 3, pp. 335–353
http://dx.doi.org/10.7858/eamj.2014.024

MULTIPLE EXISTENCE OF POSITIVE GLOBAL SOLUTIONS FOR PARAMETERIZED NONHOMOGENEOUS ELLIPTIC EQUATIONS INVOLVING CRITICAL EXPONENTS

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Abstract. We establish multiple extence of positive solutions for parameterized nonhomogeneous elliptic equations involving critical Sobolev exponent. The approach to the problem is variational method.

1. Introduction

Let $N \geq 3$ and $2^* := 2N/(N-2)$. Let consider a Hilbert space

$$H^1(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with the inner product

$$(u, v) := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv) dx$$

and the corresponding norm

$$||u|| := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx \right)^{1/2}.$$

Let Ω be an open subset of $\mathbb{R}^N$. The space $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\mathbb{R}^N)$. By $H^{-1}(\Omega)$, we denote its dual with the dual norm $||\cdot||_*$ and, by $\langle , \rangle$, the pairing of $H^1(\mathbb{R}^N)$ with its dual. We denote by $||\cdot||_p$ the usual norm of $L^p(\mathbb{R}^N)$ for $p \in [1, \infty]$.

The space

$$D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with the inner product

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx$$

and the corresponding norm

$$\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{1/2}$$

is also a Hilbert space. The space $D_0^{1,2}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $D^{1,2}(R^N)$. We note that $D^{1,2}(\mathbb{R}^N) = D_0^{1,2}(\mathbb{R}^N)$ and $H_0^1(\Omega) \subset D_0^{1,2}(\Omega)$. And, by the Poincare inequality, $H_0^1(\Omega) =$

Received March 25, 2014; Accepted May 26, 2014.
2010 *Mathematics Subject Classification.* 34B16; 34B18; 34C23; 35J20.
Key words and phrases. elliptic equations; critical exponents; bifurcation; multiplicity; positive solutions; super-sub subsolution; mountain pass lemma.
[1] Corresponding author.
[1] This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2013057580).

(pISSN 1226-6973, eISSN 2287-2833)

$D_0^{1,2}(\Omega)$ if $|\Omega| < \infty$. If $N \geq 3$, then we also have a continuous embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N),\ 2 \leq p \leq 2^*$ and $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)(cf.\ [19])$.

In this paper, we are concerned with the existence of multiple solutions of the following problem:

$$(P_\mu) \qquad \begin{cases} -\Delta u + u = u^{2^*-1} + \mu f \quad \text{in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N,\ N \geq 3 \end{cases}$$

where $\mu \in \mathbb{R}^+$, $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$ and $f \not\equiv 0$ in $\mathbb{R}^N$.

A well-known result for the homoneneous case is that all positive regular solution of

$$-\Delta u = u^{2^{*-1}} = 0$$

in $\mathbb{R}^N$ are given by

$$\omega_\epsilon = \left(\frac{\epsilon\sqrt{N(N-2)}}{\epsilon^2 + |x|^2} \right)^{(N-2)/2}$$

with $\epsilon > 0(cf.\ [10])$. Every ω_ϵ is a minimizer for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Namely, the Sobolev constant

$$S = \inf_{0 \neq u \in D^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}}$$

is achived by ω_ϵ and

$$(1,1) \qquad ||\nabla \omega_\epsilon||_2^2 = ||\omega_\epsilon||_{2^*}^{2^*} = S^{N/2}.$$

For convenience, we omit "$\mathbb{R}^N$" and "dx" in integration and, throughout this paper, we will use the letter $C > 0$ to denote the natural various contents independent of u.

Our attempt to show multiplicity of positive solutions for problem (P_μ) relies on the Ekeland's variational principle in [9] and the Mountain Pass Theorem in [4]. Since our problem (P_μ) posesses the critical nonlinearity and the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact, in taking the opportunity of variational structure of problem, the (PS) condition is no longer valid and so the Mountain Pass Theorem in [1] could not be applied directly. However, we can use the Mountain Pass Theorem *without* the (PS) condition in [4] to get some $(PS)_c$ sequence of the variational functional for the second solution with $c > 0$.

In the last decade, the existence and properties of solutions of the problem:

$$(P_0) \qquad \begin{cases} -\Delta u + u = g(x,u), u > 0 \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N),\ N \geq 2 \end{cases}$$

has been stuied by Struss[18], Lions[16, 17], Ding and Ni[8], Cao[5], Zhu[20]$(cf.$ [15]) and other authors for the case where $g(x,0) = 0$ on $\mathbb{R}^N$ and $g(x,t)$ has a subcritical superlinear growth. On the other hand, the nonhomogeneous problem with $1 < p < 2^* - 1$:

$$(P) \qquad \begin{cases} -\Delta u + u = |u|^{p-2}u + \mu f, u > 0 \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N),\ N \geq 2, \end{cases}$$

where $\mu \in \mathbb{R}^+$, $f \geq 0$, $f \in L^2(\mathbb{R}^N)$ with an exponential decay on $\mathbb{R}^N$, was studied by Zhu[21]$(cf.$ also [11]). In [21], the existence of at least two solutions of (P) was proved was proved for positive functions $f \in L^2(\mathbb{R}^N)$ with a small L^2-norm and exponential decay $f(x) \leq Cexp\{-(1+\epsilon)|x|\}$, for $x \in \mathbb{R}^N$. The multiplicity of positive solutions for problem (P)

for the subcritical case was stuied by Deng and Li[7]. In [12], the existence of at least four solutions of (P) with $N \geq 3$ was established. In the critical case $p = 2^*$, the problem is much more difficult than the subcritical case. As we mentioned, the Palais-Smale condition does not hold at some critical levels and the effect of the nonhomogeneous term f to the multiple existence of solutions is delicate. The multiplicity of the solutions of (P), also (P_μ), depends not only on the norm of f, but also the decay rate and the shape of f. In [6], it has shown that if $N < 6$ and $|x|^{N-2}f$ is bounded, then there exists $\mu^* > 0$ such that problem (P) has at least two positive solutions with $\mu \in (0, \mu^*)$. In case that $N \geq 6$, there exist $\mu^{**},\ \mu_* > 0$ with $\mu_* < \mu^{**}$ such that for each $\mu \in (\mu^{**}, \mu^*)$, problem (P) possesses two positive solutions and for $\mu \in (0, \mu_*)$, problem (P) has a unique solution(See also [7] for subcritical case). For nonhomogeneous case with critical growth nonlinearity, we refer [2]. The effect of the shape of the multiplicity of (P) was investigated in [14]. In [13], the authors consider the multiplicity of solutions of (P) with $-\Delta + I$ replaced by $-\Delta + \alpha I$ and $\alpha > 0$. Authors assume that $p = 2^*$, $3 \leq N \leq 5$, $f \in L^{2^*/(2^*-1)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $f \geq 0$ and $f \not\equiv 0$, and $|x|^{N-2}f$ is bounded. It was shown that there exist μ_* and a function $\alpha : (0, \mu_*) \to \mathbb{R}^+$ such that for each $\alpha \in (0, \alpha(\mu))$, problem (P) posesses at least three solutions; if we assume there exist exactly two positive solutions then the third solution is sign-changing. In our results we do not assume the decay rate on f but uniform boundedness of f which is independent of solution u and $x \in \mathbb{R}^N$. There seems to have been a little progress on existence theory.

We can now state our main results:

PROPOSITION 2.3. *Assume $f \in H^{-1}(\mathbb{R}^N)$, $f(x) \geq 0$, $f(x) \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$, then problem (P_μ) has at least one positive solution u_μ such that*

$$I_\mu(u_\mu) := c_1 = \inf\{I_\mu : u \in \bar{B}_{R_0}\}, \tag{2.1}$$

where $\bar{B}_{R_0} = \{u \in H^1(R^N) : ||u|| \leq R_0\}$.

PROPOSITION 2.5. *Suppose that $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$. Then there exist $\tilde{\mu} \geq \bar{\mu} > 0$ such that (P_μ) possesses a positive solution for $0 < \mu \leq \tilde{\mu}$ and no positive solution for $\mu > \tilde{\mu}$.*

PROPOSITION 3.3. *For $\mu = \mu^*$, the problem (P_μ) has a positive solution u_{μ^*} and $\lambda_1(\mu^*) = 1$. Moreover, the solution u_{μ^*} is unique in $H^1(\mathbb{R}^N)$.*

THEOREM 3.8. *Suppose $3 \leq N \leq 5$. Assume $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$. Then there exists a positive constant $\mu^* > 0$ such that (P_μ) possesses at least two positive solutions for $0 < \mu < \mu^*$, a unique solution for $\mu = \mu^*$ and no positive solution if $\mu > \mu^*$.*

2. Existence of minimal positive solutions

LEMMA 2.1. *The operator $-\Delta + I$ has the maximum principle in $H^1(\mathbb{R}^N)$.*

Proof. Let $h \geq 0$ and $-\Delta u + u = h$. Suppose that $u_- \not\equiv 0$, where $u_+ = \max\{u(x), 0\}$ and $u_- = \min\{u(x), 0\}$. then $0 < \int |\nabla u_-|^2 + |u_-|^2) = \int hu_- dx$ which leads a contradiction. This completes the proof. ∎

In order to get the existence of positive solutions for (P_μ), we consider the energy functional I_μ of the problem (P_μ) defined by

$$I_\mu(u) = \frac{1}{2}\int(|\nabla u|^2 + |u|^2) - \frac{1}{2^*}\int(u^+)^{2^*} - \mu\int fu, \quad \text{for} \quad u \in H^1(\mathbb{R}^N).$$

First, we study the existence of a local mininum for energy functional I_μ and its properities. We denote

$$C_N^* = \frac{1}{2}\left(\frac{4}{N+2}\right)\left(\frac{N}{N+2}\right)^{(N-2)/4} S^{N/4}. \tag{2,1}$$

Lemma 2.2. *Assume $f \in H^{-1}(\mathbb{R}^N)$, $f(x) \geq 0$, $f(x) \not\equiv 0$ and $||\mu f||_* \leq C_N^*$, then there exits a positive const $R_0 > 0$ such that $I_\mu(u) \geq 0$ for any $u \in \partial B_{R_0} = \{u \in H^1(\mathbb{R}^N) : ||u|| = R_0\}$.*

Proof. We consider the function $h(t) : [0, +\infty) \to \mathbb{R}^N$ defined by

$$h(t) = \frac{1}{2}t - \frac{1}{2^*}S^{-2^*/2}t^{2^*-1}.$$

Note that $h(0) = 0$, $2^* - 1 > 1$ and $h(t) \to -\infty$ as t $\to \infty$. We can show easly there a unique $t_0 > 0$ achieving the maxinum of $h(t)$ at t_0. Since

$$h'(t_0) = \frac{1}{2} - \frac{2^*-1}{2^*}S^{-2^*/2}t_0^{2^*-2} = 0,$$

we have

$$t_0 = \left[\frac{2^*}{2(2^*-1)}\right]^{1/(2^*-2)} S^{2^*/2(2^*-2)}.$$

Hence, we have

$$h(t_0) = \frac{1}{2}\left(\frac{4}{N+2}\right)\left(\frac{N}{N+2}\right)^{(N-2)/4} S^{N/4}. \tag{2,2}$$

Taking $R_0 = t_0$, for all $u \in \partial B_{R_0}$,

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int(|\nabla u|^2 + |u|^2) - \frac{1}{2^*}\int(u^+)^{2^*} - \mu\int fu \\ &\geq \frac{1}{2}||u||^2 - \frac{1}{2^*}S^{-2^*/2}||u||^{2^*} - ||\mu f||_*||u|| \\ &= t_0\left[h(t_0) - ||\mu f||_*\right] \end{aligned} \tag{2,3}$$

From $(2,2)$ and $(2,3)$, we have $I_\mu(u)|_{\partial B_{R_0}} \geq 0$. ∎

Proposition 2.3. *Assume $f \in H^{-1}(\mathbb{R}^N)$, $f(x) \geq 0$, $f(x) \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$, then problem (P_μ) has at least one positive solution u_μ such that*

$$I_\mu(u_\mu) := c_1 = \inf\{I_\mu : u \in \bar{B}_{R_0}\}, \tag{2.1}$$

where $\bar{B}_{R_0} = \{u \in H^1(R^N) : ||u|| \leq R_0\}$.

Proof. By Sobolev inequality, the generalized Hölder and Young's inequality with $\epsilon > 0$, there exists $C_\epsilon > 0$, we have

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int(|\nabla u|^2 + |u|^2) - \frac{1}{2^*}\int(u^+)^{2^*} - \mu\int fu \\ &\geq \frac{1}{2}||u||^2 - \frac{1}{2^*}S^{-2^*/2}||u||^{2^*} - ||\mu f||_*||u|| \\ &\geq \left(\frac{1}{2} - \epsilon\right)||u||^2 - \frac{1}{2^*}S^{-2^*/2}||u||^{2^*} - C_\epsilon||\mu f||_*^2. \end{aligned}$$

Taking $\epsilon < \frac{1}{2}$, then, for $R_0 = t_0$ as in Lemma 2,2, we can find a $C_{R_0} > 0$ small enough such that

$$I_\mu(u)|_{\partial B_{R_0}} \geq C_{R_0} \text{ for } ||\mu f||_* \leq C_N^*. \tag{2.2}$$

Since there exists a $\tilde{C}_{R_0} > 0$ such that $|I_\mu(u)| \leq \tilde{C}_{R_0}$ for all $u \in \bar{B}_{R_0}$ and $\bar{B}_{R_0}$ is a complete metric space with respect to the metric $d(u,v) = ||u - v||, u, v \in \bar{B}_{R_0}$, by using the Ekeland's variational principle, from (2.2), we can prove that there exists a sequence $\{u_n\} \subset \bar{B}_{R_0}$ and $u_\mu \in \bar{B}_{R_0}$ such that

$$I_\mu(u_n) \to c_1, \tag{2.3}$$

$$I'_\mu(u_n) \to 0, \tag{2.4}$$

$$\begin{aligned} u_n \to u_\mu \quad &\text{weakly in} \quad H^1(\mathbb{R}^N), \\ u_n \to u_\mu \quad &\text{a.e. in} \quad \mathbb{R}^N, \\ \nabla u_n \to \nabla u_\mu \quad &\text{a.e. in} \quad \mathbb{R}^N \end{aligned} \tag{2.5}$$

and

$$u_n{}^{2^*-1} \to u_\mu{}^{2^*-1} \quad \text{weakly in} \quad \left(L^{2^*}(\mathbb{R}^N)\right)^* \quad \text{as } n \to \infty.$$

Therefore, u_μ is a weak solution of (P_μ). Hence,

$$\langle I'_\mu(u_\mu), \varphi\rangle = 0 \quad \forall\varphi \in H^1(\mathbb{R}^N). \tag{2.6}$$

Moreover, by Lemma 2.1, u_μ is positive on $\mathbb{R}^N$, where I'_μ is the Fréchlet derivative of I_μ.

Next, we are going to prove (2.1). In fact, by the definition of c_1, we know that $I_\mu(u_\mu) \geq c_1$ since $u_\mu \in \bar{B}_{R_0}$, that is,

$$I_\mu(u_\mu) = \frac{1}{2}\int(|\nabla u_\mu|^2 + |u_\mu|^2) - \frac{1}{2^*}\int|u_\mu|^{2^*} - \mu\int fu_\mu \geq c_1 \tag{2.7}$$

By (2.6) and (2.7), we have

$$\left(\frac{1}{2} - \frac{1}{2^*}\right)\int(|\nabla u_\mu|^2 + |u_\mu|^2) - \left(1 - \frac{1}{2^*}\right)\mu\int fu_\mu \geq c_1 \tag{2.8}$$

On the other hand, by (2.3) - (2.5) and Fatou's lemma, we get

$$\begin{aligned} c_1 &= \liminf_n \left(\frac{1}{2} - \frac{1}{2^*}\right)\int(|\nabla u_n|^2 + |u_n|^2) - \limsup_n(1 - \frac{1}{2^*})\mu\int fu_n \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\int(|\nabla u_\mu|^2 + |u_\mu|^2) - \left(1 - \frac{1}{2^*}\right)\mu\int fu_\mu. \end{aligned} \tag{2.9}$$

Thus, (2.7) and (2.9) imply (2.1) holds. This completes the proof. ∎

Remark. *(1)* $c_1 < 0$, *(2)* c_1 *is bounded below, (3)* $||u_\mu|| = o(1)$ *as* $\mu \to 0^+$.

Indeed: (1) For $t > 0$ and $\varphi > 0$, we have

$$I_\mu(t\varphi) = \frac{t^2}{2}\int(|\nabla\varphi|^2 + |\varphi|^2) - \frac{t^{2^*}}{2^*}\int|\varphi|^{2^*} - t\mu\int f\varphi \le \frac{t^2}{2}||\varphi||^2 - t\mu\int f\varphi.$$

By taking $t > 0$ sufficiently small, we can see $c_1 < 0$.

(2) By (2.9) with $\varphi = u_\mu$, and $c_1 = I_\mu(u_\mu)$, we have

$$\begin{aligned} c_1 &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int(|\nabla u_\mu|^2 + |u_\mu|^2) - \left(1 - \frac{1}{2^*}\right)\mu\int f u_\mu \\ &\ge \left(\frac{1}{2} - \frac{1}{2^*}\right)||u_\mu||^2 - \left(1 - \frac{1}{2^*}\right)||\mu f||_*||u_\mu|| \\ &\ge -\frac{1}{22^*}\left[\frac{(2^*-1)^2}{2^*-2}\right]||\mu f||_*^2 \end{aligned} \tag{2.10}$$

by Young's inequality.

(3) Since $c_1 < 0$, from (2.10), we see that $||u_\mu|| \to 0$ as $\mu \to 0^+$. Hence, $||u_\mu|| = o(1)$ as $\mu \to 0^+$. We also have that $||u_\mu||_\mu$ is uniformly bounded with respect to μ. We will restate results relating to this remark in Proposition 3.4 more precisely.

Proposition 2.4. *Problem* (P_μ) *possesses at least one minimal positive solution of* (P_μ).

Proof. Let $\mathcal{N}$ be the Nehari manifold ($cf.$ [19]):

$$\mathcal{N} = \left\{u \in H^1(\mathbb{R}^N) : \int|\nabla u|^2 + |u|^2 = \int|u|^{2^*} + \int\mu f u\right\} \setminus \{0\}.$$

Note that $||\mu f||_* \ll 1$ for μ small enough and for each $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that

$$t_u^2\int|\nabla u|^2 + |u|^2 - t_u^{2^*}\int|u|^{2^*} - t_u\int\mu f u = 0$$

and $I_\mu(t_u u) > 0$. Then

$$\mathcal{N} = \left\{t_u u \ : \ u \in H^1(\mathbb{R}^N) \setminus \{0\}\right\}$$

and

$$\mathcal{N} \cong S^{N-1} = \left\{u \in H^1(\mathbb{R}^N) \ : \ ||u|| = 1\right\}.$$

Hence,

$$H^1(\mathbb{R}^N) = H_1 \cup H_2 \cup \mathcal{N}, \quad H_1 \cap H_2 = \phi \text{ and } 0 \in H_1,$$

where

$$H_1 = \left\{tu \ : \ u \in H^1(\mathbb{R}^N) \setminus \{0\}, \ t \in [0, t_u)\right\}$$
$$H_2 = \left\{tu \ : \ u \in H^1(\mathbb{R}^N) \setminus \{0\}, \ t > t_u\right\}.$$

This implies that for $t > 0$ with $t < t_u$, $tu \in H_1$.

Here, we need to switch our view point, by associating with v a mapping

$$v : [0, \infty[\to H^1(\mathbb{R}^N)$$

defined by

$$[v(t)]x = v(x,t), \quad x \in \mathbb{R}^N, t \in [0, \infty[.$$

In other words, we consider v not as a function of x and t together, but rather as a mapping v of t into the space $H^1(\mathbb{R}^N)$ of functions of x.

We have, for any $v_0 \in H_1$, the solution v of the initial value problem

$$\begin{cases} \dfrac{dv}{dt} - \Delta v + v = \quad v^{2^*-1} + \mu f(x), \\ v(0) = v_0, \end{cases}$$

converges to u_μ as $t \to \infty$,

Indeed, in the proof of Proposition 2.2, we know that $I_\mu(v(t))$ is decreasing and $\lim_{t\to\infty} I_\mu(v(t)) = I_\mu(u_\mu)$, where $I_\mu(u_\mu)$ is the local minimum.
Since

$$\begin{aligned} I_\mu(v(t)) - I_\mu(v(s)) &= \int_s^t \frac{d}{dt} I_\mu(v(t))dt \\ &= \int_s^t \left\langle \frac{d}{dt} v, \nabla I_\mu(v(t)) \right\rangle dt \\ &= -\int_t^s \left\| \frac{d}{dt} v \right\|^2 dt, \end{aligned}$$

we have, $\lim_{s,t\to\infty} \left\|\frac{d}{dt}v\right\|^2 = 0$. Thus, $v' \to 0$ a.e. in $\mathbb{R}^N$ as $t \to \infty$ and hence, $\langle I'_\mu(v), \varphi \rangle \to 0$, $\forall \varphi \in C^\infty(\mathbb{R}^N)$. Therefore, we have $v \to u_\mu$ as $t \to \infty$, since $I_\mu(v(t))$ is decreasing and converges to the local minimum $I_\mu(u_\mu)$.
Now, let $v_0 = tu$, where $t \in (0,1)$ and u is a positive solution. Then $u \in \mathcal{N}$ and $v_0 \in H_1$. Since $v_0 \le u$ and the solution v converges u_μ as $t \to \infty$, by the order preserving principle, $u_\mu \le u$. This completes the proof.■

Remark. We see that minimal solution of (P_μ) is unique from Proposition 2.3 and Proposition 2.4.

PROPOSITION 2.5. *Suppose that $f \in H^{-1}(\mathbb{R}^N)$, $f \ge 0$, $f \not\equiv 0$ and $||\mu f||_* \le C_N^*$. Then there exist $\tilde{\mu} \ge \bar{\mu} > 0$ such that (P_μ) possesses a positive solution for $0 < \mu \le \tilde{\mu}$ and no positive solution for $\mu > \tilde{\mu}$.*

Proof. By Proposition 2.3, (P_μ) has a positive solution if $\mu \le C_N^*/||f||_*$. Suppose (P_μ) has a positive solution $\bar{u}$ for some $\mu = \bar{\mu}$. We show that (P_μ) has a positive solution for any $0 < \mu < \bar{\mu}$. For fixed $0 < \mu < \bar{\mu}$, using the Lax-Milgram Theorem, we construct a positive sequence $\{u_n\}$ as following;

Let

$$-\Delta u_1 + u_1 = \mu f$$

and

$$-\Delta u_n + u_n = u_{n-1}^{2^*-1} + \mu f \quad \text{for} \quad n \ge 2. \tag{2.11}$$

Then, by the maximum principle, we have $0 < u_n < u_{n+1} < \cdots < \bar{u}$ for $n \ge 1$. And $||u_1|| \le \mu ||f||_*$ and $||u_1||_{2^*} \le S^{-1/2}||u_1|| \le S^{-1/2}\mu||f||_*$. Multiplying (2.11) by u_n, we have $||u_n|| \le S^{-2^*/2}||\bar{u}||^{2^*-1} + \mu||f||_*$. Therefore, there exists $\tilde{u}$ in $H^1(\mathbb{R}^N)$ such that

$$u_n \to \tilde{u} \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty,$$

$$u_n \to \tilde{u} \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty,$$

$$\nabla u_n \to \nabla \tilde{u} \text{ a.e. in } \mathrm{R}^{\mathrm{N}},$$

$$u_n^{2^*-1} \to \tilde{u}^{2^*-1} \text{ weakly in } (L^{2^*}(\mathbb{R}^N))^* \text{ as } n \to \infty.$$

Thus, $\tilde{u}$ is a positive solution of (P_μ).

Next, let u be a positive solution of (P_μ). Then, for any $\epsilon > 0$, multiplying (P_μ) by $\omega_\epsilon^{2^*}$, we have

$$-\Delta u\omega_\epsilon^{2^*} + u\omega_\epsilon^{2^*} = u^{2^*-1}\omega_\epsilon^{2^*} + \mu f(x)\omega_\epsilon^{2^*}. \tag{2.12}$$

Since $2^* > 2$, for any $M > 0$, there exists a constant $C > 0$ such that

$$u^{2^*-1} \geq Mu - C \quad \forall u > 0.$$

Hence, we have, from (2.12),

$$-\int \Delta u\omega_\epsilon^{2^*} + \int u\omega_\epsilon^{2^*} \geq \int \left((Mu - C)\omega_\epsilon^{2^*} + \mu f(x)\omega_\epsilon^{2^*})\right).$$

By Green's formular, we have

$$\int \Delta u\omega_\epsilon^{2^*} = \int u\Delta\omega_\epsilon^{2^*}.$$

Thus,

$$\mu \int f(x)\omega_\epsilon^{2^*} \leq C\int \omega_\epsilon^{2^*} + \int \left(1 - M - \frac{\Delta\omega_\epsilon^{2^*}}{w_\epsilon^{2^*}}\right)\omega_\epsilon^{2^*} u. \tag{2.13}$$

Since

$$\begin{aligned}\frac{\Delta w_\epsilon^{2^*}}{\omega_\epsilon^{2^*}} = \frac{\Delta(\epsilon + |x|^2)^{-N}}{(\epsilon + |x|^2)^{-N}} &= 2N(N+1)(\epsilon + |x|^2)^{-2}\left(\frac{N+2}{N+1}|x|^2 - \frac{N}{N+1}\epsilon\right)\\ &= 2N(N+1)(\epsilon + 0^2)^{-2}\left(\frac{N+2}{N+1}0^2 - \frac{N}{N+1}\epsilon\right)\\ &= -2N^2\epsilon^{-1},\end{aligned}$$

we get, from (2.13),

$$\mu \int f(x)\omega_\epsilon^{2^*} \leq C\int \omega_\epsilon^{2^*} + \left(2N^2\epsilon^{-1} + 1 - M\right)\int \omega_\epsilon^{2^*} u.$$

If we choose $M = 2N^2\epsilon + 1$, then, by (1.1), we have

$$\mu \leq \frac{C\omega_\epsilon^{2^*}}{\int f(x)\omega_\epsilon^{2^*}} = \frac{CS^{N/2}}{\int f(x)\omega_\epsilon^{2^*}}.$$

Hence, there exists $\bar{\mu} > 0$ such that

$$\bar{\mu} \leq \tilde{\mu} \doteqdot \inf_{\epsilon>0} \frac{C\int w_\epsilon^{2^*}}{\int f(x)\omega_\epsilon^{2^*}} = \inf_{\epsilon>0} \frac{CS^{N/2}}{\int f(x)\omega_\epsilon^{2^*}}. \tag{2.14}$$

Therefore, if $\mu > \tilde{\mu}$, then (P_μ) has no solution and this completes the proof.∎

3. Multiplicity of positive solutions

From now on, we assume that $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and f satisfies $||\mu f||_* \ll 1$ for μ small enough.

We set

$$\mu^* := \sup\{\mu \in \mathbb{R}^+ : (P_\mu) \text{ has at least one positive solution in } H^1(\mathbb{R}^N)\}.$$

Then, by Proposition 2.5, we have $0 < \bar{\mu} \leq \mu^* < \infty$.

Remark. The minimal solution u_μ of (P_μ) is monotonic increasing with respect to μ. Indeed, suppose $\mu^* > \nu > \mu$. Since

$$-\Delta u_\nu + u_\nu - u_\nu^{2^*-1} - \mu f(x) = (\nu - \mu) f \geq 0,$$

$u_\nu > 0$ is a supersolution of (P_μ). Since $f(x) \geq 0$ and $f(x) \not\equiv 0$, $u \equiv 0$ is a subsolution of (P_μ) for any $\mu > 0$. By the standard barrier method, we can obtain a solution u_μ of (P_μ) such that $0 \leq u_\mu \leq u_\nu$ on $\mathbb{R}^N$. We note that 0 is not a solution of (P_μ), $\nu > \mu$ and u_μ is a minimal solution of (P_μ) since u_μ can be derived by an iteration scheme with initial value $u_{(0)} = 0$. Therefore, by the maximal principle, $0 < u_\mu < u_\nu$ on $\mathbb{R}^N$ which completes the proof.

Now, consider the corresponding eigenvalue problem:

$$(3.1)_\mu \qquad \begin{cases} -\Delta\varphi + \varphi = \lambda(\mu)(2^* - 1) u_\mu^{2^*-2}\varphi, \\ \qquad \varphi \text{ in } H^1(\mathbb{R}^N). \end{cases}$$

Let λ_1 be the first eigenvalue of $(3.1)_\mu$;i.e.,

$$\lambda_1 = \lambda_1(\mu) := \inf\{\int \left(|\nabla\varphi|^2 + |\varphi|^2\right) : \varphi \in H^1(\mathbb{R}^N), (2^* - 1)\int u_\mu^{2^*-2}\varphi^2 dx = 1\}.$$

Then, $0 < \lambda_1 < \infty$ and we can achieve the minimum by some function $\varphi_1 = \varphi_1(\mu) \in H^1(\mathbb{R}^N)$ and $\varphi_1 > 0$ in $\mathbb{R}^N$ if $\mu \in (0, \mu^*)$ $(cf.$ [22]).

We summarize basic properties for $\lambda_1(\mu)$.

LEMMA 3.1. *(1) For $\mu \in (0, \mu^*)$, $\lambda_1(\mu) > 1$;*
(2) If $0 < \mu < \nu \leq \mu^*$, then $\lambda_1(\nu) < \lambda_1(\mu)$;
(3) $\lambda_1(\mu) \to +\infty$ as $\mu \to 0^+$.

Proof. (1) For given $0 < \mu < \nu \leq \mu^*$, every solution u_ν of (P_μ) with $\nu \in (\mu, \mu^*)$ is a supersolution of (P_μ). By Taylor expansion, we have

$$\begin{aligned} -\Delta(u_\nu - u_\mu) + u(u_\nu - u_\mu) &= u_\nu^{2^*-1} - u_\mu^{2^*-1} + (\nu - \mu) f \\ &> (2^* - 1) u_\mu^{2^*-2}(u_\nu - u_\mu) \end{aligned}$$

and moreover, we get

$$\begin{aligned} \int \nabla(u_\nu - u_\mu)\nabla\varphi_1 + \int (u_\nu - u_\mu)\varphi_1 &= \int \left(u_\nu^{2^*-1} - u_\mu^{2^*-1}\right)\varphi_1 + \int (\nu - \mu) f \varphi_1 \\ &> (2^* - 1)\int u_\mu^{2^*-2}(u_\nu - u_\mu)\varphi_1. \end{aligned}$$

Therefore, from $(3.1)_\mu$, we have

$$\int \nabla(u_\nu - u_\mu)\nabla\varphi_1 + \int (u_\nu - u_\mu)\varphi_1 = \lambda_1(\mu)(2^* - 1)\int u_\mu^{2^*-2}(u_\nu - u_\mu)\varphi_1,$$

which implies $\lambda_1(\mu) > 1$.

(2) Since, for $0 < \mu < \nu \leq \mu^*$, $u_\mu < u_\nu$ and

$$\begin{aligned} \lambda_1(\mu)(2^* - 1)\int u_\mu^{2^*-2}\varphi_1(\mu)\varphi_1(\nu) &= \int \nabla\varphi_1(\mu)\nabla\varphi_1(\nu) + \int \varphi_1(\mu)\varphi_1(\nu) \\ &= \lambda_1(\nu)(2^* - 1)\int u_\nu^{2^*-2}\varphi_1(\nu)\varphi_1(\mu), \end{aligned}$$

we have $\lambda_1(\mu) > \lambda_1(\nu)$.

(3) First, we show that $||u_\mu|| \to 0$ as $\mu \to 0^+$. Multiplying (P_μ) by u_μ, we have,

$$\int \left(|\nabla u_\mu|^2 + |u_\mu|^2\right) = \int u_\mu^{2^*} + \int \mu f u_\mu$$

and hence, for $\epsilon > 0$, we have, by Young's inequality with ϵ,

$$\left(1 - \frac{1}{\lambda_1(2^* - 1)} - \frac{\epsilon}{2}\right) ||u_\mu||^2 \le \frac{\mu^2}{2\epsilon} ||f||_*^2 \quad \text{for } \epsilon > 0.$$

Thus, for $\epsilon > 0$ small, we have $||u_\mu|| \le C_\epsilon \mu^2$ for some constant $C_\epsilon > 0$, and hence, $||u_\mu|| = o(1)$ as $\mu \to 0^+$. Next, Multiplying (P_μ) by $\varphi_1(\mu)$, we have, by Hölder's inequality, that

$$\begin{aligned}
\int \left(|\nabla \varphi_1|^2 + |\varphi_1|^2\right) &= \lambda_1(2^* - 1) \int u_\mu^{2^*-2} \varphi_1^2 \\
&\le \lambda_1(2^* - 1) \left(\int u_\mu^{2^*}\right)^{(2^*-2)/2^*} \left(\int \varphi_1^{2^*}\right)^{2/2^*} \\
&\le \lambda_1(2^* - 1) \left(\int u_\mu^{2^*}\right)^{(2^*-2)/2^*} \left(\int |\nabla \varphi_1|^2\right) \\
&\le \lambda_1(2^* - 1) S^{-(2^*-2)/2} ||u_\mu||^{2^*-2} ||\varphi_1||^2
\end{aligned}$$

and thus, $S^{(2^*-2)/2} \le \lambda_1 \cdot (2^* - 1)||u_\mu||^{2^*-2}$. Therefore, we have the desired result. This completes the proof. ∎

LEMMA 3.2. *Let u_μ be a positive solution of $(1.3)_\mu$ for which $\lambda_1(\mu) > 1$. Then, for any $g \in H^1(\mathbb{R}^N)$, the problem:*

$$-\Delta w + w = (2^* - 1) u_\mu^{2^*-2} w + g(x), \quad w \in H^1(\mathbb{R}^N) \tag{3.2}$$

has a solution.

Proof. Consider the functional defined by

$$J(w) = \frac{1}{2} \int \left(|\nabla w|^2 + |w|^2\right) - \frac{1}{2}(2^* - 1) \int u_\mu^{2^*-2} w^2 - \int g w, \quad w \in H^1(\mathbb{R}^N).$$

From Hölder's inequality and Young's inequality, we have, for any $\epsilon > 0$,

$$\begin{aligned}
J(w) &\ge \left(\frac{1}{2} - \frac{1}{2\lambda_1(\mu)}\right) ||w||^2 - \frac{\epsilon}{2} ||w||^2 - \frac{1}{2\epsilon} ||g||_*^2 \\
&= \left(\frac{1}{2} - \frac{1}{2\lambda_1(\mu)} - \frac{\epsilon}{2}\right) ||w||^2 - \frac{1}{2\epsilon} ||g||_*^2
\end{aligned}$$

and hence, for small $\epsilon > 0$, there exist $C_{1,\epsilon} > 0$ and $C_{2,\epsilon} > 0$ such that

$$J(w) \ge C_{1,\epsilon} ||w||^2 - C_{2,\epsilon} ||g||_*^2. \tag{3.3}$$

Let $\{w_n\} \subset H^1(\mathbb{R}^N)$ be the minimizing sequence of variational problem

$$d = \inf\{J(w) | w \in H^1(\mathbb{R}^N)\}.$$

From (3.3), we can also deduce that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. So we may suppose that

$$w_n \to w \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty,$$

$$w_n \to w \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty$$

Here, we also note that

$$\nabla w_n \to \nabla w \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty.$$

And

$$u_n^{2^*-1} \to \tilde{u}^{2^*-1} \text{ weakly in } (L^{2^*}(\mathbb{R}^N))^* \text{ as } n \to \infty.$$

By Fatou's Lemma

$$||w||^2 \le \liminf_{n\to\infty} ||w_n||^2.$$

The weak convergence and the fact that $\int u_\mu^{2^*-2} w_n^2 < \infty$ for $n \ge 1$ imply

$$\lim_{n\to\infty} \int g w_n = \int g w, \quad \lim_{n\to\infty} \int u_\mu^{2^*-2} w_n = \int u_\mu^{2^*-2} w$$

and hence,

$$J(w) \le \lim_{n\to\infty} J(w_n) = d.$$

Then, $J(w) = d$ and w is a minimizer of J. Therefore, w is a critical point of J and w is a solution of (3.2). This completes the proof. ∎

Proposition 3.3. *For $\mu = \mu^*$, the problem (P_μ) has a positive solution u_{μ^*} and $\lambda_1(\mu^*) = 1$. Moreover, the solution u_{μ^*} is unique in $H^1(\mathbb{R}^N)$.*

Proof. For $\mu \in (0, \mu^*)$, multiplying (P_μ) by u_μ, we have, by $(3.1)_\mu$,

$$\begin{aligned}\int \left(|\nabla u_\mu|^2 + |u_\mu|^2\right) &= \int u_\mu^{2^*} + \mu \int f u_\mu \\ &\le \frac{1}{\lambda_1(\mu)(2^*-1)} \int (|\nabla u_\mu|^2 + |u_\mu|^2) + \mu^* ||f||_* ||u_\mu|| \\ &= \left(\frac{1}{\lambda_1(\mu)(2^*-1)} + \frac{\epsilon \mu^*}{2}\right) ||u_\mu||^2 + \frac{\mu^*}{2\epsilon} ||f||_*^2.\end{aligned}$$

By taking $\epsilon > 0$ small enough, there exists an constant $C_\epsilon > 0$ such that $||u_\mu|| \le C_\epsilon$ for all $\mu \in (0, \mu^*)$. Then, there exists u_{μ^*} in $H^1(\mathbb{R}^N)$ such that u_μ monotonically increasing to u_{μ^*} as $\mu \to \mu^*$ and $u_\mu \to u_{\mu^*}$ weakly in $H^1(\mathbb{R}^N)$ as $\mu \to \mu^*$. Hence, u_{μ^*} is a positive solution of (P_μ) with $\mu = \mu^*$. We note that $\lambda_1(\mu)$ is a continuous function of $\mu \in (0, \mu^*]$.

Define $F : \mathbb{R}^1 \times H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ by

$$F(\mu, u) = \Delta u - u + (u^+)^{2^*-1} + \mu f(x).$$

Since $u_\mu \to u_{\mu *}$ weakly as $\mu \to \mu^*$, from Lemma 3.1, $\lambda(\mu^*) \ge 1$. If $\lambda_1(\mu^*) > 1$, then $F_u(\mu^*, u_{\mu^*})\varphi = \Delta\varphi - \varphi + (2^*-1)u_{\mu^*}^{2^*-2}\varphi = 0$ has no nontrivial solution. From Lemma 3.2, $F(\mu^*, u_{\mu^*})$ is an isomorphism of $\mathbb{R}^1 \times H^1(\mathbb{R}^N)$ onto $H^{-1}(\mathbb{R}^N)$, and by the implicitly function theorem to F, we find a neighborhood $(\mu^* - \delta,\ \mu^* + \delta)$ of u^* such that (P_μ) possesses a positive solution if $\mu \in (\mu^* - \delta,\ \mu^* + \delta)$, which contradicts the definition of μ^*. Therefore, $\lambda_1(\mu^*) = 1$.

Suppose U_{μ^*} is a positive solution of (P_{μ^*}). Then $U_{\mu *} \ge u_{\mu^*}$ since u_{μ^*} is minimal. Let $w = U_{\mu^*} - u_{\mu^*}$. Then, since $\lambda_1(\mu^*) = 1$, we have

$$-\Delta w - w \ge (2^*-1) u_{\mu^*}^{2^*-2} w.$$

Let $\varphi_1 = \varphi_1(\mu^*)$ be the eigenfunction of the problem $(3,1)_{\mu^*}$. Then,

$$(2^*-1) \int u_{\mu^*}^{2^*-2} \varphi_1 w = \int \nabla w \nabla \varphi_1 + \int w \varphi_1 \ge (2^*-1) \int u_{\mu^*}^{2^*-1} w \varphi_1$$

and hence, $w \equiv 0$. This completes the proof. ∎

PROPOSITION 3.4. *The minimal solution u_μ of (P_μ) increasing continuously to u_{μ^*} as $\mu \to \mu^*$ and uniformly bounded in $H^1(\mathbb{R}^N)$ for all $\mu \in (0, \mu^*]$. Moreover, $||u_\mu|| \le O(\mu^2)$ as $\mu \to 0^+$.*

Proof. It suffices to find the uniform bound of u_μ. Multiplying (P_μ) by u_μ, we have

$$\int \left(|\nabla u_\mu|^2 + |u_\mu|^2\right) = \int u_\mu^{2^*} + \int \mu f u_\mu$$

and hence, for $\epsilon > 0$, we have

$$\left(1 - \frac{1}{\lambda_1(2^* - 1)} - \frac{\epsilon}{2}\right) ||u_\mu||^2 \le \frac{\mu^2}{2\epsilon} ||f||_*^2 \ \text{ for } \epsilon > 0.$$

Therefore, for $\epsilon > 0$ small, we have $||u_\mu|| \le C_\epsilon \mu^2$ for some constant $C_\epsilon > 0$. Next, fix $\tau \in (0, \mu^*]$. If μ increasing to τ, then, by the first Remark in section 3, u_μ converges monotonically increasing way up to u_τ in $H^1(\mathbb{R}^N)$. If it is not the case, then, by multiplying u_μ on (P_μ) again, we have

$$||u_\mu||^2 \le \left\langle u_\tau^{2^*-1} u_\mu \right\rangle + \tau \left\langle f, u_\mu \right\rangle$$

and so

$$||u_\mu|| \le C S^{-(2^*-1)/2} ||u_\tau||^{2^*-1} + \tau ||f||_*$$

for some $C > 0$. Hence, there exists a sequence $\{u_{\mu_j}\}$ in $H^1(\mathbb{R}^N)$ conversing weakly to a solution $\tilde{u}$ of (P_τ). Then, by the maximum principle, $u_{\mu_j} \le \tilde{u} < u_\tau$ which leads a contradiction to the minimality of u_τ. This completes the proof. ∎

Next, we are going to find the second solution. In order to get another positive solution of (P_μ), we consider the following problem:

$$(Q_\mu) \qquad \begin{cases} -\Delta v + v = (v + u_\mu)^{2^*-1} - u_\mu^{2^*-1} & \text{in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N), \ v > 0 \text{ in } \mathbb{R}^N \end{cases}$$

and the corresponding variational functional:

$$J_\mu(v) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int |v|^2 - \frac{1}{2^*}\int [(v^+ + u_\mu)^{2^*} - u_\mu^{2^*} - 2^* u_\mu^{2^*-1} v^+]$$

for $v \in H^1(\mathbb{R}^N)$.

Clearly, we can have another positive solution $U_\mu = u_\mu + v_\mu$ if we show the problem $(Q)_\mu$ possesses a positive solution v_μ. We look for a critical point of J_μ which is a weak solution of (Q_μ) by employing standard argument of the Mountain Pass method without the (PS) condition.

We set

$$\psi_\epsilon(x) = \varphi(x) w_\epsilon(x), \tag{3.5}$$

where $\varphi(x) \in C_c^\infty(\mathbb{R}^N)$ is a cut off function and w_ϵ as in (1.1). Because u_μ is the critical point of $I_\mu(u)$, we can prove that

$$J_\mu(v) = K_\mu(v) - K_\mu(0) = I_\mu(v) - I_\mu(u_\mu), \tag{3.6}$$

where, for $v \in H^1(\mathbb{R}^N)$,

$$K_\mu(v) = \frac{1}{2}\int (|\nabla(v + u_\mu)|^2 + (v + u_\mu)^2 - \frac{1}{2}\int (v^+ + u_\mu) - \mu \int f(x)(v + u_\mu).$$

By using the following estimations in [4], we know

$$||\nabla\psi_\epsilon||_2^2 = S^{N/2} + O(\epsilon^{(N-2)/2}), \tag{3.7}$$

$$||\psi_\epsilon||_{2^*}^{2^*} = S^{N/2} + O(\epsilon^{N^2/(2N-2)}), \tag{3.8}$$

$$||\psi_\epsilon||_2^2 = \begin{cases} C_1\epsilon + O(\epsilon^{(N-2)/2}), & \text{for} \quad N \geq 5, \\ C_1\epsilon|\ln\epsilon| + O(\epsilon^{(N-2)/2}), & \text{for} \quad N = 4, \\ O(\epsilon^{1/2}), & \text{for} \quad N = 3, \end{cases} \tag{3.9}$$

where C_1 is a positive constant independent of ϵ.

Lemma 3.5. *Let $v \in H^1(\mathbb{R}^N) \setminus \{0\}, v \geq 0$.*
(1) For sufficiently small $\epsilon > 0$, there exist $\rho > 0$, $\alpha > 0$ such that

$$J_\mu(v)|_{\partial B_\rho} \geq \alpha > 0, \ \textit{and}$$

(2) For $t > 0$,

$$J_\mu(tv) \to -\infty \quad \textit{as} \quad t \to \infty.$$

Proof. (1) Let $v \in H^1(\mathbb{R}^N) \setminus \{0\},\ v \geq 0$ Then, for $\epsilon > 0$, by Young's inequality,

$$\begin{aligned} J_\mu(v) &= \frac{1}{2}\int \left(|\nabla v|^2 + |v|^2\right) - \int\int_0^{v^+} [(u_\mu + s)^{2^*-1} - u_\mu^{2^*-1}] \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)\int \left(|\nabla v|^2 + |v|^2\right) - \\ &\qquad - \int\int_0^{v^+} [(u_\mu + s)^{2^*-1} - u_\mu^{2^*-1} - (2^*-1)u_\mu^{2^*-2}s] \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)\int \left(|\nabla v|^2 + |v|^2\right) - \int\int_0^{v^+} [\epsilon u_\mu^{2^*-2}s + C_\epsilon s^{2^*-1}] \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)||v||^2 - \frac{\epsilon}{2}\int u_\mu^{2^*-2}(v^+)^2 - \frac{C_\epsilon}{2^*+1}\int (v^+)^{2^*} \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1} - \frac{\epsilon}{2(2^*-1)\lambda_1}\right)||v||^2 - \frac{C_\epsilon}{2^*}S^{-2^*/2}||v||^{2^*} \end{aligned}$$

for some constant $C_\epsilon > 0$. Hence, for sufficiently small $\epsilon > 0$, there exist $\rho > 0, \alpha > 0$ such that

$$J_\mu(v)|_{\partial B_\rho} \geq \alpha > 0,$$

where $B_\rho = \{u \in H^1(\mathbb{R}^N) : ||u|| < \rho\}$.

(2) Let $v \in H^1(\mathbb{R}^N),\ v \geq 0$ and $v \not\equiv 0$, then, for $t > 0$, we have

$$\begin{aligned} J_\mu(tv) &= \frac{t^2}{2}\int (|\nabla v|^2 + |v|^2) - \frac{1}{2^*}\int \left((u_\mu + tv)^{2^*} - u_\mu^{2^*} - 2^* u_\mu^{2^*-1} tv\right) \\ &\leq \frac{t^2}{2}\int (|\nabla v|^2 + |v|^2) - \frac{t^{2^*}}{2^*}\int |v|^{2^*} \\ &\leq \frac{t^2}{2}||v||^2 - \frac{t^{2^*}}{2^*}||v||_{2^*}^{2^*}. \end{aligned}$$

Therefore, we deduce

$$J_\mu(tv) \to -\infty$$

as $t \to \infty$. This completes the proof. ■

LEMMA 3.6. *Suppose $3 \le N \le 6$. Then there exists some constant $t_\epsilon > 0$, $0 < k_1 \le t_\epsilon \le k_2 < +\infty$ such that $\sup_{t\ge 0} J_\mu(t\psi_\epsilon) = J_\mu(t_\epsilon\psi_\epsilon)$ and*

$$J_\mu(t_\epsilon\psi_\epsilon) \le \frac{1}{N}S^{N/2} - mk_1^{2^*-1}\int_{B_{2\eta}} \psi_\epsilon^{2^*-1} + \begin{cases} O(\epsilon), & \text{for } N \ge 5, \\ O(\epsilon|\ln\epsilon|), & \text{for } N = 4, \\ O(\epsilon^{1/2}), & \text{for } N = 3, \end{cases}$$

where $\mu \in (0, \mu^*)$ and $m = \inf\{u_\mu(x) | x \in B_{2\eta}\} > 0$. Moreover,

$$J_\mu(t_\epsilon\psi_\epsilon) < \frac{1}{N}S^{N/2}.$$

Proof. By Lemma 3.5 and the fact $3 \le N \le 6$, we can easely show that there exist $t_\epsilon > 0$ such that $J_\mu(t_\epsilon\psi_\epsilon) = \sup_{t\ge 0} J_\mu(t\psi_\epsilon)$, we claim that there exist some constants $k_1 > 0$, $k_2 > 0$ such that $0 < k_1 \le t_\epsilon \le k_2 < +\infty$. In fact, since

$$J_\mu(t_\epsilon\psi_\epsilon) = \sup_{t\ge 0} J_\mu(t\psi_\epsilon),$$

$$\frac{dJ_\mu(t\psi_\epsilon)}{dt}|_{t=t_\epsilon} = 0, t_\epsilon > 0 \text{ and}$$

$$\int |\nabla\psi_\epsilon|^2 + |\psi_\epsilon|^2 = \int [[(t_\epsilon\psi_\epsilon + u_\mu)^{2^*-1} - u_\mu^{2*-1})]/t_\epsilon]\psi_\epsilon.$$

Therefore, we have

$$\text{(3.10)} \qquad \frac{||\nabla\psi_\epsilon||_2^2 + ||\psi_\epsilon||_2^2}{||\psi_\epsilon||_{2^*}^{2^*}} - t_\epsilon^{2^*-2} = \frac{\int [[(t_\epsilon\psi_\epsilon + u_\mu)^{2^*-1} - u_\mu^{2^*-1} - (t_\epsilon\psi_\epsilon)^{2^*-1}/t_\epsilon]\psi_\epsilon]}{||\psi_\epsilon||_{2^*}^{2^*}} \ge 0$$

From (3.7) - (3.9), we have

$$t_\epsilon^{2^*-2} \le \frac{||\nabla\psi_\epsilon||_2^2 + ||\psi_\epsilon||_2}{||\psi_\epsilon||_{2^*}^{2^*}} \le c_2 < +\infty$$

for ϵ small enough, and thus $t_\epsilon \le k_2$ for some $k_2 > 0$.

On the other hand, it is easy to check that

$$\lim_{u\to\infty} \frac{(u + u_\mu)^{2^*-1} - u_\mu^{2^*-1} - u^{2^*-1}}{u^{2^*-1}} = 0.$$

Put $u = t_\epsilon\psi_\epsilon$. Then for any $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\begin{aligned}
&\int \frac{(t_\epsilon\psi_\epsilon + u_\mu)^{2^*-1} - u_\mu^{2*-1} - (t_\epsilon\psi_\epsilon)^{2^*-1}}{||t_\epsilon\psi_\epsilon||_{2^*}^{2^*}} \\
&= [||\psi_\epsilon||_{2^*}^{2^*}]^{-1} \int \frac{[(t_\epsilon\psi_\epsilon + u_\mu)^{2^*-1} - u_\mu^{2^*-1} - (t_\epsilon\psi_\epsilon)^{2^*-1}]\psi_\epsilon}{t_\epsilon} \\
&= [||\psi_\epsilon||_{2^*}^{2^*}]^{-1} \int \frac{(\delta t_\epsilon^{2^*-1}\psi_\epsilon^{2^*-1} + t_\epsilon C_\delta\psi_\epsilon)\psi_\epsilon}{t_\epsilon} \\
&\le [||\psi_\epsilon||_{2^*}^{2^*}]^{-1}[\delta t_\epsilon^{2^*-2}||\psi_\epsilon||_{2^*}^{2^*} + C_\delta||\psi_\epsilon||_2^2] \\
&= \delta t_\epsilon^{2^*-2} + O(\epsilon^{1/2}).
\end{aligned}$$

Again, by (3.7) - (3.10),

$$1 - t_\epsilon^{2^*-2} \le ||\psi_\epsilon||_{2^*}^{2^*} \int [[(t_\epsilon\psi_\epsilon) + u_\mu)^{2^*-1} - u_\mu^{2^*-1} - (t_\epsilon\psi_\epsilon)^{2^*-1}]/t_\epsilon]\psi_\epsilon$$
$$\le \delta t_\epsilon^{2^*-2} + O(\epsilon^{1/2}),$$

and thus, we have

$$1 - t_\epsilon^{2^*-2} - \delta t_\epsilon^{2^*-2} + O(\epsilon^{1/2}) \le 0.$$

Choosing δ, ϵ small enough, we find a constant $k_1 > 0$ such that $t_\epsilon \ge k_1$. Moreover, from the definition of J_μ and the inequality:

$$(v + u_\mu)^p - u_\mu^p - v^p \ge p u_\mu v^{p-1} \quad \text{for every } v \ge 0,\ p > 2,$$

we have

$$J_\mu(v) = \frac{1}{2}\int (|\nabla v|^2 + v^2) - \frac{1}{2}\int ((v^+ + u_\mu)^{2^*} - u_\mu^{2^*} - 2^* u_\mu^{2^*-1} v)$$
$$\le \frac{1}{2}\int (|\nabla v|^2 + v^2) - \frac{1}{2^*}\int v^{2^*} - 2^* u_\mu v^{2^*-1}.$$

Hence,

$$J_\mu(t_\epsilon\psi_\epsilon) = \frac{t_\epsilon^2}{2}\int (|\nabla\psi_\epsilon|^2 + |\psi_\epsilon|^2) - \frac{1}{2^*}\int (t_\epsilon\psi_\epsilon)^{2^*} + 2^* u_\mu (t\psi_\epsilon)^{2^*-2}$$
$$= \frac{t_\epsilon^2}{2}(||\nabla\psi_\epsilon||_2^2 + ||\psi_\epsilon||_2^2) - \frac{t_\epsilon^{2^*}}{2^*}||\psi_\epsilon||_{2^*}^{2^*} - 2^* t^{2^*-2}\int u_\mu \psi_\epsilon^{2^*-2}$$
$$\le \left(\frac{t_\epsilon^2}{2} - \frac{t_\epsilon^{2^*}}{2^*}\right)||\nabla\psi_\epsilon||_2^2 + \frac{t_\epsilon^2}{2}||\psi_\epsilon||_2^2 - 2^* t^{2^*-2}\int_{B_{2\eta}} u_\mu \psi_\epsilon^{2^*-2}.$$

From (3.7) - (3.9), we have

$$J_\mu(t\psi_\epsilon) \le \frac{1}{N}S^{N/2} + O(\epsilon^{(N-2)/2}) + \begin{cases} K_1\epsilon + O(\epsilon^{(N-2)/2}) & \text{for } N \ge 5,\\ K_1\epsilon|\ln\epsilon| + O(\epsilon^{(N-2)/2}) & \text{for } N = 4,\\ O(\epsilon^{1/2}) & \text{for } N = 3, \end{cases}$$
$$- 2^* t^{2^*-1}\int_{B_{2\eta}} u_\mu \psi_\epsilon^{2^*-1}$$
$$\le \frac{1}{N}S^{N/2} - 2^* t^{2^*-1}\int_{B_{2\eta}} u_\mu \psi_\epsilon^{2^*-1}$$
$$+ \begin{cases} O(\epsilon), & \text{for } N \ge 5,\\ O(\epsilon|\ln\epsilon|), & \text{for } N = 4,\\ O(\epsilon^{1/2}), & \text{for } N = 3. \end{cases}$$

And, we have: for $N = 5$,

$$\lim_{\epsilon\to 0^+} \epsilon^{-1}\int_{B_{2\eta}} \psi_\epsilon^{2^*-1}$$
$$\ge \lim_{\epsilon\to 0^+} \epsilon^{-1}\int_{B_\eta} \psi_\epsilon^{2^*-1}$$
$$= \lim_{\epsilon\to 0^+} (N(N-2))^{(N+2)/4}\alpha(N)\epsilon^{-1}\int_0^{\eta\epsilon^{-1/2}} \left(\frac{\epsilon^{-(N-2)/4}}{(1+z^2)^{(N-2)/2}}\right)^{2^*-1} \epsilon^{N/2}\xi^{N-1}dz$$
$$= \lim_{\epsilon\to 0^+} \epsilon^{(N-6)/4}\int_0^{\eta\epsilon^{-1/2}} \alpha(N)\left(\frac{1}{1+z^2}\right)^{(N+2)/2} \xi^{N-1}dz \to \infty,$$

where $\xi = r\epsilon^{-1/2}$, $r = |x|$ and $\alpha(N)$ denote the area of unit sphere, and for $N = 4$,

$$\lim_{\epsilon\to 0^+} \epsilon^{-1}|\ln^\epsilon|^{-1} \int_{B_{2\eta}} \psi_\epsilon^{2^*-1}$$
$$\geq \lim_{\epsilon\to 0^+} \epsilon^{-1}|\ln^\epsilon|^{-1} \int_{B_\eta} \psi_\epsilon^{2^*-1}$$
$$= \lim_{\epsilon\to 0^+} (N(N-2))^{(N+2)/4}\rho(N)\epsilon^{-1}|\ln^\epsilon|^{-1} \int_0^{\eta|\ln^\epsilon|} \left(\frac{\epsilon^{-(N-2)/4}}{(1+z^2)^{(N-2)/2}}\right)^{2^*-1} \epsilon^{N/2}\xi^{N-1}dz$$
$$= \lim_{\epsilon\to 0^+} \epsilon^{(N-6)/4}|\ln^\epsilon|^N \int_0^{\eta|\ln^\epsilon|} \rho(N)\left(\frac{1}{1+z^2}\right)^{(N+2)/2} r^{N-1}dz \to \infty,$$

where $\xi = r|\ln^\epsilon|$, $r = |x|$ and $\rho(N)$ denote the area of unit sphere, and
for $N = 3$,

$$\lim_{\epsilon\to 0^+} \epsilon^{-1}\epsilon^{1/2} \int_{B_{2\eta}} \psi_\epsilon^{2^*-1}$$
$$\geq \lim_{\epsilon\to 0^+} \epsilon^{-1/2} \int_{B_\eta} \psi_\epsilon^{2^*-1}$$
$$= \lim_{\epsilon\to 0^+} (N(N-2))^{(N+2)/4}\alpha(N)\epsilon^{-1/2} \int_0^{\eta\epsilon^{-1/2}} \left(\frac{\epsilon^{-(N-2)/4}}{(1+z^2)^{(N-2)/2}}\right)^{2^*-1} \epsilon^{N/2}\xi^{N-1}dz$$
$$= \lim_{\epsilon\to 0^+} \epsilon^{(N-4)/4} \int_0^{\eta\epsilon^{-1/2}} \alpha(N)\left(\frac{1}{1+z^2}\right)^{(N+2)/2} \xi^{N-1}dz \to \infty,$$

where $\xi = r\epsilon^{-1/2}$, $r = |x|$ and $\alpha(N)$ denote the area of unit sphere. Consequently, we deduce

$$J_\mu(t_\epsilon\psi_\epsilon) < \frac{1}{N}S^{N/2}.$$

This completes the proof.■

Theorem 3.7. *Suppose $3 \leq N \leq 5$. Then the problem (P_μ) possesses at least two positive solutions for all $\mu \in (0, \mu^*)$.*

Proof. Let

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], H^1); \gamma(0) = 0,\ \gamma(1) = t_\epsilon\psi_\epsilon\}$$

and

$$c_\mu = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} J_\mu(\gamma(s)).$$

Then, we have, from Lemma 3.6,

$$0 < \alpha \leq c_\mu \leq \sup_{t\geq 0} J_\mu(t_\epsilon\psi_\epsilon) < \frac{1}{N}S^{N/2}. \tag{3.11}$$

We now applying the Mountain Pass Theorem without Palais-Smale condition in [4] to get a subsequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ such that

$$J_\mu(v_n) \to c_\mu, \quad J'_\mu(v_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N). \tag{3.12}$$

Since

$$
\begin{aligned}
1+c_\mu+||v_n||+||u_\mu|| &\geq 1+c_\mu+||v_n+u_\mu|| \\
&\geq J_\mu(v_n)-\frac{1}{2^*}J'_\mu(v_n)(v_n^++u_\mu) \\
&\geq \left(\frac{1}{2}-\frac{1}{2^*}\right)||v_n||^2-\frac{2}{2^*}||v_n||||u_\mu||-\left(1-\frac{1}{2^*}\right)||u_\mu||_{2^*}^{2^*},
\end{aligned}
$$

we see that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence, there exists a subsequence $\{v_n\}$ such that

$$v_n \to v_\mu \text{ weakly in } H^1(\mathbb{R}^N),$$

$$v_n \to v_\mu \text{ a.e. in } \mathbb{R}^N,$$

$$\nabla v_n \to \nabla v_\mu \text{ a.e. in } \mathbb{R}^N,$$

and

$$(v_n+u_\mu)^{2^*-1}-u_\mu^{2^*-1} \to (v^++u_\mu)^{2^*-1}-u_\mu^{2^*-1} \text{ weakly in } (L^{2^*}(\mathbb{R}^N))^*.$$

Then v_μ is a weak solution of $-\Delta v+v=(v^++u_\mu)^{2^*-1}-u_\mu^{2^*-1}$.

Using the maximal principle, we get $v_\mu \geq 0$ in $\mathbb{R}^N$. Set $u_n=v_n+u_\mu$, $u=v_\mu+u_\mu$. Then

$$u_n \to u \text{ weakly in } H^1(\mathbb{R}^N),$$

$$u_n \to u \text{ a.e. in } \mathbb{R}^N,$$

$$\nabla u_n \to \nabla u \text{ a.e. in } \mathbb{R}^N.$$

From (3.6),

$$J_\mu(v_n)=K_\mu(v_n)-K_\mu(0)=I_\mu(v_n)-I_\mu(u_\mu)\to c_\mu \text{ as } n\to\infty \tag{3.13}$$

and u is a solution of

$$-\Delta u+u=u^{2^*}+\mu f(x). \tag{3.14}$$

Now, we are going to show that $u \not\equiv u_\mu$. In fact, if $u\equiv u_\mu$, i.e., $v_\mu\equiv 0$, then $u_n \not\to u$ strongly in $H^1(\mathbb{R}^N)$, since $J_\mu(0)=0<u_\mu$. Let $c_1=c_\mu+I_\mu(u_\mu)$. By the Brezis-Lieb Lemma(*cf.* [3]) we have

$$
\begin{cases}
||u_n||^2=||u_\mu||^2+||v_n||^2+o(1), \\
|u_\mu^+|^{2^*}=|u_\mu|^{2^*}+|v_\mu^+|^{2^*}+o(1), \\
\displaystyle\int fu_n=\int fu_n+o(1) \text{ as } n\to\infty.
\end{cases} \tag{3.15}
$$

By (3.13), (3.14), we have

$$\int|\nabla u_\mu|^2+u_\mu^2=\int(u_\mu^+)^{2^*}+\mu\int f(x)u_\mu+o(1),$$

$$\int|\nabla u_\mu|^2+u_\mu^2=\int(u_\mu^+)^{2^*}+\mu\int f(x)u_\mu.$$

Hence,

$$\int|\nabla v_n|^2+v_n^2=\int(v_n^+)^{2^*}+o(1),$$

by substracting the two identities above and by (3.15).

Using (3.13), (3.14), (3.15) and (3.16), we have that, as $n\to\infty$

$$\begin{aligned} c_1 &= c_\mu + I_\mu(u_\mu) \\ &= J_\mu(v_n) + I_\mu(u_\mu) + o(1) \\ &= I_\mu(u_n) + o(1) \\ &= I_\mu(u_\mu) + \frac{1}{2}\int |\nabla v_\mu|^2 + v_\mu^2 - \frac{1}{2^*}\int v_n^{2^*} + o(1) \\ &= I_\mu(u_\mu) + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int (v_n)^2 + o(1) \\ &= I_\mu(u_\mu) + \frac{1}{N}\int (v_n)^{2^*} + o(1). \end{aligned}$$

By Sobolev inequality ($cf.$ [4], [7], [6]):

$$S||v_n||_{2^*}^2 \leq ||v_n||^2 = ||v_n||_{2^*}^{2^*} + o(1),$$

we have $||w_n||_{2^*}^{2^*} \geq S^{N/2}$. Thus,

$$c_1 = c_\mu + I_\mu(u_\mu) \geq I_\mu(u_\mu) + \frac{1}{N}S^{N/2}(cf.\ ).$$

This leads a contradiction to (3.11). Therefore, we have $v_\mu > 0$. This completes the proof.∎

Consequently, we have

THEOREM 3.8. *Suppose $3 \leq N \leq 5$. Assume $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$. Then there exists a positive constant $\mu^* > 0$ such that (P_μ) possesses at least two positive solutions for $0 < \mu < \mu^*$, a unique solution for $\mu = \mu^*$ and no positive solution if $\mu > \mu^*$.*

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