East Asian Math. J.
Vol. 30 (2014), No. 3, pp. 327–334
http://dx.doi.org/10.7858/eamj.2014.023

EXTRAPOLATED EXPANDED MIXED FINITE ELEMENT APPROXIMATIONS OF SEMILINEAR SOBOLEV EQUATIONS

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Abstract. In this paper, we construct extrapolated expanded mixed finite element approximations to approximate the scalar unknown, its gradient and its flux of semilinear Sobolev equations. To avoid the difficulty of solving the system of nonlinear equations, we use an extrapolated technique in our construction of the approximations. Some numerical examples are used to show the efficiency of our schemes.

1. Introduction

In this paper, we consider the following semilinear Sobolev equation

$$u_t - (u_x + u_{tx})_x = f(x,t,u), \qquad \text{in } (0,L)\times(0,T], \tag{1.1}$$

$$u_x(0,t) + u_{tx}(0,t) = u_x(L,t) + u_{tx}(L,t) = 0, \quad \text{in } (0,T], \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad \text{in } (0,L), \tag{1.3}$$

where $0 < T < \infty$ and $u_0(x)$ and $f(x,t,u)$ are sufficiently smooth functions so that (1.1)-(1.3) has a sufficiently smooth unique solution. The problem (1.1)-(1.3) arises in the various areas, for examples, in the flow of fluids through fissured materials [1] and thermodynamics [4]. For details about the physical significance and the existence and uniqueness of the solutions of the Sobolev equations, we refer to [1, 2, 3, 4, 6, 7, 12].

The advantages of mixed finite element formulations can be summarized as follows: one can simultaneously approximate both the displacement and the flux(or the stress or the pressure) and one can approximate the flux to the same order of convergence as the unknown scalar $u(x,t)$. Due to these advantages, the mixed finite element method has been applied to some types of Sobolev equations by the authors in [8, 10, 11].

Received February 25, 2014; Accepted May 16, 2014.
2010 *Mathematics Subject Classification.* 65M15, 65N30.
Key words and phrases. Semilinear Sobolev equation, expanded mixed finite element approximation, extrapolated technique.
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* This work was supported by a Research Grant of Pukyong National University(2013).

(pISSN 1226-6973, eISSN 2287-2833)

However the mixed finite element method is not useful to approximate the gradient from the flux because the flux term in (1.1)-(1.3) contains the mixed derivative with respect to the spatial and temporal variables. Therefore an expanded mixed finite element method generalizes the mixed finite element method in the sense that the unknown scalar, its gradient, and its flux are separately treated so that three variables can be approximated directly. In the several literatures such as [5,9], the authors tried to apply an expanded mixed finite element method to approximate the three variables corresponding to elliptic equations and semilinear reaction-diffusion equations.

In this paper, we construct two types of extrapolated expanded mixed finite element approximations of u, $-u_x$ and $-u_x - u_{tx}$, respectively, of (1.1)-(1.3) based on the backward Euler method and the Crank-Nicolson method. To avoid the difficulty of solving the system of nonlinear equations, we use an extrapolated technique in our constructions of the approximations. In section 2, we construct the extrapolated mixed finite element approximations of u, $-u_x$ and $-u_x - u_{tx}$, respectively, of (1.1)-(1.3) and some numerical experiments are given in section 3 to show the effectiveness of our proposed schemes.

2. Expanded mixed finite element approximations

To construct expanded mixed finite approximations for the scalar unknown, its gradient, and its flux, we let $\lambda = -u_x$, $\sigma = -(u_x + u_{tx}) = \lambda + \lambda_t$, $V = L^2(0, L)$, $\Lambda = L^2(0, L)$, and $W = H_0^1(0, L) = \{v \in H^1(0, L) : v(0) = v(L) = 0\}$. Then the weak form of (1.1)-(1.3) is given as follows: find a triple $(u(\cdot, t), \lambda(\cdot, t), \sigma(\cdot, t)) \in V \times \Lambda \times W$ such that

$$(\lambda, w) - (u, w_x) = 0, \qquad \forall\, w \in W, \tag{2.1}$$

$$(\lambda, \mu) + (\lambda_t, \mu) - (\sigma, \mu) = 0, \qquad \forall\, \mu \in \Lambda, \tag{2.2}$$

$$(u_t, v) + (\sigma_x, v) = (f(u), v), \qquad \forall\, v \in V, \tag{2.3}$$

where $u(x, 0) = u_0(x)$ and $\lambda(x, 0) = -(u_0)_x(x)$. Letting V_h, Λ_h, and W_h be the finite element subspaces of V, Λ, and W, respectively, semidiscrete expanded mixed finite element approximations of (2.1)-(2.3) are defined as follows: find a triple $(u_h(\cdot, t), \lambda_h(\cdot, t), \sigma_h(\cdot, t)) \in V_h \times \Lambda_h \times W_h$ such that

$$(\lambda_h, w) - (u_h, w_x) = 0, \qquad \forall w \in W_h, \tag{2.4}$$

$$(\lambda_h, \mu) + ((\lambda_h)_t, \mu) - (\sigma_h, \mu) = 0, \qquad \forall \mu \in \Lambda_h, \tag{2.5}$$

$$((u_h)_t, v) + ((\sigma_h)_x, v) = (f(u_h), v), \qquad \forall v \in V_h, \tag{2.6}$$

where

$$(u_h(\cdot, 0), v) = (u_0(x), v), \qquad \forall v \in V_h, \tag{2.7}$$

and

$$(\lambda_h(\cdot, 0), \mu) = (-(u_0)_x(x), \mu), \qquad \forall \mu \in \Lambda_h. \tag{2.8}$$

Now, to construct fully discrete expanded mixed finite element approximations of u, λ and σ, respectively, we let $\Delta t = T/M$, $t^n = n\Delta t$ for $n = 0, 1, \cdots,$ M and $t^{n,\theta} = \alpha_1 t^n + \alpha_2 t^{n-1}$ with $\alpha_1 = (1+\theta)/2$ and $\alpha_2 = (1-\theta)/2$ where M is a given positive integer and $\theta = 1$ or 0. Notice that $\theta = 1$ and $\theta = 0$ correspond to the backward Euler method and the Crank-Nicolson method, respectively, which are used to approximate the temporal derivative. Then the fully discrete expanded mixed finite element approximations of (2.1)-(2.3) are defined as follows: find $(U^n, \Lambda^n, \Sigma_\theta^n) \in V_h \times \Lambda_h \times W_h$, $U^n \cong u(t^n)$, $\Lambda^n \cong \lambda(t^n)$, $\Sigma_\theta^n \cong \sigma(t^{n,\theta})$, $n = 1, 2, \cdots, M$ such that

$$(\Lambda^n, w) - (U^n, w_x) = 0, \qquad \forall\, w \in W_h, \tag{2.9}$$

$$(\Lambda^{n,\theta} + \partial_t \Lambda^n, \mu) - (\Sigma_\theta^n, \mu) = 0, \qquad \forall\, \mu \in \Lambda_h, \tag{2.10}$$

$$(\partial_t U^n, v) + (\Sigma_{\theta\, x}^n, v) = (f(x, t^{n,\theta}, U^{n,\theta}), v), \quad \forall\, v \in V_h, \tag{2.11}$$

where

$$(U^0, v) = (u_0(x), v), \qquad \forall v \in V_h, \tag{2.12}$$

$$(\Lambda^0, \mu) = (-(u_0)_x(x), \mu), \qquad \forall \mu \in \Lambda_h, \tag{2.13}$$

and

$$U^{n,\theta} = \alpha_1 U^n + \alpha_2 U^{n-1}, \Lambda^{n,\theta} = \alpha_1 \Lambda^n + \alpha_2 \Lambda^{n-1}, \partial_t \Lambda^n = \frac{\Lambda^n - \Lambda^{n-1}}{\Delta t}.$$

To avoid the difficulty of solving the system of nonlinear equations (2.9)-(2.11), we use the extrapolated value $EU^n = \beta_1 U^{n-1} + \beta_2 U^{n-2}$ in (2.11) instead of $U^{n,\theta}$ for $n = 2, 3, \cdots, M$, where $\beta_1 = (3+\theta)/2$ and $\beta_2 = (-1-\theta)/2$. Therefore the extrapolated fully discrete expanded mixed finite element approximations of (2.1)-(2.3) are defined as follows: find $(U^n, \Lambda^n, \Sigma_\theta^n) \in V_h \times \Lambda_h \times W_h$, $U^n \cong u(t^n)$, $\Lambda^n \cong \lambda(t^n)$, $\Sigma_\theta^n \cong \sigma(t^{n,\theta})$, $n = 1, 2, \cdots, M$ such that

$$(\Lambda^1, w) - (U^1, w_x) = 0, \qquad \forall\, w \in W_h, \tag{2.14}$$

$$(\Lambda^{1,\theta} + \partial_t \Lambda^1, \mu) - (\Sigma^1, \mu) = 0, \qquad \forall\, \mu \in \Lambda_h, \tag{2.15}$$

$$(\partial_t U^1, v) + (\Sigma_x^1, v) = (f(x, t^{1,\theta}, U^{1,\theta}), v), \quad \forall\, v \in V_h, \tag{2.16}$$

and for $n = 2, 3, \cdots, M$

$$(\Lambda^n, w) - (U^n, w_x) = 0, \qquad \forall\, w \in W_h, \tag{2.17}$$

$$(\Lambda^{n,\theta} + \partial_t \Lambda^n, \mu) - (\Sigma_\theta^n, \mu) = 0, \qquad \forall\, \mu \in \Lambda_h, \tag{2.18}$$

$$(\partial_t U^n, v) + (\Sigma_{\theta\, x}^n, v) = (f(x, t^{n,\theta}), EU^n, v), \quad \forall\, v \in V_h, \tag{2.19}$$

where

$$(U^0, v) = (u_0(x), v), \qquad \forall v \in V_h, \tag{2.20}$$

$$(\Lambda^0, \mu) = (-(u_0)_x(x), \mu), \qquad \forall \mu \in \Lambda_h. \tag{2.21}$$

To solve the system of nonlinear equations (2.14)-(2.16), we use the following iterative technique: with $U^{(0)} = U^0$ and $\Lambda^{(0)} = \Lambda^0$, we solve the following linear systems: for $k = 0, 1, 2, \cdots$

(2.22)
$$(\Lambda^{(k+1)}, w) - (U^{(k+1)}, w_x) = 0, \qquad \forall w \in W_h,$$

(2.23)
$$(c_1\Lambda^{(k+1)}, \mu) - (\Sigma^{(k+1)}, \mu) = (c_2\Lambda^0, \mu), \qquad \forall \mu \in \Lambda_h,$$

(2.24)
$$d(U^{(k+1)}, v) + (\Sigma_x^{(k+1)}, v) = d(U^0, v) + (f(x, t^{1,\theta}, D^k), v), \ \ \forall v \in V_h,$$

where $c_1 = \alpha_1 + d$, $c_1 = -\alpha_2 + d$, $d = (1/\Delta t)$ and $D^k = \alpha_1 U^{(k)} + \alpha_2 U^0$.

3. Numerical examples and results

In this section, we consider the following semilinear Sobolev equation

(3.1) $$u_t - u_{xx} - u_{txx} = f(x, t, u), \qquad \text{in } (0,1) \times (0,1],$$

(3.2) $$u_x(0,t) + u_{tx}(0,t) = u_x(1,t) + u_{tx}(1,t) = 0, \qquad \text{in } (0,1],$$

(3.3) $$u(x,0) = u_0(x), \qquad \text{in } (0,1).$$

We set the function $f(x,t,u) = u + u^2 + 2e^t(2x-1) - e^{2t}(\frac{1}{2}x^2 - \frac{1}{3}x^3)^2$ and the initial function $u_0(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$ so that the exact solution can be given by $u(x,y,t) = e^t(\frac{1}{2}x^2 - \frac{1}{3}x^3)$. We divide the interval $[0,1]$ into N subintervals of equal length $h = 1/N$ by the grid points x_i such that $0 = x_0 < x_1 < \cdots < x_N = 1$. For a given subinterval (x_{i-1}, x_i), let $P_r(x_{i-1}, x_i)$ be the set of all polynomials of total degree less than or equal to r on the interval (x_{i-1}, x_i). Let V_h, Λ_h, and W_h be the finite element subspaces of V, Λ, and W, respectively, such that

$$V_h = \{v \in V : v \in P_{r_1}(x_{i-1}, x_i), \quad \forall i = 1, 2, \cdots, N\}$$
$$\Lambda_h = \{\mu \in \Lambda : \mu \in P_{r_2}(x_{i-1}, x_i), \quad \forall i = 1, 2, \cdots, N\}$$
$$W_h = \{w \in W : w \in P_{r_3}(x_{i-1}, x_i), \quad \forall i = 1, 2, \cdots, N\}$$

for given integers $r_1, r_2 \geq 0$ and $r_3 \geq 1$. Notice that V_h and Λ_h are the spaces of piecewise polynomials and W_h is the space of continuous piecewise polynomials. For the sake of our computations, we take the possible least integers $r_1 = 0$, $r_2 = 0$, and $r_3 = 1$.

To show the effectiveness of our proposed schemes, we first take $\theta = 1$ in the extrapolated fully discrete expanded mixed finite element approximations of (2.1)-(2.3), corresponding to the backward Euler method, to approximate the temporal derivative. In Figure 1, we plot the graphs of the exact and approximate values of $u(\cdot, 1.0)$, $\lambda(\cdot, 1.0)$, and $\sigma(\cdot, 1.0)$, respectively, when $h = \Delta t = 0.05$. In Figure 1, the green, blue, and black colored solid lines represent the exact values of $u(\cdot, 1.0)$, $\lambda(\cdot, 1.0)$, and $\sigma(\cdot, 1.0)$, and the red colored dotted lines represent their approximate values, respectively. And to show the rate of convergence in L^2 norm for the errors of the approximate values, we present our numerical results in Tables 1-3 for $h = \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}, \frac{1}{640}$, i.e., $N = 20,40,80,160, 320, 640$ and $\Delta t = h$. From Tables 1-3, we know that the approximate values of $u, -u_x$ and $-(u_x + u_{xt})$ converge with the convergence order 1 in L^2 norm.

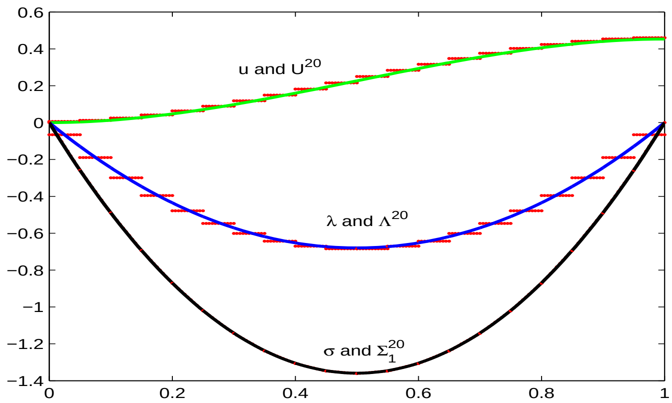

FIGURE 1. The graphs of the exact and approximate values of $u(\cdot, 1.0)$, $\lambda(\cdot, 1.0)$, and $\sigma(\cdot, 1.0)$, respectively, when $\theta = 1$ and $h = \Delta t = 0.05$.

$h = \Delta t$	$\|u(\cdot, 1.0) - U(\cdot, 1.0)\|$	convergence order
1/20	0.009657820019237	
1/40	0.004965059700427	0.960
1/80	0.002516496316038	0.980
1/160	0.001266664553715	0.990
1/320	6.354230505154782e-04	0.995
1/640	3.182323169651559e-04	0.998

TABLE 1. The rate of convergence for the approximate values of $u(\cdot, 1.0)$ when $h = \Delta t$.

Next, we take $\theta = 0$ in the extrapolated fully discrete expanded mixed finite element approximations of (2.1)-(2.3), corresponding to the Crank-Nicolson method, to approximate the temporal derivative. In Figure 2, we plot the graphs of the exact and approximate values of $u(\cdot, 1.0)$, $\lambda(\cdot, 1.0)$, and $\sigma(\cdot, t^{30,0})$, respectively, when $h = \Delta t = 1/30$. In Figure 2, the green, blue, and black colored solid lines represent the exact values of $u(\cdot, 1.0)$, $\lambda(\cdot, 1.0)$, and $\sigma(\cdot, t^{30,0})$,

$h = \Delta t$	$\|\lambda(\cdot, 1.0) - \Lambda(\cdot, 1.0)\|$	convergence order
1/20	0.023160942630080	
1/40	0.011644491186366	0.992
1/80	0.005838139762464	0.996
1/160	0.002923025012886	0.998
1/320	0.001462498701971	0.999
1/640	7.314955586062670e-04	1.000

TABLE 2. The rate of convergence for the approximate values of $\lambda(\cdot, 1.0)$ when $h = \Delta t$.

$h = \Delta t$	$\|\sigma(\cdot, 1.0) - \Sigma_1(\cdot, 1.0)\|$	convergence order
1/20	0.001472578856528	
1/40	5.268272512336551e-04	1.483
1/80	3.564958486136229e-04	0.563
1/160	2.112144554112193e-04	0.755
1/320	1.144881553739165e-04	0.884
1/640	5.952318343515723e-05	0.944

TABLE 3. The rate of convergence for the approximate values of $\sigma(\cdot, 1.0)$ when $h = \Delta t$.

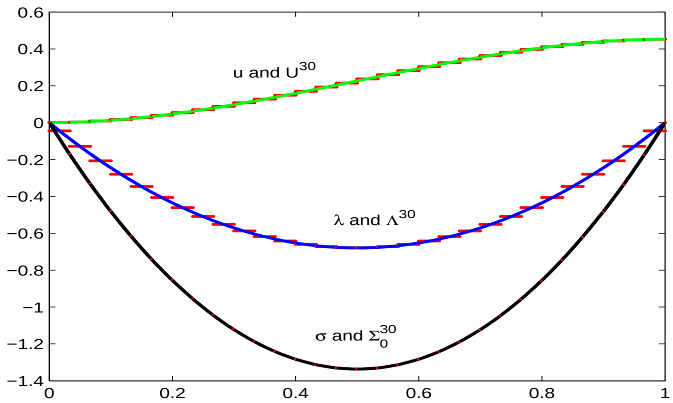

FIGURE 2. The graphs of the exact and approximate values of $u(\cdot, 1.0)$, $\lambda(\cdot, 1.0)$, and $\sigma(\cdot, t^{30,0})$, respectively, when $\theta = 0$, $t^{30,0} \approx 0.983333333333333$, and $h = \Delta t = 1/30$.

and the red colored dotted lines represent their approximate values, respectively. And to show the rate of convergence in L^2 norm for the errors of the approximate values, we present our numerical results in Tables 4-6 for $h = \frac{1}{10}, \frac{1}{30}, \frac{1}{90}, \frac{1}{270}, \frac{1}{810}$, i.e., $N = 10, 30, 90, 270, 810$ and $\Delta t = h$. From Tables 4-6, we know that the approximate values of u and $-u_x$ converge with the convergence order 1 in L^2 norm but the approximate values of $-(u_x + u_{xt})$ converge with the convergence order 2 in L^2 norm.

$h = \Delta t$	$\Vert u(\cdot, 1.0) - U(\cdot, 1.0)\Vert$	convergence order
1/10	0.014647128608439	
1/30	0.004788817960887	1.018
1/90	0.001592352182808	1.002
1/270	5.306339006865376e-04	1.000
1/810	1.768723486915579e-04	1.000

TABLE 4. The rate of convergence for the approximate values of $u(\cdot, 1.0)$ when $h = \Delta t$.

$h = \Delta t$	$\Vert \lambda(\cdot, 1.0) - \Lambda(\cdot, 1.0)\Vert$	convergence order
1/10	0.045328151984329	
1/30	0.015102510877604	1.000
1/90	0.005033891000119	1.000
1/270	0.001677953078907	1.000
1/810	5.593172981223941e-04	1.000

TABLE 5. The rate of convergence for the approximate values of $\lambda(\cdot, 1.0)$ when $h = \Delta t$.

$h = \Delta t$	$\Vert \sigma(\cdot, 0.95) - \Sigma_0(\cdot, 0.95)\Vert$	convergence order
1/10	0.009956460444873	
1/30	0.001110868251889	1.996
1/90	1.235636442052294e-04	1.999
1/270	1.373369876939529e-05	2.000
1/810	1.526122470730851e-06	2.000

TABLE 6. The rate of convergence for the approximate values of $\sigma(\cdot, 0.95)$ when $h = \Delta t$.

From Tables 1-6, we expect that our proposed schemes to (2.1)-(2.3), i.e., the problem (1.1), approximate the scalar unknown, its gradient, and its flux, separately and present the convergence of optimal order in L^2 norm. We will prove these results theoretically in the future.

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