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CIRCULANT DECOMPOSITIONS OF CERTAIN MULTIPARTITE GRAPHS INTO GREGARIOUS CYCLES OF A GIVEN LENGTH

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Abstract. It is shown that, for an even positive integer m with $m \geq 4$ and arbitrary positive integer k and t, the complete multipartite graph $K_{km+1(2t)}$ can be decomposed into edge-disjoint gregarious m-cycles in such a way that the decomposition is circulant.

1. Introduction

Decomposition of graphs into various shapes is an interesting area. Especially, decomposition into cycles of a fixed length have been considered in a number of different ways. With a series of developments, conditions for complete graphs of odd order or a complete graph of even order minus an 1-factor to have decomposition into cycles of some fixed length are recently obtained ([1], [11], [12], and the references there). The key fact used for all these works was the decomposition of complete bipartite graphs obtained by Sotteau ([14]). Many people then began considering cycle decompositions of multipartite graphs with special properties, such as resolvable decompositions or gregarious decompositions ([2], [3], [4], [10]).

Throughout the paper, $K_{n(k)}$ will denote the complete multipartite graph with n partite sets, each with k elements. A *gregarious* cycle in a multipartite graph is a cycle whose vertices belong to different partite sets. Thus, a gregarious cycle visits each partite set at most once. A *cycle decomposition* or a *decomposition into cycles* is partition of the edge set of the graph such that the edges in each equivalence class form a cycle in the graph. A cycle decomposition of a multipartite graph is called *circulant* if every circular permutation of the partite sets sends any cycle in the decomposition into another cycle in the decomposition. That is, the decomposition is invariant under all circular permutation of the partite sets. This notion will be clear later.

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Šajna ([12]) showed that $K_{km+1(2)}$ has a decomposition into m-cycles if and only if m divides the number of edges. However, the cycles in the decomposition were not necessarily gregarious. It seems that the requirement of gregariousness make the problem much more complicated and difficult. Billington and Hoffman ([2], [5]) and Cho and et el. ([9], [7]) independently produced gregarious 4-cycle decompositions and gregarious 6-cycle decompositions for certain complete multipartite graphs. More development followed soon ([6], [13], [8]). It should be noted that the decompositions of the author using difference set method are circulant ([6], [7], [9]), while decompositions of others are usually not.

For simplicity, we will say that a graph has a γ_m*-decomposition* if there is a decomposition of the graph by gregarious m-cycles.

In this paper, we will show the following theorem.

Theorem 1.1. *Let k and t be positive integers and m be an even integer with $m \geq 4$. Then the graph $K_{km+1(2t)}$ has a circulant γ_m-decomposition.*

The following theorem is a special case of Theorem 1.1, and this result is very helpful in proving Theorem 1.1.

Theorem 1.2. *Let k be a positive integer and m be an even integer with $m \geq 4$. Then the graph $K_{km+1(2)}$ has a circulant γ_m-decomposition.*

The theorem will be proved in the subsequent sections. However, we prove Theorem 1.1 using Theorem 1.2.

Proof of Theorem 1.1. We adopt the method used in [4] and [9]. Replace each vertex a of $K_{km+1(2)}$ by t new vertices and label them $a_1, a_2, \ldots, a_t$. We now join the vertex a_i to the vertex b_j if ab is an edge in $K_{km+1(2)}$. Obviously, this new graph is $K_{km+1(2t)}$. Now, by Theorem 1.2, we have a circulant γ_m-decomposition Φ of $K_{km+1(2)}$. If $\lambda = \langle a^{(1)}, a^{(2)}, \ldots, a^{(m)} \rangle$ is a gregarious m-cycle in Φ, then

$$\lambda_{ij} = \langle a_i^{(1)}, a_j^{(2)}, a_i^{(3)}, a_j^{(4)}, \ldots, a_i^{(m-1)}, a_j^{(m)} \rangle,$$

for $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, t$, are t^2 edge-disjoint gregarious m-cycles of $K_{km+1(2t)}$. The collection of all such cycles of $K_{km+1(2t)}$ obtained in this way constitutes a circulant γ_m-decomposition of $K_{km+1(2t)}$. □

For certain k, Theorem 1.2 can be obtained by the following result on decomposition of a complete graph.

Lemma 1.3. *([1], [12]) Let n be an odd integer and m be any integer at least 3. Then, K_n has a decomposition into m-cycles if and only if m divides $\frac{n(n-1)}{2}$, the number of edges in K_n.*

Theorem 1.4. *If k is even and $m \geq 4$, then $K_{km+1(2)}$ has a γ_m-decomposition.*

Proof. If k is even, $km+1$ is odd and m divides $\frac{(km+1)km}{2}$. By the preceding lemma, K_{km+1} has a decomposition Φ into gregarious m-cycles. For each $i = 0, 1, \ldots, km$, duplicate the vertex i of K_{km+1} into two vertices i and $\bar{i}$. Then the resulting graph will be a $K_{km+1(2)}$. Now, if $\lambda = \langle i_1, i_2, \ldots, i_m \rangle$ is an gregarious m-cycle in Φ, we produce the following four m-cycles of $K_{km+1(2)}$ from λ:

$$\lambda_1 = \langle i_1, i_2, i_3, i_4, \ldots, i_{m-1}, i_m \rangle, \qquad \lambda_2 = \langle i_1, \overline{i_2}, i_3, \overline{i_4}, \ldots, i_{m-1}, \overline{i_m} \rangle,$$
$$\lambda_3 = \langle \overline{i_1}, i_2, \overline{i_3}, i_4, \ldots, \overline{i_{m-1}}, i_m \rangle, \qquad \lambda_4 = \langle \overline{i_1}, \overline{i_2}, \overline{i_3}, \overline{i_4}, \ldots, \overline{i_{m-1}}, \overline{i_m} \rangle.$$

Clearly, they are disjoint gregarious m-cycle of $K_{km+1(2)}$. The collection of all such gregarious m-cycles obtained from all gregarious m-cycles in Φ is a γ_m-decomposition of $K_{km+1(2)}$. □

However, if k is odd and m is even, then m does not divide $\frac{(km+1)km}{2}$, the number of edges in $K_{km+1(2)}$, and so Lemma 1.2 can not be applied. Thus we need to find a new method. Furthermore, there is no guarantee that the decomposition is circulant.

The method we develop in the subsequent sections can be applied to all cases, regardless of the parity of k. Moreover, in all cases, the decompositions we obtain in this article are circulant.

In section 2, we introduce feasible sequences of differences of numbers in $\mathbb{Z}_{km+1}$ and explain the method of producing gregarious m-cycles from feasible sequences. In section 3, we prove Theorem 1.2 by producing appropriate feasible sequences and generating gregarious m-cycles.

2. Cycles from feasible sequences of differences

From now on, m and k will be positive integers with m even and $m \geq 4$. Let the partite sets of $K_{km+1(2)}$ be $A_0 = \{0, \overline{0}\}$, $A_1 = \{1, \overline{1}\}$, …, and $A_{km} = \{km, \overline{km}\}$. Thus, the numbers in $\mathbb{Z}_{km+1} = \{0, 1, 2, \ldots, km\}$ are used as indices of the partite sets and as vertices of the graph as well. Let $\mathcal{D}_{km+1} = \{\pm 1, \pm 2, \ldots, \pm \frac{km}{2}\}$. Then, $\mathcal{D}_{km+1}$ is a complete set of differences of two distinct numbers in $\mathbb{Z}_{km+1}$. For d with $0 < d \leq \frac{km}{2}$, an edge between a vertex in A_i and a vertex in A_j is called an *edge of distance* d if $|i-j| \equiv d \pmod{km+1}$. For example, the edges $0\overline{4}$, 73, $\overline{7}\,\overline{2}$ and $\overline{8}3$ are all edges of distance 4 in $K_{9(2)}$.

A sequence $\rho = (r_1, r_2, \ldots, r_m)$ of differences in $\mathcal{D}_{km+1}$ is called a *feasible sequence*, or an *f-sequence* for simplicity, if

(i) $\sum_{i=1}^{m} r_i = 0$, i.e., the total sum of the sequence is zero, and
(ii) $\sum_{i=p}^{q} r_i \neq 0$ if $1 \leq p \leq q < m$ or $1 < p \leq q \leq m$, i.e., any proper partial sum of consecutive entries is nonzero,

where all the additions are taken in $\mathbb{Z}_{km+1}$ modulo $km+1$.

Let $\rho = (r_1, r_2, \ldots, r_m)$ be a sequence of differences and $\eta_\rho = (s_0, s_1, s_2, \ldots, s_{m-1})$ be a member of $\mathbb{Z}_{km+1}^m$. Then, η_ρ is called the *sequence of initial sums*, or a *s-sequence* for simplicity, of ρ if $s_0 = 0$ and $s_i = \sum_{j=0}^{i} r_j$ for $i = 1, \ldots, m-1$. Thus, $s_0 = 0$ and $s_i = s_{i-1} + r_i$ for $i = 1, \ldots, m-1$.

In fact, the sequence η_ρ will be a sequence of indices of the partite sets where an m-cycle traverses stating from the first partite set A_0, and the feasibility of ρ guarantees that this m-cycle is proper and gregarious. Now, the following lemma is trivial by definitions.

Lemma 2.1. *Let $\eta_\rho = (0, s_1, s_2, \ldots, s_{m-1})$ be the s-sequence of a sequence $\rho = (r_1, r_2, \ldots, r_m)$. Then ρ is an f-sequence if and only if $\sum_{i=1}^{m} r_i = 0$ and the entries $0, s_1, s_2, \ldots, s_{m-1}$ of η_ρ are mutually distinct.*

Let ϕ^+ and ϕ^- be mappings of $\mathbb{Z}_{km+1}$ into $\bigcup_{i=0}^{km} A_i$ defined by $\phi^+(i) = i$ and $\phi^-(i) = \bar{i}$ for all i in $\mathbb{Z}_{km+1}$. A *flag* is a sequence $\phi^* = (\phi_0, \phi_1, \ldots, \phi_{m-1})$ where $\phi_i = \phi^+$ or ϕ^- for $i = 0, 1, \cdots, m-1$. Given such a flag ϕ^*, we also use the same notation ϕ^* to denote the mapping defined by

$$\phi^*(\eta_\rho) = \phi^*(s_0, s_1, \ldots, s_{m-1}) = \langle \phi_0(s_0), \phi_1(s_1), \ldots, \phi_m(s_{m-1}) \rangle$$

for any s-sequence $\eta_\rho = (s_0, s_1, \ldots, s_{m-1})$ of an f-sequence ρ. Note that $\phi^*(\eta_\rho)$ is a gregarious m-cycle, and each gregarious m-cycle is in this form.

Let Ψ be the set of all gregarious m-cycles in $K_{km+1(2)}$. We define a map $\tau : \Psi \longrightarrow \Psi$ defined by

$$\tau(\phi^*(s_0, s_1, \ldots, s_{m-1})) = \phi^*(s_0+1, s_1+1, \ldots, s_{m-1}+1).$$

Intuitively, τ is the one click rotation, i.e., $\frac{2\pi}{km+1}$ rotation, around the circle consisting of the partite sets. Note that τ^{km+1} is the identity mapping.

A γ_m-decomposition will be called *circulant* if the decomposition is invariant under τ.

Now, if we are given a gregarious m-cycle $\phi^*(\eta_\rho)$, we can produce a class $\{\tau^k(\phi^*(\eta_\rho)) \mid k \in \mathbb{Z}_{km+1}\}$ consisting of $km+1$ gregarious m-cycles. For example, if $\eta_\rho = (s_0, s_1, s_2, \ldots, s_{m-1})$ and $\phi^* = (\phi^+, \phi^-, \phi^-, \cdots, \phi^+)$, then the following gregarious m-cycles will be produced.

$$\begin{array}{rclccccc}
\tau^0(\phi^*(\eta_\rho)) & = & \langle\, 0, & \overline{s_1}, & \overline{s_2}, & \ldots, & s_{m-1} \,\rangle, \\
\tau^1(\phi^*(\eta_\rho)) & = & \langle\, 1, & \overline{s_1+1}, & \overline{s_2+1}, & \ldots, & s_{m-1}+1 \,\rangle, \\
\tau^2(\phi^*(\eta_\rho)) & = & \langle\, 2, & \overline{s_1+2}, & \overline{s_2+2}, & \ldots, & s_{m-1}+2 \,\rangle, \\
\vdots & \vdots & & & \vdots & & \\
\tau^{km-1}(\phi^*(\eta_\rho)) & = & \langle\, km-1, & \overline{s_1+km-1}, & \overline{s_2+km-1}, & \ldots, & s_{m-1}+km-1 \,\rangle, \\
\tau^{km}(\phi^*(\eta_\rho)) & = & \langle\, km, & \overline{s_1+km}, & \overline{s_2+km}, & \ldots, & s_{m-1}+km \,\rangle.
\end{array}$$

Note that every column on the right-hand side has one vertex from every partite set. Thus, each edge of the form $p\,\overline{q}$ appears as the first edge of a gregarious m-cycle above if $q - p = r_1 = s_1$. Each edge of the form $\overline{p}\,\overline{q}$ appears as the second edge of a gregarious m-cycle above if $q - p = r_2 = s_2 - s_1$. Similarly, we find edges of distances r_3, r_4, and so on. The last edges of the above gregarious m-cycles are of the form $p\,q$ with $q - p = r_m$.

This procedure explains the method which will be used to obtain a circulant γ_m-decomposition of $K_{km+1(2)}$. Then, the problem is how to choose appropriate f-sequences and flags so that the gregarious m-cycles obtained as above constitute a circulant γ_m-decomposition.

3. The Proof of Theorem 1.2.

The number of edges in $K_{km+1(2)}$ is $2(km+1)km$. In this section, we will produce $2k$ classes of edge-disjoint gregarious m-cycles, each class containing $km+1$ members. We divide the proof into two cases depending on whether m is divisible by 4 or not.

Case (1). Suppose m is divisible by 4 and put $m = 2t$. So, t is even and $\frac{km}{2} = kt$. Partition $\mathcal{D}_{km+1}$ into k subsets $E_r = \{\pm(rt+1), \pm(rt+2), \dots, \pm(rt+t)\}$ for $r = 0, 1, 2, \dots, k-1$, each subsets with $2t$ numbers.

For each fixed r, we consider the m-sequence

$$\begin{aligned}\rho_r = \big(&rt+1,\ -(rt+2),\ rt+3,\ \dots\ ,\ -(rt+(t-2)),\ rt+(t-1),\ -(rt+t),\\ &-(rt+(t-1)),\ rt+(t-2),\ -(rt+(t-3)),\ \dots\ ,\ rt+2,\ -(rt+1),\ rt+t\big).\end{aligned}$$

Since each element of E_r appears exactly once in ρ_r, the total sum of entries of ρ_r is zero. Let $\eta_{\rho_r} = (0, s_1, s_2, \dots, s_{2t-1})$ be the s-sequence of ρ_r. For $i = 1, 3, 5, \dots, t-1$, the number $i-1$ is even and so

$$\begin{aligned}s_i &= \sum_{j=1}^{i}(-1)^{j-1}(rt+j) = \sum_{j=1}^{i-1}(-1)^{j-1}(rt+j) + (rt+i)\\ &= \sum_{j=1}^{i-1}(-1)^{j-1}j + (rt+i) = -\frac{i-1}{2} + rt + i = rt + \frac{i+1}{2}.\end{aligned}$$

They are mutually distinct and constitute the interval $I_1 = \{l \mid rt+1 \le l \le (r+\frac{1}{2})t\}$. For $i = t+1, t+3, \dots, 2t-1$, we have

$$\begin{aligned}s_i &= \sum_{j=1}^{t}(-1)^{j-1}(rt+j) + \sum_{j=1}^{i-t}(-1)^{j}(rt+(t-j))\\ &= -\frac{t}{2} + \sum_{j=1}^{i-t-1}(-1)^{j}(rt+(t-j)) - (rt+(t-(i-t)))\\ &= -\frac{t}{2} - \frac{(i-t-1)}{2} - (rt + (2t-i)) = -(r+2)t + \frac{i+1}{2}\\ &= (2k-r-2)t + \frac{i+3}{2}.\end{aligned}$$

They are mutually distinct and constitute the interval $I_2 = \{l \mid (2k–r–\frac{3}{2})t+2 \leq l \leq (2k-r-1)t+1\}$. For $i = 2, 4, \ldots, t$, since i is even, we have

$$s_i = \sum_{j=1}^{i}(-1)^{j-1}(rt+j) = \sum_{j=1}^{i}(-1)^{j-1}j = -\frac{i}{2} = 2kt - \frac{i-2}{2}.$$

They are mutually distinct and constitute the interval $I_3 = \{l \mid (2k-\frac{1}{2})t+1 \leq l \leq 2kt\}$. For $i = t+2, t+4, \ldots, 2t-2$,

$$s_i = \sum_{j=1}^{t}(-1)^{j-1}(rt+j) + \sum_{j=1}^{i-t}(-1)^{j}(rt+(t-j)) = -\frac{t}{2} - \frac{i-t}{2} = -\frac{i}{2} = 2kt - \frac{i-2}{2}.$$

They are mutually distinct and constitute the interval $I_4 = \{l \mid (2k-1)t+2 \leq l \leq (2k-\frac{1}{2})t\}$. Since $1 \leq k$ and $0 \leq r \leq k-1$, all the interval are subintervals of $\{l \mid 0 < l \leq 2kt\}$, and for $i_1 \in I_1$, $i_2 \in I_2$, $i_2 \in I_3$, and $i_4 \in I_4$, we have $i_1 < i_2 < i_4 < i_3$. Therefore, the intervals I_1, I_2, I_3, and I_4 are mutually disjoint and all the entries of η_{ρ_r} are mutually distinct. By Lemma 2.1, ρ_r is an f-sequence. Now, with two appropriate flags, we produce two starter cycles from η_{ρ_r} as follows.

$$C_{\rho_r} = \langle 0, s_1, s_2, s_3, \ldots, s_{t-2}, s_{t-1}, s_t, \overline{s_{t+1}}, s_{t+2}, \overline{s_{t+3}}, s_{t+4}, \ldots, \overline{s_{2t-3}}, s_{2t-2}, \overline{s_{2t-1}}\rangle,$$
$$D_{\rho_r} = \langle \overline{0}, \overline{s_1}, \overline{s_2}, \overline{s_3}, \ldots, \overline{s_{t-2}}, \overline{s_{t-1}}, \overline{s_t}, s_{t+1}, \overline{s_{t+2}}, s_{t+3}, \overline{s_{t+4}}, \ldots, s_{2t-3}, \overline{s_{2t-2}}, s_{2t-1}\rangle.$$

Finally, we generate two classes, each with $km+1$ gregarious m-cycles, and they are

$$\Phi_r^{(1)} = \{\tau^k(C_{\rho_r}) \mid k = 0, 1, \ldots, km\} \quad \text{and} \quad \Phi_r^{(2)} = \{\tau^k(D_{\rho_r}) \mid k = 0, 1, \ldots, km\}.$$

Then, it can be seen that each edge of distance d appears exactly once in the cycles for $d = rt+1,\ rt+2, \ldots,\ rt+(t–1),\ rt+t$. For example, if $j-i = rt+4$, then the edges ij and $i\overline{j}$ appear in some cycles in $\Phi_r^{(1)}$ at the places corresponding to $s_5 s_4$ and $s_{2t-5}\overline{s_{2t-4}}$, respectively, while the edges $\overline{i}\,\overline{j}$ and $\overline{i}j$ appear in some cycles in $\Phi_r^{(2)}$ at the places corresponding to $\overline{s_5}\,\overline{s_4}$ and $\overline{s_{2t-5}}s_{2t-4}$, respectively.

For each $r = 0, 1, \ldots, k-1$, we repeat the above process and we obtain two classes $\Phi_i^{(1)}$ and $\Phi_i^{(2)}$, each class containing $km+1$ gregarious m-cycles. In these m-cycles, each edge of every nonzero distance appears exactly once. Therefore, if we put $\Phi = \bigcup_{r=0}^{k-1}\left(\Phi_r^{(1)} \bigcup \Phi_r^{(2)}\right)$, then Φ is a γ_m-decomposition of $K_{km+1(2)}$. By the way it is constructed, Φ is trivially circulant.

Example 3.1. Let $m = 8$ and $k = 3$. Then, $km + 1 = 25$ and $\mathcal{D}_{km+1} = \mathcal{D}_{25} = \{\pm 1, \pm 2, \ldots, \pm 12\}$. First, we make f-sequences

$$\rho_0 = (1, -2, 3, -4, -3, 2, -1, 4), \qquad \rho_1 = (5, -6, 7, -8, -7, 6, -5, 8)$$
$$\text{and} \qquad \rho_2 = (9, -10, 11, -12, -11, 10, -9, 12).$$

The corresponding s-sequences are

$$\eta_{\rho_0} = (0, 1, 24, 2, 23, 20, 22, 21), \qquad \eta_{\rho_1} = (0, 5, 24, 6, 23, 16, 22, 17)$$
$$\text{and} \qquad \eta_{\rho_2} = (0, 9, 24, 10, 23, 12, 22, 13),$$

respectively. The starter cycles obtained by the procedure are

$$\begin{array}{ll}
C_{\rho_0} = \langle 0,1,24,\ 2,\ 23,\overline{20},22,\overline{21}\,\rangle, & D_{\rho_0} = \langle \overline{0},\overline{1},\overline{24},\ \overline{2},\ \overline{23},20,\overline{22},21\,\rangle,\\
C_{\rho_1} = \langle 0,5,24,\ 6,\ 23,\overline{16},22,\overline{17}\,\rangle, & D_{\rho_1} = \langle \overline{0},\overline{5},\overline{24},\ \overline{6},\ \overline{23},16,\overline{22},17\,\rangle,\\
C_{\rho_2} = \langle 0,9,24,10,23,\overline{12},22,\overline{13}\,\rangle, & D_{\rho_2} = \langle \overline{0},\overline{9},\overline{24},\overline{10},\overline{23},12,\overline{22},13\,\rangle.
\end{array}$$

Finally, the six classes, each with 25 gregarious 8-cycles, generated by these starter cycles form a γ_8-decomposition of $K_{25(2)}$. Two of theses 6 classes are shown below and other 4 classes are omitted.

$$\begin{array}{cc}
\tau^k(C_{\rho_0}) & \tau^k(D_{\rho_1})\\
\langle 0,\ 1,\ 24,\ 2,\ 23,\overline{20},22,\overline{21}\,\rangle, & \langle \overline{0},\ \overline{5},\overline{24},\ \overline{6},\ \overline{23},16,\overline{22},17\,\rangle,\\
\langle 1,\ 2,\ 0,\ 3,\ 24,\overline{21},23,\overline{22}\,\rangle, & \langle \overline{1},\ \overline{6},\ \overline{0},\ \overline{7},\ \overline{24},17,\overline{23},18\,\rangle,\\
\langle 2,\ 3,\ 1,\ 4,\ 0,\ \overline{22},24,\overline{23}\,\rangle, & \langle \overline{2},\ \overline{7},\ \overline{1},\ \overline{8},\ \overline{0},\ 18,\overline{24},19\,\rangle,\\
\langle 3,\ 4,\ 2,\ 5,\ 1,\ \overline{23},\ 0,\ \overline{24}\,\rangle, & \langle \overline{3},\ \overline{8},\ \overline{2},\ \overline{9},\ \overline{1},\ 19,\ \overline{0},\ 20\,\rangle,\\
\langle 4,\ 5,\ 3,\ 6,\ 2,\ \overline{24},\ 1,\ \overline{0}\,\rangle, & \langle \overline{4},\ \overline{9},\ \overline{3},\ \overline{10},\ \overline{2},\ 20,\ \overline{1},\ 21\,\rangle,\\
\vdots & \vdots\\
\vdots & \vdots\\
\langle 22,23,21,24,20,\overline{17},19,\overline{18}\,\rangle, & \langle \overline{22},\overline{2},\overline{21},\ \overline{3},\ \overline{20},13,\overline{19},14\,\rangle,\\
\langle 23,23,22,\ 0,\ 21,\overline{18},20,\overline{19}\,\rangle, & \langle \overline{23},\overline{3},\overline{22},\ \overline{4},\ \overline{21},14,\overline{20},15\,\rangle,\\
\langle 24,\ 0,\ 23,\ 1,\ 22,\overline{19},21,\overline{20}\,\rangle, & \langle \overline{24},\overline{4},\overline{23},\ \overline{5},\ \overline{22},15,\overline{21},16\,\rangle.
\end{array}$$

Case(2). Suppose m is not divisible by 4 and put $m = 2t$. So, t is odd and $\frac{km}{2} = kt$. We proceed almost the same way as in Case (1), except that the pattern of the f-sequence and the flags are different. Partition $\mathcal{D}_{km+1}$ into k subsets $E_r = \{\pm(rt+1), \pm(rt+2), \dots, \pm(rt+t)\}$ for $r = 0, 1, 2, \dots, k-1$, each subsets with $2t$ numbers. For each fixed r, we consider the f-sequence

$$\begin{aligned}
\rho_r = \big(&rt{+}1,\ -(rt{+}2),\ rt{+}3,\ \dots\ ,\ rt{+}(t{-}2),\ -(rt{+}(t{-}1)),\ rt{+}t,\\
&rt{+}(t{-}1),\ -(rt{+}(t{-}2)),\ rt{+}(t{-}3),\ \dots\ ,\ rt{+}2,\ -(rt{+}1),\ -(rt{+}t)\big).
\end{aligned}$$

Since each element of E_r appears once in ρ_r, the total sum of entries of ρ_r is zero. Let $\eta_{\rho_r} = (0, s_1, s_2, \dots, s_{2t-1})$ be the s-sequence of ρ_r. For $i = 1, 3, \dots, t$, as in Case (1), we have

$$s_i = \sum_{j=1}^{i} (-1)^{j-1}(rt{+}j) = rt + \frac{i{+}1}{2}.$$

They are mutually distinct and constitute the interval $J_1 = \{l \mid rt{+}1 \le l \le (r{+}\frac{1}{2})t{+}\frac{1}{2}\}$. For $i = t{+}2, t{+}4, \dots, 2t-1$, the number $i{-}t$ is even and so

$$\begin{aligned}
s_i &= \sum_{j=1}^{t} (-1)^{j-1}(rt{+}j) + \sum_{j=1}^{i-t} (-1)^{j-1}(rt{+}(t{-}j))\\
&= rt{+}\frac{t{+}1}{2} + \sum_{j=1}^{i-t} (-1)^{j-1}(-j) = (r{+}\frac{1}{2})t{+}\frac{1}{2} + \frac{i{-}t}{2} = rt{+}\frac{i{+}1}{2}.
\end{aligned}$$

These numbers are mutually distinct and constitute the interval $J_2 = \{l \mid (r+\frac{1}{2})t+\frac{3}{2} \le l \le (r+1)t\}$. For $i = 2, 4, \ldots, t-1$, since i is even, we have

$$s_i = \sum_{j=1}^{i}(-1)^{j-1}(rt+j) = \sum_{j=1}^{i}(-1)^{j-1}j = -\frac{i}{2} = 2kt - \frac{i-2}{2}.$$

These numbers are mutually distinct and constitute the interval $J_3 = \{l \mid (2k-\frac{1}{2})t+\frac{3}{2} \le l \le 2kt\}$. For $i = t+1, t+3, \ldots, 2t-2$, the number $i-t-1$ is even and we have

$$\begin{aligned} s_i &= \sum_{j=1}^{t}(-1)^{j-1}(rt+j) + \sum_{j=1}^{i-t}(-1)^{j-1}(rt+(t-j)) \\ &= rt+\frac{t+1}{2} + \sum_{j=1}^{i-t-1}(-1)^{j-1}(-j) + (rt+(t-(i-t))) \\ &= (r+\frac{1}{2})t+\frac{1}{2} + \frac{i-t-1}{2} + (rt-i) = 2rt-\frac{i}{2}. \end{aligned}$$

They are mutually distinct and constitute the interval $J_4 = \{l \mid (2r-1)t+1 \le l \le (2r-\frac{1}{2})t-\frac{1}{2}\}$. It can be seen that the intervals J_1, J_2, J_3, and J_4 are mutually disjoint, and so all the entries of η_{ρ_r} are mutually distinct. Thus By Lemma 2.1, ρ_r is an f-sequence. The two starter cycles in this case as below.

$$C_{\rho_r} = \langle 0, s_1, s_2, s_3, \ldots, s_{t-4}, s_{t-3}, s_{t-2}, s_{t-1}, s_t, \overline{s_{t+1}}, s_{t+2}, \overline{s_{t+3}}, \ldots, s_{2t-3}, \overline{s_{2t-2}}, \overline{s_{2t-1}}\rangle,$$
$$D_{\rho_r} = \langle \overline{0}, s_1, \overline{s_2}, s_3, \ldots, s_{t-4}, \overline{s_{t-3}}, s_{t-2}, \overline{s_{t-1}}, \overline{s_t}, \overline{s_{t+1}}, \overline{s_{t+2}}, \overline{s_{t+3}}, \ldots, \overline{s_{2t-3}}, \overline{s_{2t-2}}, s_{2t-1}\rangle.$$

The rest of the proof is the same as in Case (1). This completes the proof of Theorem 1.2.

Example 3.2. Let $m = 10$ and $k = 2$. We have $\mathcal{D}_{mk+1} = \mathcal{D}_{21} = \{\pm 1, \pm 2, \ldots, \pm 10\}$. According to the method above, put

$$\rho_0 = (1, -2, 3, -4, 5, 4, -3, 2, -1, -5), \quad \rho_1 = (6, -7, 8, -9, 10, 9, -8, 7, -6, -10).$$

Then, the corresponding s-sequences are

$$\eta_{\rho_0} = (0, 1, 20, 2, 19, 3, 7, 4, 6, 5), \qquad \eta_{\rho_1} = (0, 6, 20, 7, 19, 8, 17, 9, 16, 10),$$

respectively. The starter cycles are as follows and other gregarious 10-cycles are omitted.

$$C_{\rho_0} = \langle 0, 1, 20, 2, 19, 3, \overline{7}, 4, \overline{6}, \overline{5}\rangle, \qquad D_{\rho_0} = \langle \overline{0}, 1, \overline{20}, 2, \overline{19}, \overline{3}, \overline{7}, \overline{4}, \overline{6}, 5\rangle,$$
$$C_{\rho_1} = \langle 0, 6, 20, 7, 19, 8, \overline{17}, 9, \overline{16}, \overline{10}\rangle, \qquad D_{\rho_1} = \langle \overline{0}, 6, \overline{20}, 7, \overline{19}, \overline{8}, \overline{17}, \overline{9}, \overline{16}, 10\rangle.$$

We omit other gregarious 10-cycles generated from these starter cycles.

We remark that there may be many different f-sequences and flags which gives a circulant γ_m decomposition. The problem is how to choose appropriate f-sequences and flags. There is no algorithm or a formula so far to produce proper of f-sequences and flags for circulant a γ_m-decomposition.

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