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ON REGULAR POLYGONS AND REGULAR SOLIDS HAVING INTEGER COORDINATES FOR THEIR VERTICES

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Abstract. We study the existence of regular polygons and regular solids whose vertices have integer coordinates in the three dimensional space and study side lengths of such squares, cubes and tetrahedra. We show that except for equilateral triangles, squares and regular hexagons there is no regular polygon whose vertices have integer coordinates. By using this, we show that there is no regular icosahedron and no regular dodecahedron whose vertices have integer coordinates. We characterize side lengths of such squares and cubes. In addition to these results, we prove Ionascu's result [4, Theorem2.2] that every equilateral triangle of side length $\sqrt{2}m$ for a positive integer m whose vertices have integer coordinate can be a face of a regular tetrahedron with vertices having integer coordinates in a different way.

1. Introduction

Eugen J. Ionascu and his colleagues did much work on regular polygons and regular solids whose vertices have integer coordinates[1, 3, 4, 5]. Some of their results are as follows:

Theorem 1.1. [3] *For an equilateral triangle ΔOPQ (O is the origin) in the three dimensionl space R^3, if the vertices P and Q have integer coordinates(x, y, z) and (u, v, w), then there exists a positive integer d satisfies*

$$a^2 + b^2 + c^2 = 3d^2, \tag{1.1}$$

where $a = yw - zv$, $b = zu - xw$. $c = xv - yu$. In particular the triangle lies on the plane $ax + by + cz = 0$. Conversely, if integers a, b, c and a positive integer d satisfies (1.1) then there exists infinitely many equilateral triangles lying on the plane $ax + by + cz = 0$ whose vertices have integer coordinates.

Theorem 1.2. [3] *An equilateral triangle of side length l whose vertices have integer coordinates in the three dimensional space R^3 exists, if and only if $l = \sqrt{2(m^2 - mn + n^2)}$ for some integers m and n (not both zero).*

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Theorem 1.3. [5] *There is no regular icosahedron and no regular dodecahedron whose vertices have integer coordinates in* R^3 *.*

Theorem 1.4. [3] *A regular tetrahedron of side length* l *whose vertices have integer coordinates exists in* R^3*, if and only if* $l = \sqrt{2}m$ *for some positive integer* m *.*

We show that among regular polygons in R^3, only equilateral triangles, squares and regular hexagons can have integer coordinates for their vertices . Using this we prove Theorem1.3 in a different way. We show that a square of side length l whose vertices have integer coordinates exists in R^3, if and only if $l = \sqrt{m^2+n^2}$ for some integers m and n (not both zero). Also we show that a cube of side length l whose vertices have integer coordinates exists in R^3, if and only if $l = m$ for some positive integer m . In [4] E.J.Ionascu showed that an equilateral triangle of side length $l = \sqrt{2}m$(m is a positive integer) whose vertices have integer coordinates can be a face of a tetrahedron having integer coordinates for its vertices by using a parametrization of equilateral triangles whose vertices have integer coordinates. In this paper we show the same result by a direct method without using the parametrization.

2. Main Results

Theorem 2.1. *If a regular* $n-$*polygon in the three dimensional space* R^3 *has rational coordinates for its vertices, then* $n = 3, 4$ *or* 6*.*

Proof. Let P_n be a regular polygon in R^3 whose vertices $X_1, X_2, \cdots, X_n$ have rational coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), \cdots, (x_n, y_n, z_n)$. Then the centroid $O = (\frac{\sum_{i=1}^n x_i}{n}, \frac{\sum_{i=1}^n y_i}{n}, \frac{\sum_{i=1}^n z_i}{n})$ of P_n has rational coordinates. Also the components of the vectors $\overrightarrow{OX_i} = (x_i - \frac{\sum_{i=1}^n x_i}{n}, y_i - \frac{\sum_{i=1}^n y_i}{n}, z_i - \frac{\sum_{i=1}^n z_i}{n})$, $i = 1, 2, \cdots, n$ are all rationals. So the cosine value $\cos\frac{2\pi}{n} = \frac{\overrightarrow{OX_1}\cdot\overrightarrow{OX_2}}{\|\overrightarrow{OX_1}\|\|\overrightarrow{OX_2}\|} = \frac{\overrightarrow{OX_1}\cdot\overrightarrow{OX_2}}{\|\overrightarrow{OX_1}\|^2}$ is a rational number. Thus the value can be written as $\cos\frac{2\pi}{n} = \frac{s}{t}$ for integers s and $t(\neq 0)$. Since

$$(z - \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n})(z - \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}) = z^2 - \frac{2s}{t}z + 1$$

and $n \geq 3$, the irreducible polynomoal of the complex number $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ over the rarional field Q is

$$z^2 - \frac{2s}{t}z + 1.$$

Also it is known that the irreducible polynomoal of the complex number $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ over the rarional field Q is the nth cyclotomic polynomial

$$g_n(z) = \Pi_{i=1}^{\phi(n)}(z - \xi_i),$$

where $\xi_1, \xi_2, \cdots, \xi_{\phi(n)}$ are all the primitive nth roots of unity and ϕ is the Euler function. So we have $g_n(z) = z^2 - \frac{2s}{t}z + 1$, which implies that $\phi(n) = 2$. From this it follows that $n = 3, 4$ or 6.

□

Theorem 2.2. *There is no regular icosahedron and no regular dodecahedron whose vertices have rational coordinates.*

Proof. By Theorem2.1 there is no regular pentagon whose vertices have rational coordinates. Since every face of a regular dodecahedron is a regular pentagon, it follows that there is no regular dodecahedron whose vertices have rational coordinates. Now assume that there is a regular icosahedron whose vertices have rational coordinates. Then the dual solid of this regular icosahedron is a regular dodecahedron whose vertices have rational coordinates. This is a contradiction. Thus there is no regular icosahedron whose vertices have rational coordinates. □

The following theorem implies that the study of regular hexagons whose vertices have integer coordinates can be reduced to the study of equilateral triangles whose vertices have integer coordinates.

Theorem 2.3. *A regular hexagon whose vertices have integer coordinate is divided into six equilateral triangles whose vertices have integer coordinates with the centroid of the given regular hexagon as their common vertex. Conversely, for a given equilateral triangle whose vertices have integer coordinate, we can construct a regular hexagon whose vertices have integer coordinate with one vertex of a given equilateral triangle as its centroid.*

Proof. Let H be a regular hexagon whose vertices $V_i(x_i, y_i, z_i)$, $i = 1, 2, 3, 4, 5, 6$ have integer coordinates. Then by a translation , we can see that the six vertices $U_i = V_i - V_1$, $i = 1, 2, 3, 4, 5, 6$ forms a regular hexagon. Note that U_1 is the origin O. Then nonadjacent three vertices U_1, U_3 and U_5 forms an equilateral triangle. Then the point $P(\frac{x'_3+x'_5}{3}, \frac{y'_3+y'_5}{3}, \frac{z'_3+z'_5}{3})$ is the centroid of the triangle , where $x'_i = x_i - x_1, y'_i = y_i - y_1$ and $z'_i = z_i - z_1$ for $i = 3, 5$. Since U_4 is symmetric to U_1 with respect to P , we know that $(x'_4, y'_4, z'_4) = (\frac{2(x'_3+x'_5)}{3}, \frac{2(y'_3+y'_5)}{3}, \frac{2(z'_3+z'_5)}{3})$, where $x'_4 = x_4 - x_1, y'_4 = y_4 - y_1$ and $z'_4 = z_4 - z_1$. This implies that $x'_3 + x'_5, y'_3 + y'_5$ and $z'_3 + z'_5$ are all multiples of 3. So we can see that $P(\frac{x'_3+x'_5}{3}, \frac{y'_3+y'_5}{3}, \frac{z'_3+z'_5}{3})$ is a point with integer coordinates. Therefore the hexagon with the six vertices U_i, $i = 1, 2, 3, 4, 5, 6$ is devided into six equilateral triangles

$$\Delta PU_1U_2, \Delta PU_2U_3, \Delta PU_3U_4, \Delta PU_4U_5, \Delta PU_5U_6 \text{ and } \Delta PU_6U_1$$

whose vertices have integer coordinates. By another translation by V_1 we can observe the hexagon H is divided into such equilateral triangles. Conversely assume that three points $O(0,0,0), P(x,y,z)$ and $Q(u,v,w)$ having integer coordinates forms an equilateral triangle. Then the six points

$$(u,v,w),(x,y,z),(x-u,y-v,z-w),(-u,-v,-w),(-x,-y,-z)$$

and

$$(u-x,v-y,w-z)$$

forms a regular hexagon with the origin its centroid. □

Suppose that the vertices P,Q and R of a square $\square OPQR$ (O is the origin) have integer coordinates $(x,y,z),\ (u,v,w)$ and $(x+u,y+v,z+w)$, respectively. Then the numbers $a=yw-zv,\ b=zu-xw$ and $c=xv-yu$ satisfy the equation

$$a^2+b^2+c^2=d^2,$$

where $d=x^2+y^2+z^2=u^2+v^2+w^2$. In particular the square $\square OPQR$ lies on the plane $ax+by+cz=0$. Conversely consider a plane $ax+by+cz=0$ with integer coefficients. And suppose that a,b and c satisfy $a^2+b^2+c^2=d^2$ for an integer d . Consider a point $P(u,v,w)$ with integer coordinates lying on the plane $ax+by+cz=0$. Then the point $R(x,y,z)$ with rational coordinates

$$x=\frac{cv-bw}{d},\ y=\frac{aw-cu}{d},\ z=\frac{bu-av}{d}$$

lies on the plane $ax+by+cz=0$. And we can see that the four points O(the origin) , $(du,dv,dw),\ (d(u+x),d(v+y),d(w+z))$ and (dx,dy,dz) forms a square with vertices having integer coordinates. We have the following theorem.

Theorem 2.4. *For a square $\square OPQR$ (O is the origin) if the vertices P,R and Q have integer coordinates $(x,y,z),(u,v,w)$ and $(x+u,y+v,z+w)$, respectively. then there exists a positive integer d satisfying*

$$a^2+b^2+c^2=d^2, \tag{2.1}$$

where $a=yw-zv,\ b=zu-xw$ and $c=xv-yu$. In particular the square lies on the plane $ax+by+cz=0$. Conversely, if integers a,b,c and a positive integer d satisfies (2.1) *then there exists infinitely many squares whose vertices have integer coordinates on the plane $ax+by+cz=0$.*

Theorem 2.5. *A square of side length l whose vertices have integer coordinates in the three dimensional space R^3 exists, if and only if $l=\sqrt{m^2+n^2}$ for some integers m and n(not both zero).*

Proof. For sufficiency, consider the square whose vertices are the four points $O(0,0,0),\ P(m,n,0),\ R(n,-m,0)$ and $Q(m+n,n-m,0)$ for given integers m and n(not both zero). The side length of this square is $\sqrt{m^2+n^2}$. For necessity, consider a square $\square OPQR$ (O is the origin) whose vertices P,R and Q have integer coordinates $(x,y,z),\ (u,v,w)$ and $(x+u,y+v,z+w)$, respectively. The square lies on the plane $ax+by+cz=0$, where $a=yw-zv,\ b=zu-xw$

and $c = xv - yu$. Since $au + bv + cw = 0$, the side lengh l of the square $\square OPQR$ satisfies

$$\begin{aligned} l^2 &= u^2 + v^2 + w^2 = (\frac{bv + cw}{a})^2 + v^2 + w^2 \\ &= \frac{(a^2 + b^2)v^2 + 2bcvw + (a^2 + c^2)w^2}{a^2} \\ &= \frac{((a^2 + b^2)v + bcw)^2 + (awd)^2}{(a^2)^2 + (ab)^2}, \end{aligned} \tag{2.2}$$

where $d = \sqrt{a^2 + b^2 + c^2} = x^2 + y^2 + z^2 = l^2$. Since the necessary and sufficient condition for that an integer N can be written as the sum of two squares is that the prime-power decomposition of N does not contain a prime factor congruent to 3 (mod 4) to an odd power[2], the prime-power decompositions of two integers $(a^2 + b^2)v + bcw)^2 + (awd)^2$ and $(a^2)^2 + (ab)^2$ do not contain a prime congruent to 3(mod 4) to an odd power. By (2.2), the integer l^2 is a ratio of two integers $(a^2 + b^2)v + bcw)^2 + (awd)^2$ and $(a^2)^2 + (ab)^2$, which implies l^2 is a sum of two squares. So l can be written as $\sqrt{m^2 + n^2}$ for some integers m and n . □

Theorem 2.6. *A square $\square OPQR$ (O is the origin) whose vertices have integer coordinates can be a face of a cube whose vertices have integer coordinates if and only if the side length of this square is equal to an integer value m .*

Proof. Suppose that the vertices P, R and Q have integer coordinates (x, y, z), (u, v, w) and $(x + u, y + v, z + w)$, respectively and that the side length $l = \sqrt{x^2 + y^2 + z^2} = \sqrt{u^2 + v^2 + w^2}$ satisfies

$$l^2 = x^2 + y^2 + z^2 = u^2 + v^2 + w^2 = m^2$$

for an integer m. Then, since $xu + yv + zw = 0$, we have

$$\begin{aligned} m^2 &= u^2 + v^2 + w^2 = (\frac{yv + zw}{x})^2 + v^2 + w^2 \\ &= \frac{(x^2 + y^2)v^2 - 2yzvw + (x^2 + z^2)w^2}{x^2} \\ &= \frac{(m^2 - z^2)v^2 - 2yzvw + (m^2 - y^2)w^2}{x^2} \end{aligned}$$

which implies that $m^2x^2 = m^2(v^2 + w^2) - (yw - xv)^2$ or $(yw - xv)^2 = m^2(v^2 + w^2 - x^2)$. From this it follows that m is a divisor of the integer $yw - xv$. By similar argumenst, we can see that m is also a divisor of the integers $zu - xw$ and $xu - yv$. So the point $S(\frac{yw-zv}{m}, \frac{zu-xw}{m}, \frac{xv-yu}{m})$ has integer coordinates and the eight points $O, P, Q, R, S, P + S, Q + S$ and $R + S$ form a cube whose vertices have integer coordinates. Conversely, suppose that a cube has integer coordinates for its vertices and a square $\square OPQR$ (O is the origin) is a face of the cube . Let (x, y, z) and (u, v, w) be coordinates of two vertices P and R . Then the point $S(\frac{yw-zv}{l}, \frac{zu-xw}{l}, \frac{xv-yu}{l})$ or $S(\frac{zv-yw}{l}, \frac{xw-zu}{l}, \frac{yu-xv}{l})$ must be a vertex of the cube , where l is the side length of the cube , which implies the point $S(\frac{yw-zv}{l}, \frac{zu-xw}{l}, \frac{xv-yu}{l})$ must have integer coordinates, which implies that l is an rational numer. By theorem2.5 l is equal to $\sqrt{m^2+n^2}$ for some integers m and n. So the number l must be an integer. □

In [5] Ionascu showed that every equilateral triangle of side length $\sqrt{2}m$ for a positive integer m whose vertices have integer coordinates can be a face of a regular tetrahedron with vertices having integer coordinates for its vertices. He proved it by using his parametrization of equilateral triangles whose vertices have integer coordinates[3]. We prove it in a elementary way without using the parametrization.

Theorem 2.7. *If ΔOPQ is an equilateral triangle of side length $\sqrt{2}m$ for a positive integer m , have integer coordinates for its vertices. Then it can be a face of a tetrahedron whose vertices have integer coordinates.*

Proof. By assumption, the vertices P and Q have integer coordinates (x, y, z) and (u, v, w), respectively. Also the side length l satisfies

$$l^2 = x^2 + y^2 + z^2 = u^2 + v^2 + w^2 = 2m^2 \tag{2.3}$$

for an integer m. And we see that with any of two points

$$R(\frac{u+x}{3} + \frac{2(yw-zv)}{3m}, \frac{v+y}{3} + \frac{2(zu-xw)}{3m}, \frac{w+z}{3} + \frac{2(xv-yu)}{3m})$$

and

$$R'(\frac{u+x}{3} - \frac{2(yw-zv)}{3m}, \frac{v+y}{3} - \frac{2(zu-xw)}{3m}, \frac{w+z}{3} - \frac{2(xv-yu)}{3m})$$

the triangle ΔOPQ forms a tetrahedron. We will show that one of two points R and R' must be a point with integer coordinates. From (2.3) and the fact that ΔOPQ is an equilateral triangle it follows that

$$xu + yv + zw = m^2. \tag{2.4}$$

By Theorem1.1 we know that both of u and v cannot be zeros. Without loss of generality assume that $x \neq 0$. Then from (2.4) we have

$$u = \frac{m^2 - yv - zw}{x}.$$

Substituting this into $u^2 + v^2 + w^2 = 2m^2$ and using $x^2 + y^2 + z^2 = 2m^2$ and (2.4) we have

$$\begin{aligned}
2m^2 &= u^2 + v^2 + w^2 = (\frac{m^2 - yv - xw}{x})^2 + v^2 + w^2 \\
&= \frac{1}{x^2}\{m^4 - 2m^2(yv + zw) + (x^2 + y^2)v^2 + 2yzvw + (x^2 + z^2)w^2\} \\
&= \frac{1}{x^2}\{m^4 + 2m^2(xu - m^2) + (2m^2 - z^2)v^2 + 2yzvw + (2m^2 - y^2)w^2\} \\
&= \frac{1}{x^2}\{-m^4 + 2m^2xu + 2m^2(2m^2 - u^2) - (yw - xv)^2\},
\end{aligned}$$

which implies that

$$2m^2x^2 = 3m^4 + 2m^2xu - 2m^2u^2 - (yw - zv)^2$$

or

$$(yw - zv)^2 = m^2(3m^2 + 6xu - 2(x + u)^2).$$

From this we see that $yw - zv$ is a multiple of m . By similar arguments we see that $zu - xw$ and $xv - yu$ are also multiples of 3 . So the points

$$(u + x \pm \frac{2(yw - zv)}{m}, v + y \pm \frac{2(zu - xw)}{m}, w + z \pm \frac{2(xv - yu)}{m})$$

have integer coordinates. By a computation, we have

$$(\frac{2(yw - zv)}{m})^2 + (\frac{2(zu - xw)}{m})^2 + (\frac{2(xv - yu)}{m})^2 = 12m^2. \qquad (2.5)$$

We proceed with two cases separately

Case 1. $u + x \equiv 0 \pmod 3$
Since $(u + x)^2 + (v + y)^2 + (w + z)^2 = 6m^2 \equiv 0 \pmod 3$, The condition $u + x \equiv 0 \pmod 3$ implies that $v + y \equiv 0 \pmod 3$ and $w + z \equiv 0 \pmod 3$. From $(yw - zv)^2 = m^2(3m^2 + 6xu - 2(x + u)^2)$ and $u + x \equiv 0 \pmod 3$ we know $\frac{2(yw-zv)}{m}$ is a multiple of 3. By similar methods we can derive that $\frac{2(zu-xw)}{m}$ and $\frac{2(xv-yu)}{m}$ are also multiples of 3. So we see that

$$R(\frac{u + x}{3} + \frac{2(yw - zv)}{3m}, \frac{v + y}{3} + \frac{2(zu - xw)}{3m}, \frac{w + z}{3} + \frac{2(xv - yu)}{3m})$$

and

$$R'(\frac{u + x}{3} - \frac{2(yw - zv)}{3m}, \frac{v + y}{3} - \frac{2(zu - xw)}{3m}, \frac{w + z}{3} - \frac{2(xv - yu)}{3m})$$

have integer coordinates.

Case 2. $u + x \not\equiv 0 \pmod 3$
Since $(u + x)^2 + (v + y)^2 + (w + z)^2 = 6m^2 \equiv 0 \pmod 3$, the condition $u + x \not\equiv 0 \pmod 3$ implies that neither $v + y$ nor $w + z$ is a multiple of 3. From $(yw - zv)^2 = m^2(3m^2 + 6xu - 2(x + u)^2)$ we know that $\frac{2(yw-zv)}{m}$ is not a multiple of 3. By similar methods we can derive that neither $\frac{2(zu-xw)}{m}$ nor $\frac{2(xv-yu)}{m}$ is a multiple of 3. Since two points $P(x, y, z)$ and $Q(u, v, w)$ lies on the plane $ax + by + cz = 0(a = yw - zv,\ b = zu - xw,\ c = xv - yu)$, the equation (2.5) implies that

$$(u + x \pm \frac{2(yw - zv)}{m})\frac{2(yw - zv)}{m} + (v + y \pm \frac{2(zu - xw)}{m})\frac{2(zu - xw)}{m}$$
$$+(w + z \pm \frac{2(xv - yu)}{m})\frac{2(xv - yu)}{m} \equiv 0 \pmod 3 \qquad (2.6)$$

From the fact that no element of the set

$$\{u+x, y+v, z+w, \frac{2(yw-zv)}{m}, \frac{2(zu-xw)}{m}, \frac{2(xv-yu)}{m}\}$$

is a multiple of 3 it follows that at least two elements of the set

$$\{u+x+\frac{2(yw-zv)}{m}, v+y+\frac{2(zu-xw)}{m}, w+z+\frac{2(xv-yu)}{m}\}$$

or at least two elements of the set

$$\{u+x-\frac{2(yw-zv)}{m}, v+y-\frac{2(zu-xw)}{m}, w+z-\frac{2(xv-yu)}{m}\}$$

must be multiples of 3. If two elements of either set are multiples of 3, then from (2.6) it follows that all three elements of the set are multiples of 3. So we can conclude that either R or R' has integer coordinates. □

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