CERTAIN FRACTIONAL INTEGRAL INEQUALITIES INVOLVING HYPERGEOMETRIC OPERATORS

Junesang Choi* and Praveen Agarwal

Abstract. A remarkably large number of inequalities involving the fractional integral operators have been investigated in the literature by many authors. Very recently, Baleanu *et al.* [2] gave certain interesting fractional integral inequalities involving the Gauss hypergeometric functions. Using the same fractional integral operator, in this paper, we present some (presumably) new fractional integral inequalities whose special cases are shown to yield corresponding inequalities associated with Saigo, Erdélyi-Kober and Riemann-Liouville type fractional integral operators. Relevant connections of the results presented here with those earlier ones are also pointed out.

1. Introduction and preliminaries

Throughout the present investigation, we shall (as usual) denote $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_0^-$ by the sets of positive integers, real numbers, complex numbers, and nonpositive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Consider the following functional:

$$
\begin{aligned}
T(f, g, p, q) = & \int_a^b q(x)\, dx \int_a^b p(x)\, f(x)\, g(x)\, dx \\
& + \int_a^b p(x) dx \int_a^b q(x)\, f(x)\, g(x) dx \\
& - \left(\int_a^b q(x)\, f(x) dx \right) \left(\int_a^b p(x)\, g(x) dx \right) \\
& - \left(\int_a^b p(x)\, f(x) dx \right) \left(\int_a^b q(x)\, g(x) dx \right),
\end{aligned}
\tag{1.1}
$$

where f, $g : [a, b] \to \mathbb{R}$ are two integrable functions on $[a, b]$ and $p(x)$ and $q(x)$ are positive integrable functions on $[a, b]$. If f and g are *synchronous* on $[a, b]$, *i.e.*,

$$(f(x) - f(y))\,(g(x) - g(y)) \geq 0, \tag{1.2}$$

for any x, $y \in [a, b]$, then we have (see, *e.g.*, [11] and [14]):

$$T(f, g, p, q) \geq 0. \tag{1.3}$$

The inequality in (1.2) is reversed if f and g are *asynchronous* on $[a, b]$, *i.e.*,

$$(f(x) - f(y))\,(g(x) - g(y)) \leq 0, \tag{1.4}$$

for any x, $y \in [a, b]$. If $p(x) = q(x)$ for any x, $y \in [a, b]$, we get the Chebyshev inequality (see [4]). Ostrowski [16] established the following generalization of the Chebyshev inequality: If f and g are two differentiable and synchronous functions on $[a, b]$, and p is a positive integrable function on $[a, b]$ with $|f'(x)| \geq m$ and $|g'(x)| \geq r$ for $x \in [a, b]$ and nonnegative real constants m and r, then we have

$$T(f, g, p) = T(f, g, p, p) \geq m\,r\,T(x - a, x - a, p) \geq 0. \tag{1.5}$$

If f and g are asynchronous on $[a, b]$, then we have

$$T(f, g, p) \leq m\,r\,T(x - a, x - a, p) \leq 0. \tag{1.6}$$

If f and g are two differentiable functions on $[a, b]$ with $|f'(x)| \leq M$ and $|g'(x)| \leq R$ for $x \in [a, b]$, and p is a positive integrable function on $[a, b]$, then we have

$$|T(f, g, p)| \leq M\,R\,T(x - a, x - a, p) \leq 0. \tag{1.7}$$

The functional (1.1) has attracted many researchers' attention due mainly to diverse applications in numerical quadrature, transform theory, probability and statistical problems. Among those applications, the functional (1.1) has also been employed to yield a number of integral inequalities (see, *e.g.*, [1, 3, 8, 9, 10, 12, 15, 18]; for a very recent work, see also [2]). Very recently Dumitru *et al.* [2] gave certain interesting fractional integral inequalities involving the Gauss hypergeometric functions. In the present sequel to these recent works, we propose to derive certain (presumably) new fractional integral inequalities involving the Gauss hypergeometric functions whose special cases are shown to yield corresponding inequalities associated with Saigo fractional integral operator (3.1), Riemann-Liouville fractional integral operator (3.2) and Erdélyi-Kober fractional integral operator (3.3). Relevant connections of some of the results presented here with those earlier ones are also pointed out.

For our purpose, we also need to recall the following definitions and some earlier works.

Definition 1. A real-valued function $f(t)$ $(t > 0)$ is said to be in the space C_μ^n $(n, \mu \in \mathbb{R})$, if there exists a real number $p > \mu$ such that $f^{(n)}(t) = t^p\,\phi(t)$,

where $\phi(t) \in C(0, \infty)$. Here, for the case $n = 1$, we use a simpler notation $C_\mu^1 = C_\mu$.

Definition 2. Let $\alpha > 0$, $\mu > -1$ and $\beta, \eta \in \mathbb{R}$. Then a generalized fractional integral $I_t^{\alpha,\beta,\eta,\mu}$ (in terms of the Gauss hypergeometric function) of order α for a real-valued continuous function $f(t)$ is defined by [6] :

$$I_t^{\alpha,\beta,\eta,\mu} \{f(t)\} = \frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_0^t \tau^\mu (t-\tau)^{\alpha-1} \, {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau) d\tau,$$

(1.8)

where the function ${}_2F_1(\cdot)$ is the Gaussian hypergeometric function defined by (see, *e.g.*, [17, Section 1.5]):

$$\,{}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{t^n}{n!}$$

(1.9)

and $\Gamma(\alpha)$ is the familiar Gamma function. Here $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see, *e.g.*, [17, p. 2 and pp. 4-6]):

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases}$$

$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad \left(\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

(1.10)

2. Certain inequalities involving generalized fractional integral operator

Here we start with presenting two inequalities involving generalized fractional integral (1.8) stated in Lemmas 1 and 2 below.

Lemma 1. *Let f and g be two continuous and synchronous functions on $[0, \infty)$ and $u, v : [0, \infty) \to [0, \infty)$ be continuous functions. Then the following inequality holds true*:

$$I_t^{\alpha,\beta,\eta,\mu}\{u(t)\} \, I_t^{\alpha,\beta,\eta,\mu}\{v(t)\,f(t)\,g(t)\} + I_t^{\alpha,\beta,\eta,\mu}\{v(t)\} \, I_t^{\alpha,\beta,\eta,\mu}\{u(t)\,f(t)\,g(t)\}$$

$$\geq I_t^{\alpha,\beta,\eta,\mu}\{u(t)\,f(t)\} \, I_t^{\alpha,\beta,\eta,\mu}\{v(t)\,g(t)\} + I_t^{\alpha,\beta,\eta,\mu}\{v(t)\,f(t)\} \, I_t^{\alpha,\beta,\eta,\mu}\{u(t)\,g(t)\},$$

(2.1)

for all $t > 0$, $\alpha > 0$, $\mu > -1$ and $\beta, \eta \in \mathbb{R}$ with $\alpha + \beta + \mu \geqq 0$ and $\eta \leqq 0$.

Proof. Let f and g be two continuous and synchronous functions on $[0, \infty)$. Then, for all $\tau, \rho \in (0, t)$ with $t > 0$, we have

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0,$$

(2.2)

or, equivalently,

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau).$$

(2.3)

Consider the following function $F(t, \tau)$ defined, $0 < \tau < t$, by

$$
\begin{aligned}
F(t, \tau) &= \frac{t^{-\alpha-\beta-2\mu}\tau^{\mu}(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \; {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{\tau}{t}\right) \\
&= \frac{\tau^{\mu}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{t^{\alpha+\beta+2\mu}} + \frac{\tau^{\mu}(\alpha+\beta+\mu)(-\eta)}{\Gamma(\alpha+1)}\frac{(t-\tau)^{\alpha}}{t^{\alpha+\beta+2\mu+1}} \\
&\quad + \frac{\tau^{\mu}(\alpha+\beta+\mu)(\alpha+\beta+\mu+1)(-\eta)(-\eta+1)}{\Gamma(\alpha+2)}\frac{(t-\tau)^{\alpha+1}}{t^{\alpha+\beta+2\mu+2}} + \cdots.
\end{aligned} \tag{2.4}
$$

We observe that each term of the above series is nonnegative under the conditions in Lemma 1, and hence, the function $F(t, \tau)$ remains nonnegative for all $\tau \in (0, t)$ $(t > 0)$.

Now, multiplying both sides of (2.3) by $F(t, \tau)\, u(\tau)$ defined by (2.4) and integrating the resulting inequality with respect to τ from 0 to t, and using (1.8), we get

$$
\begin{aligned}
I_t^{\alpha,\beta,\eta,\mu} &\{u(t)\, f(t)\, g(t)\} + f(\rho)g(\rho)I_t^{\alpha,\beta,\eta,\mu}\{u(t)\} \\
&\geq g(\rho)I_t^{\alpha,\beta,\eta,\mu}\{u(t)\, f(t)\} + f(\rho)I_t^{\alpha,\beta,\eta,\mu}\{u(t)\, g(t)\}.
\end{aligned} \tag{2.5}
$$

Next, multiplying both sides of (2.5) by $F(t, \rho)\, v(\rho)$ $(0 < \rho < t)$, where $F(t, \rho)$ is given when τ is replaced by ρ in (2.4), and integrating the resulting inequality with respect to ρ from 0 to t, and using (1.8), we are led to the desired result (2.1). $\qquad\square$

Lemma 2. *Let f and g be two continuous and synchronous functions on $[0, \infty)$ and let $u, v : [0, \infty) \to [0, \infty)$ be continuous functions. Then the following inequality holds true*:

$$
\begin{aligned}
&I_t^{\gamma,\delta,\zeta,\nu}\{v(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{u(t)\, f(t)\, g(t)\} + I_t^{\gamma,\delta,\zeta,\nu}\{v(t)\, f(t)\, g(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{u(t)\} \\
&\geq I_t^{\gamma,\delta,\zeta,\nu}\{v(t)\, g(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{u(t)\, f(t)\} + I_t^{\gamma,\delta,\zeta,\nu}\{v(t)\, f(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{u(t)\, g(t)\},
\end{aligned} \tag{2.6}
$$

for all $t > 0$, $\alpha > 0$, $\mu > -1$, $\gamma > 0$, $\nu > -1$ and $\beta, \eta, \delta, \zeta \in \mathbb{R}$ with $\alpha+\beta+\mu \geqq 0$, $\eta \leqq 0$, $\gamma+\delta+\nu \geqq 0$ and $\zeta \leqq 0$.

Proof. Multiplying both sides of (2.5) by

$$
\frac{t^{-\gamma-\delta-2\nu}\rho^{\nu}(t-\rho)^{\gamma-1}}{\Gamma(\gamma)} \; {}_2F_1\left(\gamma+\delta+\nu, -\zeta; \gamma; 1-\frac{\rho}{t}\right)\, v(\rho) \quad (0 < \rho < t),
$$

which remains nonnegative under the conditions in (2.6), and integrating the resulting inequality with respect to ρ from 0 to t, and using (1.8), we get the desired result (2.6). $\qquad\square$

Theorem 1. *Let f and g be two continuous and synchronous functions on $[0, \infty)$ and let $l, m, n : [0, \infty) \to [0, \infty)$ be continuous functions. Then the*

following inequality holds true:

$$2I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\right\}\left[I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,f(t)\,g(t)\right\}\right.$$

$$\left.+I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,f(t)\,g(t)\right\}\right]$$

$$+2I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\,f(t)\,g(t)\right\}$$

$$\geq I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\right\}\left[I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,f(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,g(t)\right\}\right.$$

$$\left.+I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,f(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,g(t)\right\}\right] \qquad (2.7)$$

$$+I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\right\}\left[I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\,f(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,g(t)\right\}\right.$$

$$\left.+I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,f(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\,g(t)\right\}\right]$$

$$+I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\right\}\left[I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\,f(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,g(t)\right\}\right.$$

$$\left.+I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,f(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\,g(t)\right\}\right],$$

for all $t > 0$, $\alpha > 0$, $\mu > -1$ *and* $\beta, \eta \in \mathbb{R}$ *with* $\alpha + \beta + \mu \geqq 0$ *and* $\eta \leqq 0$.

Proof. By setting $u = m$ and $v = n$ in Lemma 1, we get

$$I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,f(t)\,g(t)\right\}+I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,f(t)\,g(t)\right\}$$

$$\geq I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,f(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,g(t)\right\}+I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,f(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,g(t)\right\}.$$

$$(2.8)$$

Since $I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\right\} \geqq 0$ under the given conditions, multiplying both sides of (2.8) by $I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\right\}$, we have

$$I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\right\}\left[I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,f(t)\,g(t)\right\}\right.$$

$$\left.+I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,f(t)\,g(t)\right\}\right]$$

$$\geq I_t^{\alpha,\beta,\eta,\mu}\left\{l(t)\right\}\left[I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,f(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,g(t)\right\}\right. \qquad (2.9)$$

$$\left.+I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\,f(t)\right\}I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\,g(t)\right\}\right].$$

Similarly replacing u, v by l, n and u, v by l, m, respectively, in (2.1), and then multiplying both sides of the resulting inequalities by $I_t^{\alpha,\beta,\eta,\mu}\left\{m(t)\right\}$ and $I_t^{\alpha,\beta,\eta,\mu}\left\{n(t)\right\}$ both of which are nonnegative under the given assumptions,

respectively, we get the following inequalities:

$$
\begin{aligned}
I_t^{\alpha,\beta,\eta,\mu} &\{m(t)\} \left[I_t^{\alpha,\beta,\eta,\mu}\{l(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{n(t)\,f(t)\,g(t)\} \right.\\
&\left. + I_t^{\alpha,\beta,\eta,\mu}\{n(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,f(t)\,g(t)\} \right]\\
\geq I_t^{\alpha,\beta,\eta,\mu} &\{m(t)\} \left[I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,f(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{n(t)\,g(t)\} \right.\\
&\left. + I_t^{\alpha,\beta,\eta,\mu}\{n(t)\,f(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,g(t)\} \right]
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
I_t^{\alpha,\beta,\eta,\mu} &\{n(t)\} \left[I_t^{\alpha,\beta,\eta,\mu}\{l(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{m(t)\,f(t)\,g(t)\} \right.\\
&\left. + I_t^{\alpha,\beta,\eta,\mu}\{m(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,f(t)\,g(t)\} \right]\\
\geq I_t^{\alpha,\beta,\eta,\mu} &\{n(t)\} \left[I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,f(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{m(t)\,g(t)\} \right.\\
&\left. + I_t^{\alpha,\beta,\eta,\mu}\{m(t)\,f(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,g(t)\} \right].
\end{aligned}
\tag{2.11}
$$

Finally, by adding (2.9), (2.10) and (2.11), sides by sides, we arrive at the desired result (2.7). □

We present another inequality involving the Saigo fractional integral operator in (1.8) asserted by the following theorem.

Theorem 2. *Let f and g be two continuous and synchronous functions on $[0, \infty)$ and let $l, m, n : [0, \infty) \to [0, \infty)$ be continuous functions. Then the following inequality holds true:*

$$
\begin{aligned}
I_t^{\alpha,\beta,\eta,\mu} &\{l(t)\} \left[2 I_t^{\alpha,\beta,\eta,\mu}\{m(t)\}\, I_t^{\gamma,\delta,\zeta,\nu}\{n(t)\,f(t)\,g(t)\} \right.\\
&+ I_t^{\alpha,\beta,\eta,\mu}\{n(t)\}\, I_t^{\gamma,\delta,\zeta,\nu}\{m(t)\,f(t)\,g(t)\}\\
&\left. + I_t^{\gamma,\delta,\zeta,\nu}\{n(t)\}\, I_t^{\alpha,\beta,\eta,\mu}\{m(t)\,f(t)\,g(t)\} \right]\\
&+ I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,f(t)\,g(t)\} \left[I_t^{\alpha,\beta,\eta,\mu}\{m(t)\}\, I_t^{\gamma,\delta,\zeta,\nu}\{n(t)\} \right]\\[4pt]
\geq I_t^{\alpha,\beta,\eta,\mu} &\{l(t)\} \left[I_t^{\alpha,\beta,\eta,\mu}\{m(t)\,f(t)\}\, I_t^{\gamma,\delta,\zeta,\nu}\{n(t)\,g(t)\} \right.\\
&\left. + I_t^{\alpha,\beta,\eta,\mu}\{m(t)\,g(t)\}\, I_t^{\gamma,\delta,\zeta,\nu}\{n(t)\,f(t)\} \right]\\
&+ I_t^{\alpha,\beta,\eta,\mu}\{m(t)\} \left[I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,f(t)\}\, I_t^{\gamma,\delta,\zeta,\nu}\{n(t)\,g(t)\} \right.\\
&\left. + I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,g(t)\}\, I_t^{\gamma,\delta,\zeta,\nu}\{n(t)\,f(t)\} \right]\\
&+ I_t^{\alpha,\beta,\eta,\mu}\{n(t)\} \left[I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,f(t)\}\, I_t^{\gamma,\delta,\zeta,\nu}\{m(t)\,g(t)\} \right.\\
&\left. + I_t^{\alpha,\beta,\eta,\mu}\{l(t)\,g(t)\}\, I_t^{\gamma,\delta,\zeta,\nu}\{m(t)\,f(t)\} \right]
\end{aligned}
\tag{2.12}
$$

for all $t > 0$, $\alpha > 0$, $\mu > -1$, $\gamma > 0$, $\nu > -1$ *and* $\beta, \eta, \delta, \zeta \in \mathbb{R}$ *with* $\alpha + \beta + \mu \geqq 0$, $\eta \leqq 0$, $\gamma + \delta + \nu \geqq 0$ *and* $\zeta \leqq 0$.

Proof. Setting $u = m$ and $v = n$ in (2.6), we have

$$I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{m(t)\, f(t)\, g(t)\} + I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\, f(t)\, g(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{m(t)\}$$

$$\geq I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\, g(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{m(t)\, f(t)\} + I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\, f(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{m(t)\, g(t)\} \, . \tag{2.13}$$

Multiplying both sides of (2.13) by $I_t^{\alpha,\beta,\eta,\mu} \{l(t)\}$, after a little simplification, we get

$$\begin{aligned} I_t^{\alpha,\beta,\eta,\mu} \{l(t)\} \, \Big[&I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{m(t)\, f(t)\, g(t)\} \\ &+ I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\, f(t)\, g(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{m(t)\} \Big] \\ \geq I_t^{\alpha,\beta,\eta,\mu} \{l(t)\} \, \Big[&I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\, g(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{m(t)\, f(t)\} \\ &+ I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\, f(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{m(t)\, g(t)\} \Big] \, . \end{aligned} \tag{2.14}$$

Now, by replacing u, v by l, n and u, v by l, m in (2.6), respectively, and then multiplying both sides of the resulting inequalities by $I_t^{\alpha,\beta,\eta,\mu} \{m(t)\}$ and $I_t^{\alpha,\beta,\eta,\mu} \{n(t)\}$, respectively, we get the following two inequalities

$$\begin{aligned} I_t^{\alpha,\beta,\eta,\mu} \{m(t)\} \, \Big[&I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{l(t)\, f(t)\, g(t)\} \\ &+ I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\, f(t)\, g(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{l(t)\} \Big] \\ \geq I_t^{\alpha,\beta,\eta,\mu} \{m(t)\} \, \Big[&I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\, g(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{l(t)\, f(t)\} \\ &+ I_t^{\gamma,\delta,\zeta,\nu} \{n(t)\, f(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{l(t)\, g(t)\} \Big] \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} I_t^{\alpha,\beta,\eta,\mu} \{n(t)\} \, \Big[&I_t^{\gamma,\delta,\zeta,\nu} \{m(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{l(t)\, f(t)\, g(t)\} \\ &+ I_t^{\gamma,\delta,\zeta,\nu} \{m(t)\, f(t)\, g(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{l(t)\} \Big] \\ \geq I_t^{\alpha,\beta,\eta,\mu} \{n(t)\} \, \Big[&I_t^{\gamma,\delta,\zeta,\nu} \{m(t)\, g(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{l(t)\, f(t)\} \\ &+ I_t^{\gamma,\delta,\zeta,\nu} \{m(t)\, f(t)\} \, I_t^{\alpha,\beta,\eta,\mu} \{l(t)\, g(t)\} \Big] \, . \end{aligned} \tag{2.16}$$

Finally we find that the inequality (2.12) follows by adding the inequalities (2.14), (2.15) and (2.16), sides by sides. $\qquad \square$

Remark 1. It may be noted that the inequalities (2.7) and (2.12) in Theorems 1 and 2, respectively, are reversed if the functions are asynchronous on $[0, \infty)$. The special case of (2.12) in Theorem 2 when $\alpha = \gamma$, $\beta = \delta$, $\eta = \zeta$ and

$\mu = \nu$ is easily seen to yield the inequality (2.7) in Theorem 1.

3. Special Cases and Concluding Remarks

Here we briefly consider some special cases of Theorems 1 and 2 which can easily be derived by setting (for example) $\mu = 0$; $\mu = \beta = 0$; $\mu = 0$ and $\beta = -\alpha$. Such interesting consequences of our results would involve the Saigo fractional integral operators $I_t^{\alpha,\beta,\eta}$, Erdélyi-Kober fractional integral operators $\varepsilon_t^{\alpha,\eta}$ and the Riemann-Liouville fractional integral operator $R_t^{\alpha,\eta}$. Those relatively simpler fractional integral inequalities involving the Saigo fractional integral operators $I_t^{\alpha,\beta,\eta}$, Erdélyi-Kober fractional integral operators $\varepsilon_t^{\alpha,\eta}$ and the Riemann-Liouville fractional integral operator $R_t^{\alpha,\eta}$ can be deduced from Theorems 1 and 2 by appropriately applying the following relationships (see, e.g., [13]):

$$
\begin{aligned}
I_t^{\alpha,\beta,\eta}\{f(t)\} &:= \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau)d\tau \\
&= I_t^{\alpha,\beta,\eta,0}\{f(t)\},
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\varepsilon_t^{\alpha,\eta}\{f(t)\} &:= \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \, \tau^\eta f(\tau)d\tau \ (\alpha > 0, \eta \in \mathbb{R}) \\
&= I_t^{\alpha,0,\eta,0}\{f(t)\}
\end{aligned}
\tag{3.2}
$$

and

$$
R_t^\alpha\{f(t)\} := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \ (\alpha > 0) = I_t^{\alpha,-\alpha,\eta,0}\{f(t)\}.
\tag{3.3}
$$

We conclude our present investigation by remarking further that the results obtained here are useful in deriving various fractional integral inequalities involving such relatively more familiar fractional integral operators. For example, if we consider $\mu = 0$ (and $\nu = 0$ additionally for Theorem 2), and make use of the relation (3.1), Theorems 1 and 2 provide, respectively, the known fractional integral inequalities due to Choi and Agarwal [5].

Again, for $\mu = 0$ and $\beta = 0$ in Theorems 1 and $\mu = \nu = 0$ and $\beta = \delta = 0$ in Theorem 2, and make use of the relation (3.2), Theorems 1 and 2 provide, respectively, the known fractional integral inequalities due to Choi and Agarwal [5].

Finally, if we take $\mu = 0$ and $\beta = -\alpha$ in Theorem 1 and $\mu = 0$, $\beta = -\alpha$ and $\delta = -\gamma$, then Theorems 1 and 2 yield the known result due to Dahmani [7].

We may also emphasize that results derived in this paper are of general character and can specialize to give further interesting and potentially useful formulas involving fractional integral operators.

References

[1] G. A. Anastassiou, *Advances on Fractional Inequalities*, Springer Briefs in Mathematics, Springer, New York, 2011.

[2] D. Baleanu, S. D. Purohit and P. Agarwal, On fractional integral inequalities involving hypergeometric operators, Chinese J. Math. Vol. **2014** (2014), Article ID 609476, 5 pages `http://dx.doi.org/10.1155/2014/609476`

[3] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, J. Inequal. Pure Appl. Math. **10**(3)(2009), Art. 86, 5 pp (electronic).

[4] P. L. Chebyshev, *Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites*, Proc. Math. Soc. Charkov **2** (1882), 93–98.

[5] J. Choi and P. Agarwal, *Some new Saigo type fractional integral inequalities and their q-analogues*, Abstr. Appl. Anal. 2014, Vol. 2014(2014), Article ID 579260, 11 pages.

[6] L. Curiel and L. Galué, *A generalization of the integral operators involving the Gauss' hypergeometric function*, Rev. Téc. Ing. Unlv. Zulla. **19**(1) (1996), 17–22.

[7] Z. Dahmani, *New inequalities in fractional integrals*, Int. J. Nonlinear Sci. **9** (2010), 493–497.

[8] Z. Dahmani, O. Mechouar and S. Brahami, *Certain inequalities related to the Chebyshev's functional involving a type Riemann-Liouville operator*, Bull. Math. Anal. Appl. **3**(4) (2011), 38–44.

[9] S. S. Dragomir, *Some integral inequalities of Grüss type*, Indian J. Pure Appl. Math. **31**(4) (2000), 397–415.

[10] S. L. Kalla and A. Rao, *On Grüss type inequality for hypergeometric fractional integrals*, Matematiche (Catania) **66**(1) (2011), 57–64.

[11] J. C. Kuang, *Applied Inequalities*, Shandong Sciences and Technologie Press, 2004 (Chinese).

[12] V. Lakshmikantham and A. S. Vatsala, *Theory of fractional differential inequalities and applications*, Commun. Appl. Anal. **11** (2007), 395–402.

[13] A. M. Mathai, R. K. Saxena and H. J. Haubold, *The H-Function: Theory and Applications*, Springer, New York, Dordrecht, Heidelberg and London, 2010.

[14] D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[15] H. Öğünmez and U.M. Özkan, *Fractional quantum integral inequalities*, J. Inequal. Appl. 2011, Article ID 787939, 7 pp.

[16] A. M. Ostrowski, *On an integral inequality*, Aequations Math. **4** (1970), 358–373.

[17] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[18] W. T. Sulaiman, *Some new fractional integral inequalities*, J. Math. Anal. **2**(2) (2011), 23–28.

Department of Mathematics, Dongguk University
Gyeongju 780-714, Republic of Korea
E-mail address: `junesang@mail.dongguk.ac.kr`

Department of Mathematics, Anand International College of Engineering
Jaipur-303012, India
E-mail address: `praveen.agarwal@anandice.ac.in; goyal.praveen2011@gmail.com`